ON THE CONSTRUCTION OF CONVERGENT ITERATIVE SEQUENCES OF POLYNOMIALS

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Abstract

We answer two conjectures suggested by Zalman Rubinstein. We prove his Conjecture 1, that is, we construct convergent iterative sequences for \( f_m(z) = z + z^m \) with an arbitrary initial point, where \( f_m(z) = z + z^m \) with \( m \geq 2 \). We also show by several counterexamples that Rubinstein's Conjecture 2 is generally false.


1. Introduction

Zalman Rubinstein constructed convergent iterative sequences for the polynomials \( f(z) = z + z^m, \quad m \geq 2 \), with initial point in the lemniscate \( \{ z \mid |f'(z)| \leq 1 \} \) by variational methods. His main results showed that for every point \( z_0 \in \{ z \mid |f'(z)| \leq 1 \} \), the iterative sequence \( z_{n+1} = f(z_n), \quad n = 0, 1, \ldots \), converges to 0 as \( n \to \infty \). In the particular case \( m = 2 \), convergent iterative sequences were constructed also for \( f^{-1}(z) \) with an arbitrary initial point. For the case \( m > 2 \), and more generally, for polynomials with positive real coefficients, the following two conjectures were mentioned in [1].

**Conjecture 1.** Let \( f(z) = z + z^m, \quad m \geq 2 \). There exists a determination of \( f^{-1}(z) \) such that for every \( z_0 \in \mathbb{C} \) the sequence \( z_n = f^{-1}(z_{n-1}) \) tends to zero as \( n \to \infty \).

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Conjecture 2. Let \( f(z) = z + a_2z^2 + \cdots + a_mz^m \) be of degree \( m \geq 2 \), and assume that \( a_k \geq 0 \) for all \( k \). Then for every \( z_0 \) such that \( |f'(z_0)| \leq 1 \), the sequence \( z_{n+1} = f(z_n) \) converges.

In this paper, we will discuss the above two problems. We will show that Conjecture 1 is true, while Conjecture 2 is generally false, by way of several counterexamples.

2. Definitions and lemmas

We need some results of the Fatou and Julia theory of iteration ([3], [4] and [5]; also see [2]). Let \( f(z) \) be a polynomial. Denote \( f^n = f \circ f \circ \cdots \circ f \) as the \( n \)th order iteration of \( f \). The Fatou set \( F \) of \( f \) is the maximal open set in which \( \{f^n\} \) is a normal family. The Julia set \( J \) of \( f \) is the complement of \( F \). The point \( z \) is called an \( n \)th order periodic point if \( f^n(z) = z \) and \( f^k(z) \neq z \) for all \( 0 < k < n \). Such an \( n \)th order periodic point \( z \) is called attractive (repulsive or rationally indifferent respectively) if \( |(f^n)'(z)| < 1 \) (\( |(f^n)'(z)| > 1 \) or \( (f^n)'(z) \) is a root of unity respectively). We also call \( \{f^n(z)\} \) a forward orbit of \( f \) at \( z \), and denote by \( f^{-n}(z) \) the inverse images of \( f^n \) at \( z \), for \( n = 1, 2, \ldots \). Every branch of \( f^{-n}(z) \) on a domain is denoted by \( f_j^{-n}(z) \).

The following results of Fatou and Julia will be used.

1. \( F \) is open. \( J \) is perfect and non-empty. \( F \) and \( J \) are completely invariant under \( f \), that is, \( f(F) = f^{-1}(F) = F \), etc.

2. The Julia set coincides with the closure of the set of repulsive periodic points.

3. Every attractive periodic point is in \( F \) and every repulsive or rational indifferent periodic point in \( J \).

4. If \( f \) is a polynomial, then the unbounded component \( A(\infty) \) of \( F \) is exactly the set of all points whose iterative sequences tend to infinity.

5. If \( z_0 \) is not a limit point of the forward orbit of some point \( z \notin J \), then every accumulation point of \( \{f^{-n}(z)\} \) belongs to \( J \).

6. Let \( \{f_j^{-n}(z)\}_{j,n} \) be any infinite set of inverse branches which are holomorphic in a domain \( D \), and suppose that there exists an open subset of \( D \) containing no limit points of the forward orbit of any point \( z \notin J \). Then \( \{f_j^{-n}(z)\} \) is normal in \( D \) and every convergent subsequence tends to a constant.
Now suppose that $g(z) = z + a_m z^m + \cdots$ is a power series analytic at the origin. For $0 < \theta < \pi/2$ and sufficiently small $\rho > 0$, we define the domain

$$D(j, \theta, \rho) = \left\{ z \mid 0 < |z| < \rho, -\gamma - \frac{(2j - 2)\pi}{m - 1} - \frac{\pi - \theta}{m - 1} < \arg z < -\gamma - \frac{(2j - 2)\pi}{m - 1} + \frac{\pi - \theta}{m - 1} \right\}$$

for $j = 1, 2, \ldots, m - 1$ and the "star domain" $D(\theta, \rho) = \bigcup_{j=1}^{m-1} D(j, \theta, \rho)$, where $\gamma$ is a constant satisfying $-a_m \exp\{-i\gamma(m - 1)\} > 0$.

**Lemma 1** [6, Lemma 9]. Let $g(z) = z + a_m z^m + \cdots$ be analytic at the origin. Then for given $0 < \theta < \pi/2$ and sufficiently small $\rho > 0$, we have $g(D(\theta, \rho)) \subset D(\theta, \rho)$ and the iteration $g^n(z)$ converges to zero locally uniformly in $D(\theta, \rho)$.

**Lemma 2.** Let $f(z) = z + z^m$. Then $\{z \mid z^{m-1} \in \mathbb{R}\}$, which we abbreviate to $\{z^{m-1} \in \mathbb{R}\}$, and $\{z^{m-1} > 0\}$ are both invariant under $f$, and $\{z^{m-1} > 0\} \subset A(\infty)$.

**Proof.** If $z = r e^{i\pi/(m-1)} \in \{z^{m-1} \in \mathbb{R}\}$, $0 \leq r < +\infty$, then

$$f(z) = r e^{i\pi/(m-1)}(1 \pm r^{m-1}) \in \{z^{m-1} \in \mathbb{R}\}.$$  

If $z = r e^{2i\pi/(m-1)} \in \{z^{m-1} > 0\}$, $0 < r < +\infty$, then

$$(f(z))^{m-1} = ((r + r^m)e^{2i\pi/(m-1)})^{m-1} = (r + r^m)^{m-1} > 0.$$  

These show that $\{z^{m-1} \in \mathbb{R}\}$ and $\{z^{m-1} > 0\}$ are both invariant under $f$.

Because $f(z) = (|z| + |z|^m)e^{2i\pi/(m-1)}$ for $z \in \{z^{m-1} > 0\}$, and also $|f(z)| = |z| + |z|^m \geq |z|$, we have by induction that

$$f^n(z) = |f^{n-1}(z)|(1 + |f^{n-1}(z)|^{m-1})e^{2i\pi/(m-1)}$$

$$= f^{n-1}(z)(1 + |f^{n-1}(z)|^{m-1})$$

$$= z \prod_{k=0}^{n-1} (1 + |f^k(z)|^{m-1}).$$

Hence

$$|f^n(z)| \geq |z|(1 + |z|^{m-1})^n = \rho(1 + \rho^{m-1})^n,$$

which tends to infinity as $n \to \infty$, that is $\{z^{m-1} > 0\} \subset A(\infty)$, from Result 4 above.

**Lemma 3.** Let $l_k = \{z \mid z = re^{(2k+1)i\pi/(m-1)}, -\infty < r < +\infty\}$, $k = 1, 2, \ldots, m - 1$, be a straight line in $\{z^{m-1} \in \mathbb{R}\}$, and let $h_k$ be the subset of $l_k$,

$$h_k = \{z \mid z = re^{(2k+1)i\pi/(m-1)}, \rho > \rho_0\},$$
where \( \rho_0 = ((m - 1)/m)(1/m)^{1/(m-1)} \). Then if \( m \) is even, all \( m \) branches of \( f^{-1}(h_k) \) are disjoint from \( \{z^{m-1} \in \mathbb{R}\} \). If \( m \) is odd, there is a branch of \( f^{-1}(h_k) \):

\[
\{ z | z = re^{i(2k+1)\pi i/(m-1)}, -\infty < r < r_0 < -(1/m)^{1/(m-1)} \},
\]

which is contained in \( l_k \). The other \( m - 1 \) branches of \( f^{-1}(h_k) \) are disjoint from \( \{z^{m-1} \in \mathbb{R}\} \).

**Proof.** We first prove that \( f^{-1}(h_k) \cap \{z^{m-1} \in \mathbb{R}\} \subset l_k \). In fact, if \( z = re^{i\theta} \in f^{-1}(h_k) \cap \{z^{m-1} \in \mathbb{R}\} \), we have \( e^{i(m-1)\theta} = \pm 1 \) and there is \( \rho > \rho_0 \) such that \( z^m + z = \rho e^{i(2k+1)\pi i/(m-1)} \), that is,

\[
re^{i\theta}(1 \pm r^{m-1}) = \rho e^{i(2k+1)\pi i/(m-1)}.
\]

Now \( \rho \neq 0 \) implies \( r \neq 0 \) and \( (1 \pm r^{m-1}) \neq 0 \). Thus, the above equality shows that \( z = re^{i\theta} \) and \( f(z) = \rho e^{i(2k+1)\pi i/(m-1)} \) lie on the same straight line \( l_k \).

However, if \( z = re^{i(2k+1)\pi i/(m-1)} \in l_k \) with \( r \) real and \( z \in f^{-1}(h_k) \), then we have

\[
r - r^m = \rho \quad \text{where} \quad \rho > \rho_0
\]

or

\[
\varphi_\rho(r) = r^m - r + \rho = 0.
\]

It is easy to check that when \( m \) is even and \( \rho > \rho_0 \), the equation has no real root, so \( f^{-1}(h_k) \cap \{z^{m-1} \in \mathbb{R}\} = \emptyset \).

If \( m \) is odd, there is a unique real root \( r_\rho \) of equation \( \varphi_\rho(r) = 0 \) and \( r_\rho \) belongs to the interval \((-\infty, r_1)\), where \( r_1 = -(1/m)^{1/(m-1)} \). We now want to prove that the real root \( r_\rho \) is a one-to-one continuous function of \( \rho \) when \( \rho > \rho_0 \). Suppose \( \rho_0 < \rho, \rho' \). Then \( r_\rho - r^m_\rho = \rho \) and \( r_\rho' - r^m_\rho = \rho' \). We have

\[
r_\rho' - r_\rho - (r^m_\rho - r^m_\rho) = \rho' - \rho,
\]

or

\[
(r_\rho' - r_\rho) \left( 1 - \sum_{k=0}^{m-1} r^k_\rho r^{m-1-k}_\rho \right) = \rho' - \rho.
\]

Since \( r_\rho, r_\rho' \) are both less than \( r_1 = -(1/m)^{1/(m-1)} \), we have that \( r^k_\rho r^{m-1-k}_\rho \) is more than \( 1/m \) for \( k = 0, 1, \ldots, m-1 \). This means that \( \sum_{k=0}^{m-1} r^k_\rho r^{m-1-k}_\rho > 1 \) or \( 1 - \sum_{k=0}^{m-1} r^k_\rho r^{m-1-k}_\rho < 0 \). Hence \( \rho' > \rho \) implies \( r_\rho' < r_\rho \). We have thus shown that \( r_\rho \) is a strictly monotone function for \( \rho > \rho_0 \). If we fix \( \rho > \rho_0 \) and let \( \rho' \) be sufficiently close to \( \rho \), we can be sure that \( r_\rho \) and \( r_\rho' \) are all less than a constant \( c < r_1 \). Then \( \sum_{k=0}^{m-1} r^k_\rho r^{m-1-k}_\rho - 1 \) will be greater than a positive constant \( \delta \) (dependent only on \( \rho \)). Hence, from the equality

\[
|r_\rho' - r_\rho| = \frac{|\rho' - \rho|}{|1 - \sum_{k=0}^{m-1} r^k_\rho r^{m-1-k}_\rho|}
\]
it follows that \( r_\rho \) is continuous for \( \rho > \rho_0 \). We now know that the ray line \( \{r_\rho e^{(2k+1)\pi i/(m-1)} | \rho > \rho_0 \} \) is a branch of \( f^{-1}(h_k) \) contained in \( \{z | z = r_\rho e^{(2k+1)\pi i/(m-1)}, -\infty < r < r_1 \} \). By the monotonicity and continuity of \( r_\rho \), that branch is
\[
\{z | z = r e^{(2k+1)\pi i/(m-1)}, -\infty < r < r_0 < r_1 \},
\]
with endpoint \( r_0 \), the negative root of the equation \( r - r^m = \rho_0 \). And the other branches of \( f^{-1}(h_k) \) are disjoint from \( \{z^m \in \mathbb{R} \} \).

**Lemma 4.** The figure of \( f^{-1}(h_k) \) is symmetrical about the straight line \( l_k \):

**Proof.** Let \( z_1 = r e^{((2k+1)\pi/(m-1)) + \theta) i} \in f^{-1}(h_k) \). We will prove that \( z_2 = r e^{((2k+1)\pi/(m-1)) - \theta) i} \in f^{-1}(h_k) \), where \( r, \theta \) are real. In fact, there is \( \rho > \rho_0 \) such that
\[
Z_1^m + z_1 = r e^{((2k+1)\pi/(m-1)) + \theta) i} (1 + r e^{-(m-1)\theta) i}) = r e^{(2k+1)\pi i/(m-1)}.
\]
That is
\[
(r \cos \theta + r \cos m\theta) + i(r \sin \theta + r \sin m\theta) = \rho,
\]
or
\[
r \sin \theta + r \sin m\theta = 0.
\]
Hence
\[
Z_2^m + z_2 = r e^{((2k+1)\pi/(m-1)) - \theta) i} (1 + r e^{-(m-1)\theta) i})
\]
\[
= ((r \cos \theta + r \cos m\theta) - i(r \sin \theta + r \sin m\theta)) e^{(2k+1)\pi i/(m-1)}
\]
\[
= r e^{(2k+1)\pi i/(m-1) = Z_1^m + z_1}.
\]
This shows the symmetry of the figure of \( f^{-1}(h_k) \).

### 3. Theorem and its proof

Let \( f(z) = z + z^m, m \geq 2 \). The critical points (singularities) of \( f^{-1}(z) \) are
\[
c_k = \rho_0 e^{(2k+1)\pi i/(m-1)}, \quad k = 1, 2, \ldots, m - 1,
\]
where \( \rho_0 = ((m-1)/m)(1/m)^{1/(m-1)} \), and \( \infty \). Let \( L = \{z | z = \rho e^{(2k+1)\pi i/(m-1)}, \rho_0 < \rho < +\infty, k = 1, 2, \ldots, m - 1 \} \). Then we can choose a single-value analytic branch of \( f^{-1} \) on the domain \( C \setminus L \). We have

**Theorem.** Let \( f(z) = z + z^m, m \geq 2 \). Then there exists an analytic determination of \( f^{-1}(z) \) in \( C \setminus L \) which satisfies \( f^{-1}(0) = 0 \), is continuous to
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The one-sidedly, and is such that for every \( z_0 \in \mathbb{C} \) the sequence \( z_n = f^{-1}(z_{n-1}) \) tends to zero as \( n \to \infty \).

**Proof.** We choose \( f^{-1}(z) \) in \( \mathbb{C} \setminus \overline{\mathbb{L}} \) that is an inverse analytic branch of \( f \) satisfying \( f^{-1}(0) = 0 \), and choose \( f^{-1} \) on \( \overline{\mathbb{L}} = \bigcup_{k=1}^{m-1} \mathbb{h}_k \) that maps \( h_k \) onto one of the inverse branches of \( h_k \) ending at \( z_k = (1/m)^{(m-1)z/(m-1)} e^{(2k+1)x/(m-1)} \) and \( f^{-1}(c_k) = z_k \) for \( k = 1, 2, \ldots, m \). Thus \( f^{-1}(z) \) is well defined on \( \mathbb{C} \). We have \( f^{-1}(z) \) is continuous when \( z \) tends to \( \overline{\mathbb{L}} \) from one side of \( h_k \). In fact, there are two inverse branches of \( h_k \) ending at \( z_k \). By Lemma 3, they do not lie on \( l_k \) and are disjoint from \( \{z^{m-1} \in \mathbb{R}\} \). By Lemma 4, they are symmetrical about \( l_k \). So they, when \( z_k \) is added, form a curve through the point \( z_k \) which is symmetrical about \( l_k \) and separates \( \mathbb{C} \) into two regions. For \( k = 1, 2, \ldots, m \), there are \( m \) such curves separating \( \mathbb{C} \) into \( m \) regions, only one of them containing the origin. Then the region containing the origin is the image domain of \( \mathbb{C} \setminus \overline{\mathbb{L}} \) under \( f^{-1} \) as \( f^{-1}(0) = 0 \). Also \( f^{-1}(z) \) constructed as above is continuous to \( \overline{\mathbb{L}} \) one-sidedly. Moreover, \( f^{-1}(L) \cap \{z^{m-1} \in \mathbb{R}\} = \emptyset \).

Obviously, \( f^{-1}(z) \) is analytic at \( z = 0 \) with an expansion

\[
f^{-1}(z) = z - z^m + \cdots.
\]

Let \( G = \mathbb{C} \setminus \{z^{m-1} \in \mathbb{R}\} \), which is such that \( G \subset \mathbb{C} \setminus \overline{\mathbb{L}} \). By Lemma 2, \( f^{-1}(G) \subset G \subset \mathbb{C} \setminus \overline{\mathbb{L}} \). Now \( G \) is the union of \( 2(m-1) \) components \( G_j, j = 1, 2, \ldots, 2(m-1) \), each \( G_j \) being a simply connected unbounded sector. Given \( G_j \) for some \( j \), \( f^{-n}(z) \) is analytic in \( G_j \) for all \( n > 0 \). Since \( f^n(z) \) tends to infinity uniformly for \( z \) sufficiently large, there exists a region in \( G_j \) containing no limit points of the forward orbit of any \( z \in \mathbb{C} \). By Result 6 of Fatou and Julia, \( \{f^{-n}\} \) is normal in \( G_j \) and every convergent subsequence tends to a constant. By Lemma 1, with \( 0 < \theta < \pi/2 \) and sufficiently small \( \rho > 0 \), \( f^{-n}(z) \) tends to 0 locally uniformly in the domain \( D(\theta, \rho) = \bigcup_{j=1}^{m-1} D(j, \theta, \rho) \) where

\[
D(j, \theta, \rho) = \left\{ z \mid 0 < |z| < \rho, \frac{(2j-2)\pi}{m-1} - \frac{\pi - \theta}{m-1} < \arg z < -\frac{(2j-2)\pi}{m-1} + \frac{\pi - \theta}{m-1} \right\}.
\]

Since the intersection between \( G_j \) and \( D(\theta, \rho) \) is nonempty, every convergent subsequence of \( \{f^{-n}(z)\} \) tends to zero in \( D(\theta, \rho) \cap G_j \) and so tends to zero in \( G_j \) for \( j = 1, 2, \ldots, 2(m-1) \). This shows that \( \{f^{-n}(z)\} \) tends to zero in \( G = \bigcup_{j=1}^{2(m-1)} G_j \).

Next, we consider the convergence of \( f^{-n}(z) \) in the set \( \{z^{m-1} \in \mathbb{R}\} \). If \( z \in L \), then \( f^{-1}(z) \in G \) from Lemmas 3, 4 and the construction of \( f^{-1} \).
The above discussion shows that \( f^{-n}(z) \) tends to zero as \( n \to \infty \). If \( z = 0 \) then \( f^{-n}(0) \equiv 0 \) for all \( n > 0 \). We will prove that \( \{z^{m-1} \in \mathbb{R}\} \setminus (L \cup \{0\}) = \{z^{m-1} > 0\} \cup \{z^{m-1} < 0\} \setminus L \) lies in the Fatou set of \( f \).

By Lemma 2, \( \{z^{m-1} > 0\} \subset A(\infty) \subset F \). Let \( R = \{z^{m-1} < 0\} \setminus L = \{z = re^{(2k+1)i/(m-1)}, 0 < r \leq \rho_0, k = 1, 2, \ldots, m-1\} \). For \( z = re^{(2k+1)i/(m-1)} \in \mathbb{R}, f'(z) = 1 + mz^{m-1} = 1 - mr^{m-1} \) so \( |f'(z)| < 1 \) as \( 0 < r \leq \rho_0 \). This implies that \( |f'(z)| < 1 \) as \( z \in \mathbb{R} \) except for \( z = 0 \). By [1, Lemma 2 and Theorem 1], we get \( R \subset F, \) \( \overline{R} \cap \{0\} = \{0\} \) and \( R \) contains no limit points of forward orbits of points in \( C \). Also \( \{z^{m-1} > 0\} \subset A(\infty) \) contains no limit points of forward orbits. From Result 5 above the accumulation points of \( \{f^{-n}(z)\} \) belong to the Julia set, for every \( z \in \{z^{m-1} \in \mathbb{R}\} \setminus (L \cup \{0\}) \). If \( f^{-n}(z) \in \{z^{m-1} \in \mathbb{R}\} \setminus L \) for all \( n > 0, f^{-n}(z) \to 0 \) as \( n \to \infty \) since \( \{z^{m-1} \in \mathbb{R}\} \setminus L \cap J = \{0\} \). Otherwise, there exists an integer \( n > 0 \) such that \( w = f^{-n}(z) \notin \{z^{m-1} \in \mathbb{R}\} \setminus L \). But we have shown, for \( w \notin \{z^{m-1} \in \mathbb{R}\} \setminus L \), that, for \( w \in G \) or \( w \in L \), that \( f^{-n}(w) \) tends to zero as \( n \to \infty \). Hence \( f^{-n}(z) \) also tends to zero as \( n \to \infty \).

**Corollary.** Let \( f(z) = z + z^m, m \geq 2 \). Then for every \( z_0 \in \{z| |f'(z)| \leq 1\} \), there exists a sequence \( \{z_n\} \) such that \( z_{n+1} = f(z_n) \) and \( z_n \to 0, z_{-n} \to 0 \) as \( n \to \infty \).

**Proof.** This is a direct consequence of the above theorem and [1, Theorem 1].

## 4. Counterexamples

In this section we will give two examples to show that Conjecture 2 is false.

**Example 1.** Let \( f(z) = z(1 + az)^2, a > 1 \), be a polynomial with positive real coefficients. Now \( f(-1/a) = 0 \in J \) (since \( f(0) = 0 \) and \( f'(0) = 1 \) is a root of unity, from Result 3) and \(-1/a \in J \) (since the Julia set is completely invariant, from Result 1). It is easy to see that \( f'(-1/a) = 0 \), so that \(-1/a \) is in \( D \), one of the components of \( \{z||f'(z)|| < 1\} \). But \( J \) is a perfect set and the repulsive periodic points of \( f \) are dense in \( J \) from Results 1 and 2. There exists at least one repulsive periodic point \( p \in D \) with period not less than 2. Thus \( f^n(p) \) does not converge.

Since \(-1/2 \leq f'(z) < 1 \) when \( z \in [-1/a, 0) \), we have \( D \supset [-1/a, 0) \). So the origin is a boundary point of \( D \). If we restrict the initial point to be in the component of \( \{z||f'(z)|| < 1\} \) with boundary point 0, the result is also not true.

In this example, we showed that for a polynomial with positive real coefficients \( f(z) \), the set \( \{z||f'(z)|| < 1\} \) may contains some points in \( J \). The
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next example shows that there exists such a polynomial for which there is a region in \( \{z \mid |f'(z)| < 1\} \cap F \) in which iterative sequences of all points are divergent.

Let \( z_0 \in \mathbb{C} \) be a fixed point of polynomial \( f(z) \), and suppose that \( \lambda = f'(z_0) = e^{2\pi i \omega} \). Then we have

**Lemma 5** (Siegel [7]). Let \( \omega \) be an irrational. Suppose there are positive constants \( a \) and \( b \) satisfying \( |\omega - (m/n)| > a/n^b \) for all integers \( m, n \) with \( n \geq 1 \). Then there exists a neighbourhood \( U \) of \( z_0 \) and a homeomorphism \( \varphi: U \to D_r = \{z \mid |z| < r\} \), \( \varphi(z_0) = 0 \), such that \( \varphi \circ f \circ \varphi^{-1}(\zeta) = e^{2\pi i \omega} \zeta \).

The set of \( \omega \) satisfying the condition of Lemma 5 is dense in interval \([0, 1]\).

We will construct a polynomial \( f(z) \) satisfying the condition of Conjecture 2, which has a fixed point \( z_0 \) different from \( 0 \) and is such that \( \lambda = f'(z_0) = e^{2\pi i \omega} \), where \( \omega \) satisfies the condition of Lemma 5. For \( g(\zeta) = e^{2\pi i \omega} \zeta: D_r \to D_r \), when \( \zeta_1 \in D_r \) and \( \zeta_1 \neq 0 \), its iterative sequence \( \{\zeta_n\} \), \( \zeta_n = g(\zeta_{n-1}) = e^{2\pi i \omega} \zeta_1 \) is dense on circle \( \{|\zeta| = |\zeta_1|\} \). Thus \( \zeta_n \) does not converge as \( n \to \infty \), and therefore, for \( z_1 = \varphi^{-1}(\zeta_1) \), \( z_1 \neq z_0 \), \( z_{n+1} = f(z_n) \) is also not convergent as \( n \to \infty \). Since \( |f'(z_0)| = 1 \), we deduce, using the minimum principle, that there is a region \( V \) in \( U \) disjoint from \( z_0 \) such that for all \( z \in V \), \( |f'(z)| < 1 \).

This is all we need.

**Example 2.** Choose \( \omega \in [0, 1] \), satisfying the condition of Lemma 5. Let \( \theta = (2 + \omega)/4 \). Then \( \pi < 2\pi \theta < 3\pi/2 \) or \( \cos 2\pi \theta < 0 \). Let \( r = (|e^{2\pi i \omega} - 1|/|e^{4\pi i \theta} - 1|)^{1/3} \). Let

\[
\begin{align*}
f(z) &= z + z^2(z - re^{2\pi i \theta})(z - re^{-2\pi i \theta}) \left( z + r^2 z^2 - 2r \cos(2\pi \theta) z^3 + z^4 \right.
\end{align*}
\]

Then \( f(z) \) is a polynomial with positive real coefficients having nonzero fixed point \( z_0 = re^{2\pi i \theta} \).

\[
f'(z_0) = 1 + 2r^2 z_0 - 3r(e^{2\pi i \theta} + e^{-2\pi i \theta}) z_0^2 + 4z_0^3
= 1 + r^3 e^{2\pi i \theta}(e^{4\pi i \theta} - 1).
\]

Since \( e^{i\alpha} - 1 = |e^{i\alpha} - 1|e^{i(\pi + \alpha)/2} \) for real \( \alpha \),

\[
r^3 e^{2\pi i \theta}(e^{4\pi i \theta} - 1) = r^3 e^{2\pi i \theta}|e^{4\pi i \theta} - 1|e^{(4\pi \theta + \pi)i/2}.
\]

When \( \theta = (2 + \omega)/4 \) and \( r = (|e^{2\pi i \omega} - 1|/|e^{4\pi i \theta} - 1|)^{1/3} \), we get

\[
f'(z_0) = 1 + |e^{2\pi i \omega} - 1|e^{8\pi((2+\omega)/4+\pi)i/2}
= 1 + |e^{2\pi i \omega} - 1|e^{(2\pi \omega + \pi)i/2} = e^{2\pi i \omega}.
\]

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This completes the construction of our example.

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References


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