Ergod. Th. & Dynam. Sys. (1988), **8***, 301–358
Printed in Great Britain

Isospectral sets for boundary value problems on the unit interval

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Abstract. We analyse isospectral sets of potentials associated to a given 'generalized periodic' boundary condition $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2, \mathbf{R})$ for the Sturm–Liouville equation on the unit interval. This is done by first studying the larger manifold M of all pairs of boundary conditions and potentials with a given spectrum and characterizing the critical points of the map from M to the trace $a+d$. Isospectral sets appear as slices of M whose geometry is determined by the critical point structure of the trace function. This paper completes the classification of isospectral sets for all real self-adjoint boundary conditions.

0. *Introduction*

We consider the differential operators

$$L(q)y = -y'' + q(x)y, \qquad x \in [0, 1],$$

for q real-valued and square-integrable, i.e. $q \in L^2_{\mathbf{R}}[0, 1]$, with real self-adjoint boundary conditions. Considering boundary data $(y(0), y'(0), y(1), y'(1))$ as vectors in $\mathbf{R}^4$, the set of real self-adjoint boundary conditions can be identified with the set of Lagrangian planes in $\mathbf{R}^4$, i.e. the set of two-dimensional subspaces on which the bilinear form $\langle v, w\rangle = v_1 w_2 - v_2 w_1 - v_3 w_4 + v_4 w_3$ vanishes. The real self-adjoint boundary conditions split naturally into the 'separated' boundary conditions, i.e. those of the form

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad y(1)\cos\beta + y'(1)\sin\beta = 0,$$

where $(\alpha, \beta) \in [0, \pi) \times [0, \pi)$, and the 'generalized periodic' boundary conditions, i.e. those of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix},$$

where $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$. The separated boundary conditions include the Dirichlet $(\alpha = \beta = 0)$ and Neumann $(\alpha = \beta = \pi/2)$ boundary conditions, and the generalized periodic boundary conditions include the periodic $(\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) = I)$ and anti-periodic $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = -I)$ boundary conditions.

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Our objective here is an inverse spectral theory for L. Given a spectrum, i.e. the eigenvalues of L with their multiplicities for given boundary conditions, we would like as complete as possible a description of the set of $q \in L^2_{\mathbf{R}}[0, 1]$ for which L with the given boundary conditions has the given spectrum. This paper is intended as the last in a series that began with the study of the periodic boundary condition in [**5**]. The case of the Dirichlet boundary condition was discussed in some detail in [**6**]. The remaining separated cases were studied in [**1**], [**3**] and [**4**]. Finally Johan Tysk considered the generalized periodic case with $b = 0$. Since his work has not appeared elsewhere, though it was completed before this paper was begun, we have included it here as Appendix A.

The case of $b \neq 0$ in the generalized periodic boundary condition is the subject of this paper. In the following sense this is the most general case. There is a natural sequence of degenerations of these boundary conditions. One can send $b \downarrow 0$, then $a \to 1$ and finally $c \to 0$, thus moving from the boundary conditions here down to the periodic case. In the other direction one can reach the separated conditions

$$-d_0 y(0) + y'(0) = 0, \qquad a_0 y(1) + y'(1) = 0 \tag{0.1}$$

by letting $b \uparrow \infty$ in the generalized periodic conditions

$$\begin{pmatrix} a_0 b & b \\ a_0 d_0 b - b^{-1} & d_0 b \end{pmatrix} \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix},$$

and then send $a_0 \to \infty$ and finally $d_0 \to \infty$, thus moving from the boundary conditions here to the Dirichlet case. Each of the steps described above produces significant changes in the asymptotic behaviour of the eigenvalues as they tend to ∞ and/or in the structure of the isospectral sets. At one extreme the Dirichlet isospectral sets with the topology induced from $L^2_{\mathbf{R}}[0, 1]$ are homeomorphic to $l^2(\mathbf{Z})$, while at the other extreme the periodic isospectral sets are compact tori, generically of infinite dimension. In this paper we will begin to see how the isospectral sets for generalized periodic boundary conditions 'interpolate' between these extremes.

As in the study of the separated boundary conditions (0.1) in [**4**], we have found it easier to begin by considering the set of potentials *and* boundary conditions which give a fixed spectrum. Identifying boundary conditions with matrices

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we let M be the set of

$$(q, B) \in L^2_{\mathbf{R}}[0, 1] \times (\mathrm{SL}(2, \mathbf{R}) \backslash \{b = 0\})$$

such that $L(q)$ with boundary condition B has the spectrum of $L(q_0)$ with boundary condition B_0. For all choices of (q_0, B_0) in $L^2_{\mathbf{R}}[0, 1] \times (\mathrm{SL}(2, \mathbf{R}) \backslash \{b = 0\})$ the structure of M is essentially the same. On M one has a countable family of periodic commuting flows which fix the entry d in B, and d takes all real values on M. Using the flows, one can map M into an infinite product of circles, and this map is injective on each level set $M \cap \{d = \delta\}$, $\delta \in \mathbf{R}$. However, $M \cap \{d = \delta\}$ is only a subset of the infinite product of circles. The full description of $M \cap \{d = \delta\}$ is given in § 3, but one can picture it simply as follows: if we let $\theta_n \in [-n, n)$ parametrize the nth circle, then the image of $M \cap \{d = \delta\}$ is

$$\{(\theta_1, \theta_2, \ldots) : \sum (\theta_n - \theta(\delta))^2 < \infty\},$$

where the constant $\theta(\delta)$ goes from ∞ to $-\infty$ as δ goes from ∞ to $-\infty$. Note that one cannot change d on M without changing an infinite number of the θ_n as well.

Once we have the explicit description of $M \cap \{d = d_0\}$ outlined in the preceding paragraph, it remains to fix the remaining entries in B and study $M \cap \{B = B_0\}$. This brings us back to the problem of isospectral sets of q for fixed boundary conditions. Fortunately, b is constant on M, and hence, since $\det B = 1$,

$$M \cap \{B = B_0\} = M \cap \{a = a_0, d = d_0\},$$

so that we only need to study the level sets of the entry a as a function on $M \cap \{d = \delta\}$. To get information on the topology of these level sets, we need to study the critical points of a on $M \cap \{d = \delta\}$. At this point a fortunate accident occurs. It happens that p is a critical point for a as a function on $M \cap \{d = \delta\}$ if and only if p is a critical point for $a + d$ as a function on M. Moreover, p is critical for $a + d$ on M if and only if it is fixed by the involution

$$q(x) \to q(1-x), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

This involution leaves M invariant and is the extension of the involution for the boundary conditions (0.1),

$$q(x) \to q(1-x), \qquad a_0 \to d_0, \qquad d_0 \to a_0,$$

which was important in [4]. To each fixed point p on M for this involution we assign an index **I**, a finite subset of the non-negative integers. The cardinality of **I** is the Morse index of p as a critical point of a on $M \cap \{d = d(p)\}$. The main result of § 4 is that for each finite subset **I** there is a unique fixed point p_I on M of index **I**.

When **I** is the empty set $\varnothing$, $a + d$ has a strict global maximum on M at $p_\varnothing$, so that $M \cap \{B = B(p_\varnothing)\}$ is a singleton. When $a_0 + d_0$ is not a critical value of $a + d$ on M, it turns out that the topology of $M \cap \{B = B_0\}$ is determined by $a_0 + d_0$ alone. If $a_0 + d_0$ is not a critical value, then all the homotopy groups of $M \cap \{B = B_0\}$ are isomorphic to those of a rather explicit subset of M whose topology only depends on $\{I : a_0 + d_0 < a(p_I) + d(p_I)\}$; cf. theorem 6.4. The main point at which our analysis is incomplete is that we have not shown that the only accumulation point for critical values of $a + d$ on M is $-\infty$. If that were established, then, as $a_0 + d_0$ went from $a(p_\varnothing) + d(p_\varnothing)$ to $-\infty$, one would see the progression in discrete steps of the homotopy of $M \cap \{B = B_0\}$ from triviality to the full homotopy of M.

As in the case of periodic boundary conditions, Floquet theory plays a fundamental role in the spectral theory here. In this paragraph we will outline the Floquet theory for generalized periodic boundary conditions (with $b \neq 0$). In doing this we will also fix some notation. We let $y_i(x, \lambda, q)$, $i = 1, 2$, be the solutions of

$$-y'' + qy = \lambda y$$

for $\lambda \in \mathbf{C}$ such that the matrix

$$F(x, \lambda, q) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

satisfies $F(0, \lambda, q) = I$. Letting B denote the boundary condition matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as before, we define the 'discriminant'

$$\Delta(\lambda, q) = \operatorname{tr}(BF(1, \lambda, q))$$

and check that λ is an eigenvalue for the boundary condition B if and only if $\Delta(\lambda) = 2$. Moreover, $\Delta(\lambda)$ is an entire function of λ of order $\frac{1}{2}$ (cf. theorem 1.1), and for the boundary condition given by B with $b \neq 0$ one can recover b and hence the top-order asymptotic of $\Delta(\lambda)$ as $\lambda \to \infty$ from the spectrum (cf. theorems 1.1 and 1.4). Hence by Hadamard's theorem, given the spectrum of one of the problems considered here—without specifying which one—$\Delta(\lambda)$ is determined. In particular the roots of $\Delta(\lambda) = -2$, i.e. the spectrum with B replaced by $-B$, are determined. For this reason one can without loss of generality assume that both spectra are given and restrict oneself to the case $b > 0$, as we will from here on. As in the periodic case, $\Delta(\lambda) \to \infty$ on the real axis as $\lambda \to -\infty$ (assuming $b > 0$), and, as $\lambda \to \infty$, $\Delta(\lambda)$ has an infinite sequence of non-degenerate minima less than or equal to -2 followed by non-degenerate maxima greater than or equal to 2 and no other real critical values (cf. Appendix B). Hence the roots of $\Delta(\lambda) = 2$ and $\Delta(\lambda) = -2$ listed by multiplicity interlace in pairs once one is above λ_0, the first root of $\Delta(\lambda) = 2$; i.e.

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots,$$

where $\Delta(\lambda_{4k-1}) = \Delta(\lambda_{4k}) = 2$ and $\Delta(\lambda_{4k+1}) = \Delta(\lambda_{4k+2}) = -2$. We will often refer to the sequence $\{\lambda_k\}_{k=0}^{\infty}$ as the 'generalized periodic spectrum', even though strictly speaking it is the union of two spectra. These spectra are listed by multiplicity as eigenvalues, since $\lambda_{2k} = \lambda_{2k-1}$ if and only if λ_{2k} is an eigenvalue of multiplicity two (cf. Appendix B).

The spectrum of $-y'' + qy$ with the boundary conditions $y(0) = 0$ and $ay(1) + by'(1) = 0$ is given by the roots of $[ay_2 + by_2'](1, \lambda) = 0$. Since this boundary value problem is self-adjoint, all these roots are real, and they are also simple (cf. (1.2)). Hence we list them as a strictly increasing sequence $\{\mu_j\}_{j+1}^{\infty}$. We call $\{\mu_j\}_{j=1}^{\infty}$ the generalized Dirchlet spectrum, since it reduces to the Dirichlet spectrum when $b = 0$. The generalized periodic and generalized Dirichlet spectra are related by the identity (cf. lemma 4.2(iv))

$$[y_2^2 + b\Delta(\lambda)y_2 + b^2](1, \lambda) = [(ay_2 + by_2')(dy_2 + by_1)](1, \lambda).$$

Since $y_2(1, \lambda)$ is real for λ real, this implies that the μ_j lie in the 'gaps', i.e. in the set where $|\Delta(\lambda)| \geq 2$. In the case of periodic boundary conditions there is exactly one Dirichlet eigenvalue in each gap above λ_0. Since $\Delta(\lambda, q, B)$ and the μ_j, being simple roots, depend continuously on B, it follows that for $b \neq 0$ there is exactly one generalized Dirichlet eigenvalue in each gap above λ_0, i.e. $\lambda_{2j-1} \leq \mu_j \leq \lambda_{2j}$ for $j \geq 1$.

The plan of this long paper is fairly straightforward. In § 1 we determine the asymptotic behaviour of the sequences $\{\lambda_k\}$ and $\{\mu_k\}$ as $k \to \infty$ and show that the generalized Dirichlet spectra together with b, d and the 'norming constants' $\kappa_k = y_2(1, \mu_k, q)$ form an analytic coordinate system on

$$L^2_{\mathbf{R}}[0,1] \times (\mathrm{SL}(2,\mathbf{R}) \backslash \{b=0\}).$$

The point here is not only that these data determine (q, B) uniquely, but also that, with the proper Hilbert manifold structure on the space S of sequences

$$(b, d, \mu_1, \kappa_1, \mu_2, \kappa_2, \ldots),$$

the Jacobian of the coordinate mapping is boundedly invertible so that the inverse mapping is also analytic. This section is closely related to the early chapters of [**6**].

In § 2 we present flows generated by analytic vector fields V_k on $L^2_{\mathbf{R}}[0,1] \times (\mathrm{SL}(2,\mathbf{R}) \backslash \{b=0\})$ which preserve M and move (μ_k, κ_k) around the (topological) circle $\kappa_k^2 + b\Delta(\mu_k)\kappa_k + b^2 = 0$ while fixing b, d and (μ_j, κ_j), $j \neq k$. As in [**4**] and [**6**], these flows can be explicitly integrated to give formulae for q and B on the orbits. The formula for q does not play a role in the sections that follow, but we use the formula for B frequently.

In § 3 we identify M with an explicit analytic submanifold N of S by using the flows of § 2 to show that the range of the coordinate functions is dense in N, from which it follows that the range is equal to N since M is closed and the inverse coordinate map is analytic. M thus inherits the analytic structure of N.

After some preliminary lemmas, § 4 is devoted to the proof of the existence of a unique critical point of $a+d$ on M with index **I**. For the proof of uniqueness we use the explicit integration of the flows and the involution. The flows, however, do not seem to be useful in proving the existence of critical points. For our proof of existence we use another fortunate accident: for k sufficiently large the minimum of

$$\int_0^1 |k + q(x)|^2 \, dx$$

on a properly chosen subset of M is assumed at p_I. For example,

$$\min_{M \cap \{a=d\}} \int_0^1 |k + q(x)|^2 \, dx$$

is assumed at $p_\varnothing$ for k sufficiently large.

In § 5 we show that the range of B on M is

$$B(p_\varnothing) \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b = b_0, \det B = 1, a + d < a(p_\varnothing) + d(p_\varnothing) \right\} \tag{0.2}$$

and prove generalizations of this which will be used in § 6. Formula (0.2) shows that d takes all real values on M, which had not been proven earlier in the paper.

In the final section, § 6, we collect all the information which we have obtained on the isospectral sets $M \cap \{B = B_0\}$. We show that they are always connected, are non-compact when they are not singletons, and are analytic submanifolds of $L^2_{\mathbf{R}}[0,1]$ unless they contain fixed points of the involution. We conclude with the result on the homotopy groups of $M \cap \{B = B_0\}$ described earlier.

For the boundary conditions considered by Tysk in Appendix A there is a full set of isospectral flows fixing the boundary conditions, so that the description of the isospectral sets for fixed boundary conditions become analogous to the description of $M\cap\{d=\delta\}$. This makes some of the analysis nearly identical to §§ 1, 2 and 3 and we have not presented it in detail.

Finally we should point that we have not solved the problem considered in [2], [3] and [6] of determining what sequences $\{\lambda_k\}$ arise. It would be interesting to learn if anything like the result of [2] for the periodic spectrum is valid here.

1. *A coordinate system*

This section is devoted to the construction of a global real analytic coordinate system on $L^2_{\mathbf{R}}[0,1]\times(\mathrm{SL}(2,\mathbf{R})\backslash\{b=0\})$. We will also determine the asymptotic behaviour of the generalized periodic eigenvalues. The development follows chapters 2 and 3 of Pöschel and Trubowitz [6] closely, and we state two basic theorems from [6] here for future reference.

THEOREM 1.1 (theorem 1.3 of [6]). *On* $[0,1]\times\mathbf{C}\times L^2_{\mathbf{C}}[0,1]$

(i) $$|y_1(x,\lambda,q)-\cos\lambda^{1/2}x|\le\frac{1}{|\lambda^{1/2}|}\exp\left(|\mathrm{Im}\,\lambda^{1/2}|x+\|q\|x^{1/2}\right),$$

(ii) $$\left|y_2(x,\lambda,q)=\frac{\sin\lambda^{1/2}x}{\lambda^{1/2}}\right|\le\frac{1}{|\lambda|}\exp\left(|\mathrm{Im}\,\lambda^{1/2}|x+\|q\|x^{1/2}\right),$$

(iii) $$|y_1'(x,\lambda,q)+\lambda^{1/2}\sin\lambda^{1/2}x|\le\|q\|\exp\left(|\mathrm{Im}\,\lambda^{1/2}|x+\|q\|x^{1/2}\right),$$

(iv) $$|y_2'(x,\lambda,q)-\cos\lambda^{1/2}x|\le\frac{\|q\|}{|\lambda^{1/2}|}\exp\left(|\mathrm{Im}\,\lambda^{1/2}|x+\|q\|x^{1/2}\right).$$

By the Riesz representation theorem the Frechet derivative $d_qF(v)$ of a function on L^2 can be written $d_qF(v)=\int_0^1(\partial F/\partial q(t))v(t)\,dt$. The following theorem gives the 'gradient' $\partial F/\partial q(t)$ for the y.

THEOREM 1.2 (theorem 1.6 of [6]). *For* $j=1,2$

$$\frac{\partial y_j}{\partial q(t)}(x)=y_j(t)(y_1(t)y_2(x)-y_1(x)y_2(t))\chi_{[0,x]}(t),$$

$$\frac{\partial y_j'}{\partial q(t)}(x)=y_j(t)(y_1(t)y_2'(x)-y_1'(x)y_2(t))\chi_{[0,x]}(t),$$

where $\chi_E(t)$ *is the characteristic function of* E. *These gradients are continuous in* (x,λ,q). *In addition*

$$\frac{\partial y_j}{\partial\lambda}=-\int_0^1\frac{\partial y_j}{\partial q(t)}\,dt\qquad\textit{and}\qquad\frac{\partial y_j'}{\partial\lambda}=-\int_0^1\frac{\partial y_j'}{\partial q(t)}\,dt.$$

We refer the reader to [6] for proofs of theorems 1.1 and 1.2. Theorem 1.2 is an easy computation. Theorem 1.1 is also quite standard in various forms. We need the particular form given here to control y_1 and y_2 for complex λ and for q in bounded sets of L^2.

We begin by studying the generalized Dirichlet spectrum.

THEOREM 1.3. *The generalized Dirichlet eigenvalues $\mu_n(q, a, b)$, $n = 1, 2, \ldots$, are real analytic functions on $L^2_{\mathbf{R}}[0,1] \times (\mathbf{R}^2 \backslash 0)$. When $b \neq 0$ they satisfy the asymptotic estimate*

$$\mu_n(q, a, b) = (n - \tfrac{1}{2})^2 \pi^2 + 2a/b + \int_0^1 q\, dx + r_n,$$

where $\{r_n\} \in l^2(\mathbf{Z})$. For convenience here and elsewhere we will denote this by

$$\mu_n(q, a, b) = (n - \tfrac{1}{2})^2 \pi^2 + 2a/b + \int_0^1 q\, dx + l^2(n). \tag{1.1}$$

Remark. When $a \neq 0$ and $b = 0$, the set of μ_n becomes the Dirichlet spectrum and

$$\mu_n(q, a, 0) = n^2 \pi^2 + \int_0^1 q\, dx + l^2(n).$$

Thus the asymptotics change drastically when $b = 0$. However, (1.1) does hold uniformly in the sense that $\|\{r_n\}\|_2$ is uniformly bounded on sets where $\|q\|_2$ and $|a|/|b|$ are bounded.

Proof. This is precisely the analogue of parts of theorems 2.3 and 2.4 of [**6**]. We will only sketch the proof, indicating the modifications needed in the arguments of chapter 2 of [**6**].

Multiplying the identity

$$\left(-\frac{d^2}{dx^2} + q - \mu \right) \frac{\partial y_2}{\partial \mu} = y_2$$

by y_2 and integrating from 0 to 1 in x, one has

$$\int_0^1 y_2^2(x, \mu)\, dx = \left(-y_2 \frac{\partial y_2'}{\partial \mu} + y_2' \frac{\partial y_2}{\partial \mu} \right)(1, \mu).$$

Evaluating at $\mu = \mu_n$ gives

$$\int_0^1 y_2^2(x, \mu)\, dx = \begin{cases} -\dfrac{y_2(1, \mu_n)}{b} \left(a \dfrac{\partial y_2}{\partial \mu} + b \dfrac{\partial y_2'}{\partial \mu} \right)(1, \mu_n), & b \neq 0 \\[2ex] \dfrac{y_2'(1, \mu_n)}{a} \left(a \dfrac{\partial y_2}{\partial \mu} + b \dfrac{\partial y_2'}{\partial \mu} \right)(1, \mu_n), & a \neq 0. \end{cases} \tag{1.2}$$

Thus, since $a \partial y_2/\partial \mu(1, \mu_n, q) + b \partial y_2'/\partial \mu(1, \mu_n, q) \neq 0$, the analyticity of $\mu_n(q, a, b)$ follows from the implicit function theorem applied to the equation $a y_2(1, \mu_n, q) + b y_2'(1, \mu_n, q) = 0$.

The estimates of theorem 1.1 show that for $\lambda \in \mathbf{C}$

$$\left| \frac{a}{b} y_2(1, \lambda, q) + y_2'(1, \lambda, q) - \cos \lambda^{1/2} \right| \leq \frac{1}{|b|} \left[\frac{|a|}{|\lambda|^{1/2}} + \left(\frac{|a|}{|\lambda|} + \frac{|b|\, \|q\|}{|\lambda|^{1/2}} \right) e^{\|q\|} \right] e^{|\operatorname{Im} \lambda^{1/2}|}.$$

Thus, as in lemma 2.2 of [**6**], Rouché's theorem can be used to show that, for $N > N(|a|/|b|, \|q\|)$, $a y_2(1, \lambda, q) + b y_2'(1, \lambda, q)$ has exactly N roots in $\operatorname{Re} \lambda < N^2 \pi^2$ and exactly one root in $|\lambda^{1/2} - (N - \frac{1}{2})\pi| < \pi/2$. This gives

$$(\mu_n)^{1/2} = (n - \tfrac{1}{2})\pi + O(1). \tag{1.3}$$

To improve (1.3) to (1.1), one can use the identity

$$\mu_n-(n-\tfrac{1}{2})^2\pi^2=\int_0^1 \frac{d}{dt}\,\mu_n(tq, ta, b)\,dt \tag{1.4}$$

together with

$$\begin{aligned}\frac{\partial\mu_n}{\partial q(x)}&=\frac{y_2^2(x,\mu_n)}{\|y_2(\cdot,\mu_n)\|^2}=2\sin^2(\mu_n)^{1/2}x+O\left(\frac{1}{n}\right)\\&=1-\cos 2(\mu_n)^{1/2}x+O(1/n)\end{aligned} \tag{1.5}$$

and

$$\frac{\partial\mu_n}{\partial a}=\frac{y_2^2(1,\mu_n)}{b\|y_2(\cdot,\mu_n)\|^2}.$$

This gives $\mu_n=(n-\frac{1}{2})^2\pi^2+O(1)$ and $(\mu_n)^{1/2}=(n-\frac{1}{2})\pi+O(1/n)$, so that

$$\cos 2(\mu_n)^{1/2}x=\cos(2n-1)\pi x+O(1/n).$$

Using this improvement when one again substitutes (1.5) into (1.4), one derives (1.1). □

The same methods will be used to find the asymptotics of the generalized periodic spectrum.

THEOREM 1.4. *The generalized periodic eigenvalues* $\lambda_n(q, B)$ *satisfy the asymptotic estimates when* $b\neq 0$:

$$\begin{aligned}\lambda_{2n}(q,B)&=n^2\pi^2+\frac{2}{b}(a+d-2)+\int_0^1 q\,dx+l^2(n),\\ \lambda_{2n+1}(q,B)&=n^2\pi^2+\frac{2}{b}(a+d+2)+\int_0^1 q\,dx+l^2(n).\end{aligned} \tag{1.6}$$

Hence b *and* $2(a+d)+b\int_0^1 q\,dx$ are spectral invariants.

Proof. The eigenvalues $\lambda_n(q, B)$ are the roots of $\Delta^2(\lambda, q, B)-4=0$. Using

$$\Delta(\lambda)=ay_1(1,\lambda)+by_1'(1,\lambda)+cy_2(1,\lambda)+dy_2'(1,\lambda)$$

and the estimates of theorem 1.1, we have for $|\lambda|>1$

$$|\Delta^2(\lambda)-4-b^2\lambda\sin^2\lambda^{1/2}|\leq C(\|q\|, B)|\lambda|^{1/2}e^{2|\mathrm{Im}\,\lambda^{1/2}|}. \tag{1.7}$$

Thus, using Rouché's theorem on the boundary of

$$\{|\mathrm{Re}\,z|\leq(n+\tfrac{1}{4})^2\pi^2\}\cap\{|\mathrm{Im}\,z|\leq K\},$$

one sees that, for $n\geq N(\|q\|, \|B\|, 1/|b|)$, $\Delta^2(\lambda)-4$ has exactly $2n+2$ roots in the half-plane $\mathrm{Re}\,\lambda<(n+\frac{1}{4})^2\pi^2$. On the other hand, Rouché's theorem applied on the contour $|\lambda^{1/2}-n\pi|=\pi/4$ shows $\Delta^2(\lambda)-4$ has exactly two roots in the region $|\lambda^{1/2}-n\pi|<\pi/4$. These conclusions do not require $\det B=1$ and hold uniformly on sets of (q, B) where $\|q\|$, $\|B\|$ and $1/|b|$ are bounded. In the case that $\det B\neq 1$ all the roots of $\Delta^2(\lambda)-4=0$ need not be real, but again, since $\Delta^2(\lambda)-4$ is always real-valued for λ real, (1.7) implies that, for $n>N(\|q\|, 1/|b|, \|B\|)$, $\Delta^2(\lambda)-4$ always has at least two real roots in the open interval $(\pi^2(n-\frac{1}{4})^2, \pi^2(n+\frac{1}{4})^2)$. Hence, combining this with the argument from Rouché's theorem, we conclude that, for $n\geq$

$N(\|q\|, 1/|b|, \|B\|)$, $\Delta^2(\lambda)-4$ has exactly two roots in $|\lambda^{1/2}-n\pi|<\pi/4$ and they are real, and exactly $2n+2$ roots in the half-plane $\operatorname{Re}\{\lambda\}<(n+\frac{1}{4})^2\pi^2$.

Combining theorems 1.1 and 1.2, one computes for real λ

$$\frac{\partial\Delta(\lambda)}{\partial\lambda}=-\frac{b\cos\lambda^{1/2}}{2}+O\left(\frac{1}{\lambda^{1/2}}\right), \tag{1.8}$$

where again this holds uniformly on sets where $\|q\|$, $1/|b|$ and $\|B\|$ are bounded. Thus for $n\geq N(\|q\|, 1/|b|, \|B\|)$ the two roots of $\Delta^2(\lambda)-4$ in $|\lambda^{1/2}-n\pi|<\pi/4$ are smooth functions of (q, B) by the implicit function theorem, and we label them $\lambda_{2n}(q, B)<\lambda_{2n+1}(q, B)$.

Now we are in a position to finish the proof of theorem 1.4 in the same way as theorem 1.3. Since we require only $b\neq 0$, we have for $k=2n$, $2n+1$, $n\geq N(\|q\|, 1/|b|, \|B\|)$

$$\lambda_k(q, B)-\lambda_k^0=\int_0^1\frac{d}{dt}\lambda_k(tq, ta, b, tc, td)\,dt. \tag{1.9}$$

The λ_k^0 are the roots of

$$b^2\lambda^2\sin^2\lambda^{1/2}=4 \qquad \text{in } (\pi^2(n-\tfrac{1}{4})^2, \pi^2(n+\tfrac{1}{4})^2)$$

and hence

$$\lambda_{2n}^0=n^2\pi^2-\frac{4}{b}+O\left(\frac{1}{n}\right), \qquad \lambda_{2n+1}^0=n^2\pi^2+\frac{4}{b}+O\left(\frac{1}{n}\right).$$

Again combining theorems 1.1 and 1.2, one computes for real λ

$$\frac{\partial\Delta(\lambda)}{\partial q(x)}=b\left(\cos\lambda^{1/2}\frac{1+\cos 2\lambda^{1/2}x}{2}+\sin\lambda^{1/2}\frac{\sin 2\lambda^{1/2}x}{2}\right)+O\left(\frac{1}{\lambda^{1/2}}\right).$$

Hence, using (1.8), implicit differentiation and $|(\lambda_k)^{1/2}-n\pi|<\pi/4$, one computes for $k=2n$, $2n+1$

$$\begin{gathered}\frac{\partial\lambda_k}{\partial q(x)}=\frac{1}{2}+\frac{\cos 2(\lambda_k)^{1/2}x}{2}+\tan(\lambda_k)^{1/2}\frac{\sin 2(\lambda_k)^{1/2}x}{2}+O\left(\frac{1}{n}\right),\\ \frac{\partial\lambda_k}{\partial a}=\frac{2}{b}+O\left(\frac{1}{n}\right), \qquad \frac{\partial\lambda_k}{\partial c}=O\left(\frac{1}{n}\right), \qquad \frac{\partial\lambda_k}{\partial d}=\frac{2}{b}+O\left(\frac{1}{n}\right),\end{gathered} \tag{1.10}$$

uniformly for $n\geq N(\|q\|, 1/|b|, \|B\|)$. Using (1.10) in (1.9), one sees

$$\lambda_k=n^2\pi^2+O(1)$$

and hence $(\lambda_k)^{1/2}=n\pi+O(1/n)$. Thus (1.10) shows

$$\frac{\partial\lambda_k}{\partial q(x)}=\frac{1}{2}+\frac{\cos 2\pi nx}{2}+O\left(\frac{1}{n}\right).$$

Substituting this together with the rest of (1.10) into (1.9) gives (1.6). □

We now give the analogue for the generalized Dirichlet spectrum of the well known result (theorem 3.5 in [**6**]) that the spectrum plus the 'norming constants' determine the potential. In the generalized Dirichlet case these data determine the ratio a/b as well, but the method of proof is the same.

THEOREM 1.5. *If* $\mu_n(q, a, b) = \mu_n(\tilde{q}, \tilde{a}, \tilde{b}) \equiv \mu_n$, $n \geq 1$, *and* $y_2(1, \mu_n, q) = y_2(1, \mu_n, \tilde{q}) \equiv y_2(1, \mu_n)$, $n \geq 1$, *then* $q = \tilde{q}$ *and* $a/b = \tilde{a}/\tilde{b}$.

Proof. Following the method of theorems 3.3 and 3.5 in [6], we introduce $z(x, \lambda, q, c)$, the solution of

$$-z'' + qz = \lambda z$$

satisfying $z(1, \lambda, q, c) = -1$, $z'(1, \lambda, q, c) = c$. Then for $n \geq 1$

$$y_2(x, \mu_n, q) = -y_2(1, \mu_n) z(x, \mu_n, q, a/b),$$
$$y_2(x, \mu_n, \tilde{q}) = -y_2(1, \mu_n) z(x, \mu_n, \tilde{q}, \tilde{a}/\tilde{b}).$$

Consider the function

$$f(\lambda) = \frac{(y_2(x, \lambda, q) - y_2(x, \lambda, \tilde{q}))(z(x, \lambda, q, a/b) - z(x, \lambda, \tilde{q}, \tilde{a}/\tilde{b}))}{a y_2(1, \lambda, q) + b y_2'(1, \lambda, q)}$$

Note that the numerator is entire and the denominator is entire with simple zeros at the μ_n (see theorem 1.3) and no other zeros. Thus $f(\lambda)$ is meromorphic with simple poles at $\lambda = \mu_n$, $n \geq 1$, with residues

$$R_n = \frac{-y_2(1, \mu_n, q)}{a(\partial y_2/\partial \mu)(1, \mu_n, q) + b(\partial y_2'/\partial \mu)(1, \mu_n q)} (z(x, \mu_n, q, a/b) - z(x, \mu_n, \tilde{q}, \tilde{a}/\tilde{b}))^2$$

Since (1.2) implies

$$R_n = \frac{y_2^2(1, \mu_n, q)}{b \int_0^1 y_2^2(x, \mu_n, q)\, dx} (z(x, \mu_n, q, a/b) - z(x, \mu_n, \tilde{q}, \tilde{a}/\tilde{b}))^2.$$

we see $R_n \geq 0$.

The estimates of theorem 1.1 show that

$$|f(\lambda)| \leq C \frac{e^{|\text{Im}\, \lambda^{1/2}| x}}{|\lambda|} \frac{e^{|\text{Im}\, \lambda^{1/2}|(1-x)}}{|\lambda|^{1/2}} \Big/ \left[b \cos \lambda^{1/2} + O\left(\frac{e^{|\text{Im}\, \lambda^{1/2}|}}{|\lambda|^{1/2}} \right) \right]$$

Here we used

$$z(x, \lambda, q, c) = -y_1(1-x, q^*) + c y_2(1-x, q^*),$$

where $q^*(x) = q(1-x)$, in estimating $z(x, q, a/b) - z(x, \tilde{q}, \tilde{a}/\tilde{b})$. Thus, taking $r_n = (n - \frac{1}{2})^2 \pi^2$, we see $\lim_{n \to \infty} \text{Max}_{|\lambda| = r_n} |\lambda f(\lambda)| = 0$. Thus $R_n = 0$ for all n, f is entire, and we see $f(\lambda) \equiv 0$ for $x \in [0, 1]$. In particular

$$y_2(x, \mu_1, q) = y_2(x, \mu_1, \tilde{q}),$$

so that a.e. in [0, 1]

$$0 = (q(x) - \tilde{q}(x)) y_2(x, \mu_1, q).$$

Since y_2 has only a finite number of zeros, we conclude

$$q = \tilde{q}.$$

Hence $\int_0^1 q\, dx = \int_0^1 \tilde{q}\, dx$ and (1.1) implies $a/b = \tilde{a}/\tilde{b}$. □

If (q, B) is an element of $L^2[0, 1] \times (\text{SL}(2, \mathbf{R}) \backslash \{b = 0\})$, theorem 1.5 implies (q, B) is uniquely determined by

$$\{\mu_n(q, a, b)\}_{n=1}^{\infty}, \qquad \{y_2(1, \mu_n(q, a, b), q)\}_{n=1}^{\infty}, \qquad b \qquad \text{and} \qquad d.$$

As earlier, we use the notation $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus these data could be used as coordinates on $L^2[0, 1] \times (\mathrm{SL}(2, \mathbf{R}) \backslash \{b = 0\})$. However, to construct analytic coordinates from them, we will need several preliminary results. We begin with the analogue of theorem 1.3 for the norming constants.

THEOREM 1.6. *The norming constants* $y_2(1, \mu_n(q, a, b), q)$, $n = 1, 2, \ldots$, *are analytic functions on* $L^2[0, 1] \times (\mathbf{R}^2 \backslash 0)$. *When* $b \neq 0$ *they satisfy the asymptotic estimate*

$$y_2(1, \mu_n(q, a, b), q) = \frac{(-1)^{n+1}}{(n - \frac{1}{2})\pi} + \frac{1}{n^2} l^2(n). \tag{1.11}$$

Proof. Since $y_2(1, \lambda, q)$ is analytic in (λ, q) and $\mu_n(q, a, b)$ is analytic by theorem 1.3, the analyticity is immediate.

For the asymptotics we use the analogue of (1.4):

$$y_2(1, \mu_n(q, a_1 b_1), q) - \frac{(-1)^{n+1}}{(n - \frac{1}{2})\pi} = \int_0^1 \frac{d}{dt} (y_2(1, \mu_n(tq, ta, b), tq))\, dt.$$

Using theorems 1.1 and 1.2, we have

$$\begin{aligned} \frac{\partial y_2(1, \lambda, q)}{\partial q(x)} &= y_2(x, \lambda)(y_2(1, \lambda) y_1(x, \lambda) - y_1(1, \lambda) y_2(x, \lambda)) \\ &= \frac{-\cos \lambda^{1/2}}{2\lambda} + \frac{\cos \lambda^{1/2}(2x - 1)}{2\lambda} + O(\lambda^{-3/2}), \end{aligned} \tag{1.12}$$

$$\frac{\partial y_2}{\partial y} = -\int_0^1 \frac{\partial y_2}{\partial q(x)}\, dx.$$

Combining (1.12) with the estimates on μ_n, and $\partial \mu_n / \partial q(x)$ and $\partial \mu_n / \partial a$ from theorem 1.3 and its proof, gives (1.11). □

Theorems 1.3 and 1.6 show that the mapping

$\Phi : (q, B)$

$\to (b, d, \mu_1(q, a, B), y_2(1, \mu_1(q, a, b), q), \mu_2(q, a, b), y_2(1, \mu_2(q, a, b), q), \ldots)$

takes $L^2_{\mathbf{R}}(0, 1) \times \mathrm{SL}(2, \mathbf{R}) \backslash \{b = 0\}$ into S, the space of real sequences

$$s = (b, d, \mu_1, \kappa_1, \mu_2, \kappa_2, \ldots),$$

where

$$\mu_n = (n - \tfrac{1}{2})^2 \pi^2 + r + l^2(n)$$

and

$$\kappa_n = \frac{(-1)^{n+1}}{(n - \frac{1}{2})\pi} + \frac{1}{n^2} l^2(n).$$

To make S into a (trivial) real analytic manifold modelled on $l^2_{\mathbf{R}}$, we introduce the global coordinates on S, assigning

$$\tilde{s} = (b, d, r, \tilde{\mu}_1, \tilde{\kappa}_1, \tilde{\mu}_2, \tilde{\kappa}_2, \ldots) \tag{1.13}$$

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

to s, where

$$\tilde{\mu}_n = \mu_n(n-\tfrac{1}{2})^2\pi^2 - r$$

and

$$\tilde{\kappa}_n = 2\pi^2 n^2\left(\kappa_n + \frac{(-1)^n}{(n-\frac{1}{2})\pi}\right).$$

We claim that Φ is a real analytic mapping of $L^2_{\mathbf{R}}[0,1]\times(\mathrm{SL}(2,\mathbf{R})\backslash\{b=0\})$ into S. For this we need to show that given $(q_0, B_0)\in L^2_{\mathbf{R}}[0,1]\times(\mathrm{SL}(2,\mathbf{R})\backslash\{b=0\})$, the functions $\mu_n(q,a,b)$ and $y_2(1,\mu_n(q,a,b),q)$ are analytic on a (complex) neighbourhood U of (q_0,B_0) in $L^2[0,1]\times M_{22}$, independent of n, and that $\tilde{\Phi}(U)$ is a bounded set in l^2 (see theorem A.3 of [**6**]). The proofs of these facts follow the proofs of the corresponding results for the Dirichlet spectrum in theorems 3.1 and 3.6 in [**6**] so closely that we will omit them here.

As we observed earlier, theorem 1.5 implies that Φ is globally one-to-one. The final step in showing that Φ is a global coordinate function is showing its derivative Φ' is boundedly invertible at all points of $L^2_{\mathbf{R}}[0,1]\times(\mathrm{SL}(2,\mathbf{R})\backslash\{b=0\})$.

THEOREM 1.7. *In terms of the coordinates on S, $\Phi'(q,B)$ is the linear mapping*

$$(\dot{q},\dot{B})\to\left(\dot{b},\dot{d},2\left(\frac{\dot{a}}{b}-\frac{a\dot{b}}{b^2}\right)+\int_0^1\dot{q}\,dx,\langle\tilde{\mu}_1'(q,a,b),(\dot{q},\dot{B})\rangle,\langle\tilde{\kappa}_1'(q,a,b),(\dot{q},\dot{B})\rangle,\ldots\right),$$

where $\dot{q}\in L^2[0,1]$, $\dot{B}=\left(\begin{smallmatrix}\dot{a}&\dot{b}\\ \dot{c}&\dot{d}\end{smallmatrix}\right)$ with $\dot{c}=(\dot{a}d+\dot{d}a)/b+\dot{b}(1-ad)/b^2$, i.e. $\dot{B}$ is in the tangent space to SL(2, **R**) *at B. The pairings $\langle\tilde{\mu}_n'(q,a,b),(\dot{q},\dot{B})\rangle$ and $\langle\tilde{\kappa}_n'(q,a,b),(\dot{q},\dot{B})\rangle$ are given by*

$$\langle\tilde{\mu}_n'(q,a,b),(\dot{q},\dot{B})\rangle=-2\left(\frac{\dot{a}}{b}-\frac{a\dot{b}}{b^2}\right)-\int_0^1\dot{q}\,dx$$
$$+\|y_2(\mu_n)\|_2^{-2}\left[\int_0^1 y_2^2(\mu_n)\dot{q}\,dx+\left(\frac{\dot{a}}{b}-\frac{a\dot{b}}{b^2}\right)y_2^2(1,\mu_n)\right]$$

and

$$(2\pi^2n^2)^{-1}\langle\tilde{\kappa}_n'(q,a,b),(\dot{q},\dot{B})\rangle$$
$$=y_2(1,\mu_n)\int_0^1 y_1(\mu_n)y_2(\mu_n)\dot{q}\,dx-y_1(1,\mu_n)\int_0^1 y_2^2(\mu_n)\dot{q}\,dx$$
$$-\left(y_2(1,\mu_n)\int_0^1 y_1(\mu_n)y_2(\mu_n)\,dx-y_1(1,\mu_n)\int_0^1 y_2^2(\mu_n)\,dx\right)$$
$$\times\left[\langle\tilde{\mu}_n'(q,a,b),(\dot{q},\dot{B})\rangle+2\left(\frac{\dot{a}}{b}-\frac{a\dot{b}}{b^2}\right)+\int_0^1\dot{q}\,dx\right].$$

The mapping Φ' is boundedly invertible.

Proof. The formulae for $\langle\tilde{\mu}_n',(\dot{q},\dot{B})\rangle$ and $\langle\tilde{\kappa}_n',(\dot{q},\dot{B})\rangle$ are computed using (1.5) and (1.12).

Once again using the asymptotics for the μ_n in n, and y_1 and y_2 in λ, one has

$$
\begin{aligned}
w_n &\equiv \|y_2(\mu_n)\|_n^{-2} y_2^2(\mu_n) - 1 = -\cos(2n-1)\pi x + O(1/n),\\
g_n &= (1/b)(\|y_2(\mu_n)\|_2^{-2} y_2^2(1,\mu_n) - 2) = O(1/n),\\
z_n &= 2\pi^2 n^2 (y_2(1,\mu_n) y_1(\mu_n) y_2(\mu_n) - y_1(1,\mu_n) y_2^2(\mu_n))\\
&\quad -2\pi^2 n^2 \left(\int_0^1 (y_2(1,\mu_n) y_1(\mu_n) y_2(\mu_n) - y_1(1,\mu_n) y_2^2(\mu_n))\, dx \right)\\
&\quad \times \|y_2(\mu_n)\|_2^{-2} y_2^2(\mu_n)\\
&= \sin(2n-1)\pi x + O(1/n),\\
h_n &= -\frac{2\pi^2 n^2}{b} \left(\int_0^1 (y_2(1,\mu_n) y_1(\mu_n) y_2(\mu_n) - y_1(1,\mu_n) y_2^2(\mu_n))\, dx \right)\\
&\quad \times \|y_2(\mu_n)\|_2^{-2} y_2^2(1,\mu_n)\\
&= O(1/n).
\end{aligned}
\tag{1.14}
$$

Hence

$$
\begin{aligned}
\Phi'(q, B)[\dot q, \dot B] &= \dot b f_1 + \dot d f_2\\
&\quad + \left(0, 0, \frac{2\dot a}{b} + \int_0^1 \dot q\, dx, \int_0^1 w_1 \dot q\, dx - \dot a g_1, \int_0^1 z_1 \dot q\, dx + \dot a h_1, \ldots\right),
\end{aligned}
$$

where $f_i \in l^2$, $i = 1, 2$, and

$$f_1 = (1, 0, \ldots), \qquad f_2 = (0, 1, 0, \ldots).$$

Thus to show that $\Phi'(q, B)$ is boundedly invertible, it will suffice to show that

$$\psi : (\dot q, \dot a) \to \left(\frac{2\dot a}{b} + \int_0^1 \dot q\, dx, \int_0^1 w_1 \dot q\, dx + \dot a g_1, \int_0^1 z_1 \dot q\, dx + \dot a h_1, \ldots \right)$$

is boundedly invertible. Moreover, (1.14) implies

$$\infty > \sum_{n=1}^{\infty} (\|\{w_n, g_n\} - \{-\cos(2n-1)\pi, 0\}\|^2 + \|\{z_n, h_n\} - \{\sin(2n-1)\pi, 0\}\|^2),$$

where for $\{w, g\} \in L^2[0, 1] \times \mathbf{C}$

$$\|\{w, g\}\|^2 = \int_0^1 |w|^2\, dx + |g|^2.$$

Hence, applying theorem D.3 of [**6**] to the complete orthonormal sequence $\{0, 1\}$, $\{2^{1/2}\cos(2n-1)\pi x, 0\}$, $\{2^{1/2}\sin(2n-1)\pi x, 0\}$, $n = 1, \ldots$, in $L^2[0, 1] \times \mathbf{C}$, we see that to show ψ is boundedly invertible it will suffice to show that $\{\{1, 2/b\}, \{w_n, g_n\}, \{z_n, g_n\}, \{z_n, h_n\}, n = 1, 2, \ldots\}$ is linearly independent in the sense that no element is in the closed linear span of the others.

The key to establishing linear independence is the following observation which we will use again in § 2. If we let w_+ and w_- denote two solutions of $-u'' + qu = \lambda u$ with $\lambda = \mu$, and z_+ and z_- denote a second pair with $\lambda = \nu$, $\mu \neq \nu$, then, letting

$[f, g] = fg' - gf'$,

$$2\int_0^1 w_+w_-(z_+z_-)'\,dx = (w_+w_-z_+z_-)|_0^1 + \int_0^1 (w_+w_-(z_+z_-)' - (w_+w_-)'z_+z_-)\,dx$$

$$= (w_+w_-z_+z_-)|_0^1 + \int_0^1 (w_+z_+[w_-, z_-] + w_-z_-[w_+, z_+])\,dx$$

$$= (w_+w_-z_+z_-)|_0^1 + \frac{1}{\mu-\nu}\int_0^1 ([w_+, z_+][w_-, z_-])'\,dx$$

$$= (w_+w_-z_+z_-)|_0^1 + \frac{1}{\mu-\nu}[w_+, z_+][w_-, z_-]|_0^1. \tag{1.15}$$

Using (1.15) with $w_+w_- = y_2^2(\mu_n)$ and $z_+z_- = y_2^2(\mu_m)$, we have

$$0 = \int_0^1 w_n(y_2^2(\mu_n))'\,dx - g_n\frac{b}{2}y_2^2(1, \mu_m) \tag{1.16}$$

for all $m = 1, 2, \ldots$. Similarly for $m \neq n$

$$0 = \int_0^1 z_n(y_2^2(\mu_m))'\,dx - h_n\frac{b}{2}y_2^2(1, \mu_n), \tag{1.17}$$

and trivially

$$0 = \int_0^1 (y_2^2(\mu_m))'\,dx - \frac{2}{b}\frac{b}{2}y_2^2(1, \mu_n). \tag{1.18}$$

However,

$$\int_0^1 z_n(y_2^2(\mu_n))'\,dx - h_n\frac{b}{2}y_2^2(1, \mu_n)$$

$$= \pi^2n^2\int_0^1 (y_2(\mu_n)y_2(1, \mu_n)y_1(\mu_n) - y_1(1, \mu_n)y_2(\mu_n)(y_2(\mu_n)^2)')\,dx \tag{1.19}$$

$$= \pi^2n^2\int_0^1 y_2^2(\mu_1)y_2(1, \mu_n)\,dx \neq 0.$$

Formulae (1.16)-(1.19) show that no $\{z_n, h_n\}$ is in the closed linear span of the other vectors.

By the same method

$$0 = \int_0^1 z_n(z_m)'\,dx - \frac{b^2}{4}h_nh_m,$$

$$0 = \int_0^1 (z_m)'\,dx - \frac{2}{b}\frac{b^2}{4}h_m \qquad \text{for all } m,$$

and for $m \neq n$

$$0 = \int_0^1 w_n(z_m)'\,dx - \frac{b^2}{4}g_nh_m.$$

However, by (1.19)

$$\int_0^1 w_n(z_n)'\,dx - \frac{b^2}{4}g_nh_n\,dx \neq 0.$$

Thus we may conclude that no $\{w_n, g_n\}$ is in the span of the other vectors.

Finally we note that

$$\int_0^1 z_n\,dx = \int_0^1 w_n\,dx = 0,$$

so that $\{1, 0\}$ is orthogonal to the span of

$$\{\{w_n, g_n\}, \{z_n, h_n\}, n > 1\}$$

but obviously not orthogonal to $\{1, 2/b\}$. □

2. *Flows*

In this section we begin by finding vector fields $(\dot{q}, \dot{B})$ which are tangent to M. In the case of periodic boundary conditions, the vector field $(\dot{q}, \dot{B})$ with

$$\dot{q} = 2\left(\frac{\partial\Delta(\lambda)}{\partial q(x)}\right)' \tag{2.1}$$

and $\dot{B} = 0$ is tangent (cf. [**5**]). We will use (2.1) for $\dot{q}$ here, but to get a vector field tangent to M we will need to make $\dot{B}$ non-zero. One natural way to arrive at the correct expression for $\dot{B}$ is simply to compute

$$\int_0^1 \frac{\partial\Delta(\mu)}{\partial q(t)}\,\dot{q}(t, \nu)\,dt \tag{2.2}$$

using (2.1) and then choose $\dot{B}$ so that $\dot{\Delta}(\mu) = 0$. This is the approach we will take here.

From theorem 1.2 we have

$$\begin{aligned}\frac{\partial\Delta(\lambda)}{\partial q(x)} &= ay_1(x)(y_2(1)y_1(x) - y_1(1)y_2(x)) + by_1(x)(y_2'(1)y_1(x) - y_1'(1)y_2(x))\\ &\quad + cy_2(x)(y_2(1)y_1(x) - y_1(1)y_2(x)) + dy_2(x)(y_2'(1)y_1(x) - y_1'(1)y_2(x)),\end{aligned} \tag{2.3}$$

where all functions are evaluated at $(\cdot, \lambda, q)$. The derivative $(\partial\Delta(\lambda)/\partial q(x))'$ is the derivative of the right-hand side of (2.3) with respect to x. The key to evaluating (2.2) is the result of the computation (1.15), i.e. if we let w_+ and w_- denote two solutions of $-w'' + qw = \lambda w$, then

$$2\int_0^1 w_+w_-(z_+z_-)'\,dx = (w_+w_-z_+z_-)|_0^1 + \frac{1}{\mu - \nu}[w_+, z_+][w_-, z_-]|_0^1. \tag{2.4}$$

The integrand in (2.2) is clearly a linear combination of terms like the one evaluated in (2.4). Hence one could apply (2.4) directly to (2.2). However, it is better to try first to find solutions of $-u'' + qu = \lambda u$, f_+ and f_-, such that

$$\frac{\partial\Delta(\lambda)}{\partial q(x)} = h(\lambda)f_+(x, \lambda)f_-(x, \lambda).$$

Then one can evaluate (2.2) with a single application of (2.4).

The functions $f_\pm$ in the factorization turn out to be the solutions of $-u'' + qu = \lambda u$ associated with the eigenvectors of $BF(\lambda)$. Let $f_\pm(x, \lambda) = y_1(x) + c_\pm(\lambda)y_2(x)$, where

$$BF(\lambda)\begin{pmatrix}1\\ c_\pm\end{pmatrix} = \xi_\pm(\lambda)\begin{pmatrix}1\\ c_\pm\end{pmatrix}.$$

One computes easily

$$f_\pm = y_1(x,\lambda) + \frac{\xi_\pm(\lambda) - ay_1(1,\lambda) - by_1'(1,\lambda)}{ay_2(1,\lambda)+by_2'(1,\lambda)}\, y_2(x,\lambda).$$

The eigenvalues $\xi_\pm$ are the roots of $\xi^2 - \Delta(\lambda)\xi + 1 = 0$, and hence $\xi_+\xi_- = 1$ and $\xi_+ + \xi_- = \Delta$. The product

$$f_+f_- = y_1^2 + y_1y_2\left(\frac{cy_2(1,\lambda)+dy_2'(1,\lambda)-ay_1(1,\lambda)-by_1'(1,\lambda)}{ay_2(1,\lambda)+by_2'(1,\lambda)}\right)$$
$$+y_2^2\left(\frac{1-(ay_1(1,\lambda)+by_1'(1,\lambda))\Delta + (ay_1(1,\lambda)+by_1'(1,\lambda))^2}{(ay_2(1,\lambda)+by_2'(1,\lambda))^2}\right)$$

To simplify this, note that the denominator of the coefficient of $(y_2(x))^2$ equals

$$\begin{aligned}
&1-(ay_1(1,\lambda)+by_1'(1,\lambda))(cy_2(1,\lambda)+dy_2'(1,\lambda))\\
&\quad = 1 - acy_1(1,\lambda)y_2(1,\lambda) - bdy_1'(1,\lambda)y_2'(1,\lambda) - ad(1+y_2(1,\lambda)y_1'(1,\lambda))\\
&\qquad - bc(y_1(1,\lambda)y_2'(1,\lambda)-1)\\
&\quad = -(ay_2(1,\lambda)+by_2'(1,\lambda))(cy_1(1,\lambda)+dy_1'(1,\lambda))
\end{aligned}$$

since $ad-bc=1$. Thus

$$\begin{aligned}
(ay_2(1,\lambda)+by_2'(1,\lambda))f_+f_- &= (ay_2(1,\lambda)+by_2'(1,\lambda))y_1^2\\
&\quad +(cy_2(1,\lambda)+dy_2'(1,\lambda)-ay_1(1,\lambda)-by_1'(1,\lambda))y_1y_2\\
&\quad +(-cy_1(1,\lambda)-dy_2'(1,\lambda))y_2^2 = \partial\Delta(\lambda)/\partial q(x) \qquad (2.5)
\end{aligned}$$

directly from (2.3).

Now combining (2.1)-(2.5), we have

$$\begin{aligned}
&\int_0^1 \frac{\partial\Delta(\mu)}{\partial q(x)}\,\dot q(x,\nu)\,dx\\
&\quad = \frac{\partial\Delta(\mu)}{\partial q(x)}\frac{\partial\Delta(\nu)}{\partial q(x)}\bigg|_0^1\\
&\quad + \frac{(ay_2(1,\mu)+by_2'(1,\mu))(ay_2(1,\nu)+by_2'(1,\nu))}{\mu-\nu}[f_+(\mu),f_+(\nu)][f_-(\mu),f_-(\nu)]\bigg|_0^1.
\end{aligned}$$

However, since B is symplectic, by the definition of $f_\pm$

$$[f_\pm(\mu),f_\pm(\nu)]|_{x=1} = \xi_\pm(\mu)\xi_\pm(\nu)[f_\pm(\mu),f_\pm(\nu)]|_{x=0}.$$

Thus

$$0 = [f_+(\mu),f_+(\nu)][f_-(\mu),f_-(\nu)]|_0^1$$

and one is left with

$$\int_0^1 \frac{\partial\Delta(\mu)}{\partial q(t)}\,\dot q(t,\nu)\,dt = \frac{\partial\Delta(\mu)}{\partial q(t)}\frac{\partial\Delta(\nu)}{\partial q(t)}\bigg|_0^1. \qquad (2.6)$$

Note that (2.6) is also true, trivially, when $\mu=\nu$.

Now we are ready to choose $\dot B(\lambda)$ so that the flow determined by $(\dot q(t,\lambda), \dot B(t,\lambda))$ with $\dot q$ given by (2.1) will be isospectral. Since

$$\frac{\partial\Delta(\lambda)}{\partial q(0)} = ay_2(1,\lambda)+by_2'(1,\lambda), \qquad \frac{\partial\Delta(\lambda)}{\partial q(1)} = dy_2(1,\lambda)+by_1(1,\lambda),$$

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

one can use the trivial relations

$$\frac{\partial\Delta(\lambda)}{\partial a}=y_1(1,\lambda),\qquad \frac{\partial\Delta(\lambda)}{\partial b}=y_1'(1,\lambda),$$
$$\frac{\partial\Delta(\lambda)}{\partial c}=y_2(1,\lambda),\qquad \frac{\partial\Delta(\lambda)}{\partial d}=y_2'(1,\lambda)$$

to conclude from (2.6) that the correct choice is

$$\dot B=\begin{pmatrix} -b(dy_2(1,\lambda)+by_1(1,\lambda)) & 0 \\ -d(dy_2(1,\lambda)+by_1(1,\lambda))+a(ay_2(1,\lambda)+by_2'(1,\lambda)) & b(ay_2(1,\lambda)+by_2'(1,\lambda)) \end{pmatrix}. \tag{2.7}$$

Thus we have proven:

THEOREM 2.1. *For all real* λ *the vector fields* $(\dot q, \dot B)=V(q, B, \lambda)$, *given by* (2.1) *and* (2.7), *are tangent to the isospectral manifold* M *at* (q, B).

Replacing the constant λ in $V(q, B, \lambda)$ by any smooth function $\lambda(q, B)$ gives a new vector field tangent to M. If we let $\lambda=\mu_n(q, B)$, the nth generalized Dirichlet eigenvalue, we get an analytic vector field $V_n(q, B)$. The vector fields V_n give rise to commuting isospectral flows. The proof of this will be based on the uniqueness theorem (theorem 1.6) of § 1. We will show that μ_m, $m\neq n$, and $y_2(1, \mu_m, q)$, $m\neq n$, do not change under the flow of V_n and that the motion of μ_n and $y_2(1, \mu_n, q)$ is determined by $\Delta(\lambda, q, B)$ alone. Since the flow is isospectral for the generalized periodic spectrum, $\Delta(\lambda, q, B)$ is constant on the orbits. Thus it will be clear that the induced flows on $\{\mu_n\}_{n=1}^{\infty}$ and $\{y_2(1, \mu_n, q)\}_{n=1}^{\infty}$ commute, and, since all the flows fix b, d and $ad-bc$, the uniqueness theorem implies the flows commute.

We begin by showing that μ_m, $m\neq n$, is constant on integral curves of V_n. Let $\dot{}$ denote the derivative along V_n. The derivative $\dot y_2(x, \mu, q)$, with μ constant, satisfies

$$-\dot y_2''+q\dot y_2-\mu\dot y_2=-\dot q y_2.$$

Thus

$$\int_0^1 \dot q y_2^2(x, \mu)\,dx=(y_2\dot y_2'-y_2'\dot y_2)|_{x=0}^1,$$

and, evaluating at $\mu=\mu_m$, we have

$$\int_0^1 \dot q y_2^2(x, \mu_m)\,dx=\frac{y_2(1, \mu_m)}{b}(a\dot y_2(1, \mu_m)+b\dot y_2'(1, \mu_m)). \tag{2.8}$$

Hence, since $a(t)y_2(1, \mu_m(t), q(t))+by_2'(1, \mu_m(t), q(t))=0$ by definition, using the chain rule and substituting (1.2) and (2.8), we have

$$0=\dot a y_2(1, \mu_m)+\left(\frac{-b}{y_2(1, \mu_m)}\int_0^1 y_2^2(x, \mu_m)\,dx\right)\dot\mu_m+\frac{b}{y_2(1, \mu_m)}\int_0^1 \dot q y_2^2(x, \mu_m)\,dx.$$

Thus, using (2.7),

$$\left(\int_0^1 y_2^2(x, \mu_m)\,dx\right)\dot\mu_m=\int_0^1 \dot q y_2^2(x, \mu_m)\,dx-y_2^2(1, \mu_m)(dy_2(1, \mu_n)+by_1(1, \mu_n)). \tag{2.9}$$

To avoid complications in the factorization (2.5) when $\lambda=\mu_n$, we first compute $\int_0^1 \dot q y_2^2(x,\mu_m)\,dx$, assuming $\dot q$ is given by (2.1), and then pass to the limit $\lambda=\mu_n(q,B)$ at the end of the computation. From (2.4) and (2.5) we have

$$\int_0^1 \dot q y_2^2(x,\mu_m)\,dx = y_2^2(1,\mu_m)\frac{\partial\Delta(\lambda)}{\partial q(1)}+\frac{ay_2(1,\lambda)+by_2'(1,\lambda)}{\lambda-\mu_m} \times[y_2(x,\mu_m),f_+(x,\lambda)][y_2(x,\mu_m),f_-(x,\lambda)]\Big|_0^1. \qquad (2.10)$$

Note by the formula for $f_\pm$

$$[y_2(x,\mu_m),f_\pm(x,\lambda)]=-1 \qquad \text{at } x=0,$$

and at $x=1$

$$\begin{aligned}[y_2(x,\mu_m),f_\pm(x,\lambda)]&=\frac{y_2(1,\mu_m)}{b}(af_\pm(1,\lambda)+bf_\pm'(1,\lambda))\\ &=\frac{y_2(1,\mu_m)}{b}\left(\frac{a(y_2(1,\lambda)\xi_\pm(\lambda)+b)+b(y_2'(1,\lambda)\xi_\pm-a)}{ay_2(1,\lambda)+by_2'(1,\lambda)}\right)\\ &=\frac{y_2(1,\mu_m)}{b}\xi_\pm(\lambda).\end{aligned}$$

Moreover, by (2.3)

$$\frac{\partial\Delta(\lambda)}{\partial q(1)}=dy_2(1,\lambda)+by_1(1,\lambda).$$

Thus, combining (2.9) and (2.10), we conclude that along V_n

$$\begin{aligned}\left(\int_0^1 y_1^2(x,\mu_m)\,dx\right)\dot\mu_m&=\left(\lim_{\lambda\to\mu_n}\frac{ay_2(1,\lambda)+by_2'(1,\lambda)}{\lambda-\mu_m}\right)\left(\frac{y_2^2(1,\mu_m)}{b^2}-1\right)\\ &=\begin{cases}0, & n\neq m,\\ \left[a\dfrac{\partial y_2}{\partial\mu}(1,\mu_m)+b\left(\dfrac{\partial y_2'}{\partial\mu}\right)(1,\mu_m)\right]\left(\dfrac{y_2^2(1,\mu_m)}{b^2}-1\right), & n=m.\end{cases}\end{aligned}$$

To further simplify the case $n=m$, one may use (2.8). The final result is: the derivative of μ_m along V_n is given by

$$\dot\mu_m=\begin{cases}0 & \text{if } m\neq n,\\ \dfrac{b}{y_2(1,\mu_n)}-\dfrac{y_2(1,\mu_n)}{b} & \text{if } m=n.\end{cases} \qquad (2.11)$$

Since

$$\begin{aligned}\Delta(\mu_m)&=\left[ay_1+b\left(\frac{y_1y_2'-1}{y_2}\right)+cy_2+dy_2'\right]\Big|_{(1,\mu_m)}\\ &=\frac{-b}{y_2(1,\mu_m)}-\frac{y_2(1,\mu_m)}{b},\end{aligned} \qquad (2.12)$$

one sees

$$\dot{\mu}_n = \sqrt{\Delta^2(\mu_n) - 4}, \tag{2.13}$$

where $\sqrt{\ }$ denotes either the positive or negative square root. The correct choice of $\sqrt{\ }$ at $t = 0$ is given in (2.11). Subsequently the sign of $\dot{\mu}_n$ changes from + to − when μ_n hits λ_{2n}, and from − to + when μ_n hits λ_{2n-1}. Thus, since $\Delta(\lambda)$ is constant on integral curves of V_n, $\mu_n(t)$ is determined by the initial values of μ_n, $y_2(1, \mu_n, q)$ and $\Delta(\lambda)$.

Finally, since for $\mu_m = \mu_m(q, B)$

$$\Delta(\mu_m, q, B) = -\frac{b}{y_2(1, \mu_m, q)} - \frac{y_2(1, \mu_m, q)}{b},$$

we have

$$y_2(1, \mu_m, q) = -b/\xi(\mu_m),$$

where $\xi(\mu_m)$ is one of the roots of $\xi^2 - \Delta(\mu_m) + 1$. For $m \neq n$, since b, $\Delta(\lambda)$ and μ_m are constant on the integral curves of V_n, and $y_2(1, \mu_m, q(t))$ is continuous in t, we conclude $y_2(1, \mu_m, q)$ is constant on the integral curves too. For $m = n$ we still have $\Delta(\lambda)$ determined by the initial data and

$$y_2(1, \mu_n(t), q(t)) = -b/\xi_*(\mu_n(t)), \tag{2.14}$$

but now $* = \pm$ is determined by

$$\dot{\mu}_n = -\xi_*(\mu_n(t)) + \xi_*^{-1}(\mu_n(t)). \tag{2.15}$$

Thus we have proven:

THEOREM 2.2. *The flows generated by the vector fields V_n, $n = 1, \ldots,$ commute. The data μ_m and $y_2(1, \mu_m, q)$ are constant on the integral curves of V_n for $m \neq n$. On integral curves of V_n, μ_n and $y_2(1, \mu_n, q)$ are determined by* (2.13), (2.14) *and* (2.15).

As in the earlier papers in this series, the vector fields V_n can be integrated explicitly. As before, the integration is based on the following tricky but directly verifiable observation which actually goes back to Darboux (see the discussion in chapter 5 of [**6**]): if

$$-d^2f/dx^2 + qf = \alpha f \tag{2.16}$$

and

$$-d^2g/dx^2 + qg = \beta g,$$

and we set

$$\tilde{q} = q - 2\frac{d^2}{dx^2}\log|f|,$$

$\tilde{f} = 1/f$ and $\tilde{g} = (1/f)[f, g]$, then

$$-d^2\tilde{f}/dx^2 + \tilde{q}\tilde{f} = \alpha\tilde{f}, \qquad -d\tilde{g}/dx^2 + \tilde{q}\tilde{g} = \beta\tilde{g}.$$

These equations only hold where $f \neq 0$, and since f in general will have zeros, the usefulness of this observation is not immediately clear. However, one can repeat the argument with $\tilde{q}$ in place of q and $(1/f)\,[f, g]$ in place of f in (2.16). Then setting

$$\hat{q} = \tilde{q} - 2\frac{d^2}{dx^2}\log\left|\frac{1}{f}[f, g]\right| = q - 2\frac{d^2}{dx^2}\log|[f, g]|, \qquad \hat{f} = \frac{f}{[f, g]}, \tag{2.17}$$

one again has $-\hat f''(x)+\hat q(x)\hat f(x)=\beta\hat f(x)$ when $[f, g](x)\neq 0$. Moreover, if $-h''+qh=\gamma h$, then, setting

$$\hat h=\frac{f}{[f,g]}\left[\frac{[f,g]}{f},\frac{[f,h]}{f}\right]=(\gamma-\alpha)h+(\alpha-\beta)\frac{[f,h]}{[f,g]}g, \tag{2.18}$$

one has $-\hat h''(x)+\hat q(x)\hat h(x)=\gamma\hat h(x)$ when $[f, g](x)\neq 0$. As we will see, it is possible to choose f and g in the cases we require so that $[f, g](x)\neq 0$ on $[0, 1]$.

Given (q, B), we let $\mu_n(t)$ be the solution of (2.13). To integrate V_n, we must choose f and g and $\hat B$ so that

$$\mu_m(\hat q, \hat B)=\begin{cases}\mu_m(q, B), & m\neq n,\\ \mu_n(t), & m=n\end{cases}$$

$$y_2(1, \mu_n(\hat q, \hat B), \hat q)=\begin{cases}y_2(1, \mu_m(q, B), q), & m\neq n,\\ -b/\xi_*(\mu_n(t)), & m=n.\end{cases}$$

The correct choices turn out to be

$$f=y_2(x, \mu_n, q),\qquad g=f_*(x, \mu_n(t), q, B),$$

where once again $*=\pm$ is determined by (2.15). Taking (2.17) with these choices of f and g as the definition of $\hat q$, we need to show that (2.17) does define a potential in $L^2[0, 1]$, choose $\hat B$ and then show that (2.19) holds.

LEMMA 2.1. *$[y_2(\mu_n, f_*(\lambda)](x)<0$ for $(x, \lambda)\in[0, 1]\times[\lambda_{2n-1}, \lambda_{2n}]$.*

Proof. This proof follows the proof of lemma 6.1 in [**6**] closely. We have

$$[y_2(\mu_n), f_*(\lambda)](0)=-1,\qquad [y_2(\mu_n), f_*(\lambda)](1)=\frac{y_2(\mu_n)}{b}\xi_*(\lambda).$$

Since $\Delta(\mu_n)=-b/y_2(1, \mu_n)-y_2(1, \mu_n)/b$, we have $\xi_*(\mu_n)=-b/y_2(1, \mu_n)=\xi_1(\mu_n)$ or $\xi_*(\mu_n)=-y_2(1, \mu_n)/b=\xi_2(\mu_n)$. Thus $[y_2(\mu_n), f_*(\lambda)](x)<0$ for $\lambda\in[\lambda_{2n-1}, \lambda_{2n}]$ when $x=0$ or 1.

Since $ay_1(1, \mu_n)+by_1'(1, \mu_n)=-b/y_2(1, \mu_n)$,

$$\lim_{\lambda\to\mu_n}[y_2(\mu_n), f_*(\lambda)](x)=-1-\frac{\xi_*(\mu_n)+by_2^{-1}(1, \mu_n)}{a\dfrac{\partial y_2}{\partial\mu}(1, \mu_n)+b\dfrac{\partial y_2'}{\partial\mu}(1, \mu_n)}\int_0^x y_2^2(x, \mu_n)\,dx$$

(see (2.8)). Thus the limit is monotone in x and by the preceding paragraph we conclude

$$\lim_{\lambda\to\mu_n}[y_2(\mu_n), f_*(\lambda)](x)<0$$

for $x\in[0, 1]$.

Let $\bar\mu=\sup\{\mu'\le\lambda_{2n}: [y_2(\mu_n), f_*(\mu)]<0 \text{ for } (x, \mu)\in[0, 1]\times[\mu_n, \mu']\}$. If $[y_2(\mu_n), f_*(\bar\mu)]<0$ for $x\in[0, 1]$, $\bar\mu=\lambda_{2n}$ and we have part of the conclusion. If $[y_2(\mu_n, f_*(\bar\mu)](x_0)=0$, then

$$x_0\in(0, 1),\qquad 0=[y_2(\mu_n), f_*(\bar\mu)]'(x_0)=(\bar\mu-\mu_n)y_2(x_0, \mu_n)f_*(x_0, \bar\mu),$$

and $[y_2(\mu_n), f_*(\bar\mu)](x)$ does not change sign at $x=x_0$. From $[y_2(\mu_n), f_*(\bar\mu)](x_0)=0$ and either $y_2(x_0, \mu_n)=0$ or $f_*(x_0, \bar\mu)=0$, it follows that $y_2(x_0, \mu_n)=f_*(x_0, \bar\mu)=0$.

Since both these zeros must be simple, it follows that

$$[y_2(\mu_n), f_x(\bar{\mu})](x) = k(x-x_0)^3 + O((x-x_0)^4)$$

with $k \neq 0$. This contradicts $[y_2(\mu_n), f_*(\bar{\mu})](x)$ being of one sign on $[0, 1]$ and we conclude

$$[y_2(\mu_n), f_*(\lambda)](x) < 0$$

for $(x, \lambda) \in [0, 1] \times [\mu_n, \lambda_{2n}]$. The argument for $\lambda \in [\lambda_{2n-1}, \mu_n]$ is identical. □

With lemma 2.1 we know that for $\hat{\mu}_n \in [\lambda_{2n-1}, \lambda_{2n}]$

$$\hat{q} = q - 2\frac{d^2}{dx^2}\log[f_*(\hat{\mu}_n), y_2(\mu_n)]$$

is a well defined potential on $[0, 1]$: $\hat{q} - q$ is absolutely continuous. Thus by (2.17)

$$y(x) = \frac{y_2(x, \hat{\mu}_n, q)}{[f_*(\hat{\mu}_n), y_2(\mu_n)]}$$

satisfies $-y'' + \hat{q}y = \hat{\mu}_n y$. Since $y(0) = 0$ and $y'(0) = 1$, we see

$$y(x) = y_2(x, \hat{\mu}_n, \hat{q}).$$

To choose $\hat{B}$ so that $\mu_n(\hat{q}, \hat{B}) = \hat{\mu}_n$, we need to pick $\hat{a}$ so that

$$\hat{a}y_2(1, \hat{\mu}_n, \hat{q}) + by_2'(1, \hat{\mu}_n, \hat{q}) = 0.$$

We have

$$\begin{aligned} y_2(1, \hat{\mu}_n, \hat{q}) &= -b/\xi_*(\hat{\mu}_n), \\ y_2'(1, \hat{\mu}_n, \hat{q}) &= \frac{y_2'(1, \mu_n, q)}{\left(\frac{-y_2(1, \mu_n, q)}{b}\xi_*(\hat{\mu}_n)\right)} - \frac{y_2^2(1, \mu_n, q)(\hat{\mu}_n - \mu_n)f_*(1, \hat{\mu}_n, q)}{\left(\frac{y_2(1, \mu_n, q)}{b}\xi_*(\hat{\mu}_n)\right)^2} \\ &= \frac{a}{\xi_*(\hat{\mu}_n)} - \frac{(\hat{\mu}_n - \mu_n)b^2 f_*(1, \hat{\mu}_n, q)}{\xi_*^2(\hat{\mu}_n)}. \end{aligned} \tag{2.19}$$

Thus

$$\hat{a}y_2(1, \hat{\mu}_n, \hat{q}) + by_2'(1, \hat{\mu}_n, \hat{q}) = \frac{1}{\xi_*(\hat{\mu}_n)}\left(-\hat{a}b + ab - (\hat{\mu}_n - \mu_n)\frac{b^3 f_*(1, \hat{\mu}_n, q)}{\xi_*(\hat{\mu}_n)}\right).$$

Thus we require

$$\begin{aligned} \hat{a} &= a - (\hat{\mu}_n - \mu_n)\frac{b^2 f_*(1, \hat{\mu}_n, q)}{\xi_*(\hat{\mu}_n)} \\ &= a - (\hat{\mu}_n - \mu_n)b^2\frac{y_2(1, \hat{\mu}_n, q) + b/\xi_*(\hat{\mu}_n)}{ay_2(1, \hat{\mu}_n, q) + by_2'(1, \hat{\mu}_n, q)} \end{aligned} \tag{2.20}$$

and define

$$\hat{B} = \begin{pmatrix} \hat{a} & b \\ \hat{c} & d \end{pmatrix},$$

where $\hat{c}$ is determined by $\hat{a}d - b\hat{c} = 1$.

LEMMA 2.2. *$\mu_m(\hat{q}, \hat{B})$ and $y_2(1, \mu_m(\hat{q}, \hat{B}), \hat{q})$ are given by* (2.19) *for all m.*

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Proof. By (2.18), since $[y_2(\mu_n), y_2(\lambda)]$ vanishes to second order at $x=0$, we have

$$y_2(x, \lambda, \hat{q}) = y_2(x, \lambda, q) + \frac{\mu_n - \hat{\mu}_n}{\lambda - \mu_n} \frac{[y_2(\mu_n), y_2(\lambda)]}{[y_2(\mu_n), f_*(\hat{\mu}_n)]} f_*(x, \hat{\mu}_n, q).$$

Hence

$$y_2(1, \lambda, \hat{q}) = y_2(1, \lambda, q) + \frac{\mu_n - \hat{\mu}_n}{\lambda - \mu_n} \frac{a y_2(1, \lambda, q) + b y_2'(1, \lambda, q)}{\xi_*(\hat{\mu}_n)} f_*(1, \hat{\mu}, q) \quad (2.21)$$

and

$$y_2'(1, \lambda, \hat{q}) = y_2'(1, \lambda, q) + \frac{\mu_n - \hat{\mu}_n}{\lambda - \mu_n} \Bigg(\frac{b(\mu_n - \lambda) y_2(1, \lambda, q)}{\xi_*(\hat{\mu}_n)} f_*(1, \hat{\mu}_n, q)$$
$$+ \frac{a y_2(1, \lambda, q) + b y_2'(1, \lambda, q)}{\xi_*(\hat{\mu}_n)} f_*'(1, \hat{\mu}_n, q)$$
$$- \frac{b(a y_2(1, \lambda, q) + b y_2'(1, \lambda, q))}{y_2(1, \mu_n, q) \xi_*(\hat{\mu}_n)^2} [y_2(\mu_n), f_*(\hat{\mu}_n)]'(1) f_*(1, \hat{\mu}_n) \Bigg).$$

Thus, since $a y_2(1, \mu_m, q) + b y_2'(1, \mu_m, q) = 0$, it follows directly from (2.20) that for $m \neq n$

$$\hat{a} y_2(1, \mu_m, \hat{q}) + b y_2'(1, \mu_m, \hat{q}) = 0.$$

In § 1 we showed

$$\mu_m = (m - \tfrac{1}{2})^2 \pi^2 + O(1).$$

Thus μ_m must actually be the mth zero of $\hat{a} y_2(1, \lambda, \hat{q}) + b y_2(1, \lambda, \hat{q})$ for m large. Consequently, this function has no zeros other than $\{\mu\}_{m \neq n}$ and $\hat{\mu}_n$. Thus

$$\mu_m(\hat{q}, \hat{B}) = \begin{cases} \mu_m(q, B), & m \neq n, \\ \hat{\mu}_n, & m = n \end{cases}.$$

Finally we note that (2.21) implies $y_2(1, \mu_m, \hat{q}) = y_2(1, \mu_m, q)$ for $m \neq n$ and $y_2(1, \hat{\mu}_n, \hat{q}) = -b/\xi_*(\hat{\mu}_n)$. □

Combining theorem 2.2, lemma 2.2 and theorem 1.5, we have proven:

THEOREM 2.3. *The solution to* $(\dot{q}, \dot{B}) = V_n(q, B)$ *with initial data* $q(0) = q$, $B(0) = B$ *is given by*

$$q(t) = q - 2 \frac{d^2}{dx^2} \log [f_*(\mu_n(t), q), y_2(\mu_n, q)],$$

$$B(t) = \begin{pmatrix} a - (\mu_n(t) - \mu_n) b^2 \dfrac{f_*(1, \mu_n(t), q)}{\xi_*(\mu_n(t))} & b \\ c - (\mu_n(t) - \mu_n) bd \dfrac{f_*(1, \mu_n(t), q)}{\xi_*(\mu_n(t))} & d \end{pmatrix},$$

where $\mu_n = \mu_n(q, B)$, *and* $\mu_n(t)$ *and* $*$ *are determined from* (2.13) *and* (2.15).

The formula in theorem 2.3 may be iterated to give the result of moving successively on integral curves of $V_{n_1}, \ldots, V_{n_k}$. Without further computation this would involve $y_2(x, \lambda, q_i)$ and $f_*(x, \lambda, q_i, B_i)$, $i = 0, \ldots, k-1$, where (q_i, B_i) is the result of moving (q_{i-1}, B_{i-1}) along an integral curve of V_i. However, since the flows commute, it is possible to derive manageable expressions for the final (q, B) in terms of $y_2(x, \lambda, q_0)$ and $f_*(x, \lambda, q_0, B_0)$.

To derive the formula described in the preceding paragraph, we begin by computing the analogue of (2.21) for $f_*(x, \lambda, \hat{q}, \hat{B})$. For μ_n, $\hat{\mu}$ and λ distinct, and $\sigma = \pm$, $\delta = \pm$, let

$$f(x) = \frac{\lambda - \mu_n}{\lambda - \hat{\mu}_n} f_\sigma(x, \lambda, q, B) + \frac{\mu_n - \hat{\mu}_n}{\lambda - \hat{\mu}_n} \frac{[y_2(\mu_n), f_\sigma(\lambda)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]} f_\delta(x, \hat{\mu}_n, q, B).$$

Note that $f(0) = 1$ and by (2.17)

$$-f'' + \hat{q} f = \lambda f.$$

Thus to show that $f(x) = f_\sigma(x, \lambda, \hat{q})$ it suffices to show

$$\hat{B}\begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} = \xi_\sigma(\lambda)\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}.$$

We will do this without assuming $\Delta(\lambda, \hat{q}, \hat{B}) = \Delta(\lambda, q, B)$, hence proving $(\hat{q}, B)$ is isospectral to (q, B) without using theorem 2.3. We have, with $\xi_\tau(\lambda)$ equal to the root of $\xi^2 - \Delta(\lambda, q, B)\xi + 1 = 0$ used in defining $f_\tau(x, \lambda, q, B)$,

$$\hat{B}\begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} = \left(\frac{\lambda - \mu_n}{\lambda - \hat{\mu}_n}\right) \hat{B}\begin{pmatrix} f_\sigma(1, \lambda) \\ f'_\sigma(1, \lambda) \end{pmatrix} + \left(\frac{\mu_n - \hat{\mu}_n}{\lambda - \hat{\mu}_n}\right)\left(\frac{\xi_\sigma(\lambda)}{\xi_\delta(\hat{\mu}_n)}\right) \hat{B}\begin{pmatrix} f_\delta(1, \hat{\mu}_n) \\ f'_\delta(1, \hat{\mu}_n) \end{pmatrix}$$
$$+ \left(\frac{\mu_n - \hat{\mu}_n}{\lambda - \hat{\mu}_n}\right)\left(\frac{[y_2(\mu_n), f_\sigma(\lambda)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]}\right)'(1) \hat{B}\begin{pmatrix} 0 \\ f_\delta(1, \hat{\mu}_n) \end{pmatrix}.$$

Hence, since $[y_2(\beta), f_*(\lambda)](0) = 1$,

$$\hat{B}\begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} - \xi_\sigma(\lambda)\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}$$
$$= \begin{pmatrix} \hat{a} - a & 0 \\ \hat{c} - c & 0 \end{pmatrix}\left[\left(\frac{\lambda - \mu_n}{\lambda - \hat{\mu}_n}\right)\begin{pmatrix} f_\sigma(1, \lambda) \\ f'_\sigma(1, \lambda) \end{pmatrix} + \left(\frac{\mu_n - \hat{\mu}_n}{\lambda - \hat{\mu}_n}\right)\left(\frac{\xi_\sigma(\lambda)}{\xi_\delta(\hat{\mu}_n)}\right)\begin{pmatrix} f_\delta(1, \hat{\mu}_n) \\ f'_\delta(1, \hat{\mu}_n) \end{pmatrix}\right]$$
$$+ \left(\frac{\mu_n - \hat{\mu}_n}{\lambda - \hat{\mu}_n}\right)\left(\frac{(\mu_n - \lambda) f_\sigma(1, \lambda)\xi_\delta(\hat{\mu}_n) - (\mu_n - \hat{\mu}_n) f_\delta(1, \hat{\mu}_n)\xi_\sigma(\lambda)}{(\xi_\delta(\hat{\mu}_n))^2}\right)$$
$$\times \begin{pmatrix} b^2 f_\delta(1, \hat{\mu}_n) \\ bd f_\delta(1, \hat{\mu}_n) \end{pmatrix}.$$

Since

$$\begin{pmatrix} \hat{a} - a \\ \hat{c} - c \end{pmatrix} = \begin{pmatrix} (\mu_n - \hat{\mu}_n) b^2 \dfrac{f_\delta(1, \hat{\mu}_n)}{\xi_\delta(\hat{\mu}_n)} \\ (\mu_n - \hat{\mu}_n) bd \dfrac{f_\delta(1, \hat{\mu}_n)}{\xi_\delta(\hat{\mu}_n)} \end{pmatrix},$$

we have

$$\hat{B}\begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} - \xi_\sigma(\lambda)\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

as claimed. Thus

$$f_\sigma(x, \lambda, \hat{q}, \hat{B}) = \left(\frac{\lambda - \mu_n}{\lambda - \hat{\mu}_n}\right) f_\sigma(x, \lambda, q, B) + \left(\frac{\mu_n - \hat{\mu}_n}{\lambda - \hat{\mu}_n}\right) \frac{[y_2(\mu_n), f_\sigma(\lambda)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]} f_\delta(x, \hat{\mu}_n, q, B). \tag{2.22}$$

To get a compact formula for the result of moving μ_k to $\hat{\mu}_k$, $k = 1, \ldots, N$, we will need a compact formula for $[y_2(\alpha, \hat{q}), f_*(\beta, \hat{q}, \hat{B})]$. For this we note first

$$f_*(x, \hat{\mu}_n)[y_2(\mu_n), y_2(\alpha)] = y_2(x, \alpha)[y_2(\mu_n), f_*(\hat{\mu}_n)] - y_2(x, \mu_n)[y_2(\alpha), f_*(\hat{\mu}_n)].$$

Hence (2.21) may be rewritten

$$y_2(x, \alpha, \hat{q}) = \left(\frac{\alpha - \hat{\mu}_n}{\alpha - \mu_n}\right) y_2(x, \alpha, q) + \left(\frac{\hat{\mu}_n - \mu_n}{\alpha - \mu_n}\right) \frac{[y_2(\alpha, f_*(\hat{\mu}_n)]}{[y_2(\mu_n, f_*(\hat{\mu}_n)]} y_2(x, \mu_n, q). \tag{2.23}$$

Next by (2.22) and (2.23)

$$\begin{aligned}
[y_2(\alpha, \hat{q}), f_\sigma(\beta, \hat{q}, \hat{B})]' &= (\alpha - \beta) y_2(x, \alpha, \hat{q}) f_\sigma(x, \beta, \hat{q}, \hat{B}) \\
&= (\alpha - \beta)\Bigg[\left(\frac{\alpha - \hat{\mu}_n}{\alpha - \mu_n}\right)\left(\frac{\beta - \mu_n}{\beta - \hat{\mu}_n}\right) y_2(x, \alpha) f_\sigma(x, \beta) \\
&\quad + \left(\frac{\alpha - \hat{\mu}_n}{\alpha - \mu_n}\right)\left(\frac{\mu_n - \hat{\mu}_n}{\beta - \hat{\mu}_n}\right) \frac{[y_2(\mu_n), f_\sigma(\beta)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]} y_2(x, \alpha) f_\delta(x, \hat{\mu}_n) \\
&\quad + \left(\frac{\hat{\mu}_n - \mu_n}{\alpha - \mu_n}\right)\left(\frac{\beta - \mu_n}{\beta - \hat{\mu}_n}\right) \frac{[y_2(\alpha), f_\delta(\hat{\mu}_n)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]} y_2(x, \mu_n) f_\sigma(x, \beta) \\
&\quad + \left(\frac{\hat{\mu}_n - \mu_n}{\alpha - \mu_n}\right)\left(\frac{\mu_n - \hat{\mu}_n}{\beta - \hat{\mu}_n}\right) \frac{[y_2(\mu_n), f_\sigma(\beta)][y_2(\alpha), f_\delta(\hat{\mu}_n)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]^2} y_2(x, \mu_n) f_\delta(x, \hat{\mu}_n)\Bigg] \\
&= \Bigg[\left(\frac{\alpha - \hat{\mu}_n}{\alpha - \mu_n}\right)\left(\frac{\beta - \mu_n}{\beta - \hat{\mu}_n}\right)[y_2(\alpha), f_\sigma(\beta)] \\
&\quad + \left(\frac{\alpha - \beta}{\alpha - \mu_n}\right)\left(\frac{\mu_n - \hat{\mu}_n}{\beta - \hat{\mu}_n}\right) \frac{[y_2(\mu_n), f_\sigma(\beta)][y_2(\alpha), f_\delta(\hat{\mu}_n)]}{[y_2(\mu_n), f_\delta(\hat{\mu}_n)]}\Bigg]'
\end{aligned}$$

and we conclude

$$\begin{aligned}
&[y_2(\alpha, \hat{q}), f_\sigma(\beta, \hat{q}, \hat{B})] \\
&\quad = \left(\frac{\alpha - \hat{\mu}_n}{\alpha - \mu_n}\right)\left(\frac{\beta - \mu_n}{\beta - \hat{\mu}_n}\right)[y_2(\alpha, q), f_\sigma(\beta, q, \beta)] \\
&\qquad + \left(\frac{\alpha - \beta}{\alpha - \mu_n}\right)\left(\frac{\mu_n - \hat{\mu}_n}{\beta - \hat{\mu}_n}\right) \frac{[y_2(\mu_n, q), f_\sigma(\beta, q, \beta)][y_2(\alpha, q), f_\delta(\hat{\mu}_n, q, B)]}{[y_2(\mu_n, q), f_\delta(\hat{\mu}_n, q, B)]},
\end{aligned} \tag{2.24}$$

since both sides of this equation equal -1 at $x = 0$.

THEOREM 2.4. *Assume $\mu_k \neq \hat{\mu}_k$ for $k = 1, \ldots, N$. Let*

$$\theta_{jk}(q, B) = \frac{1}{\mu_j - \hat{\mu}_k}[y_2(\mu_j, q), f_{\sigma_k}(\hat{\mu}_k, q, B)]$$

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

and let $\theta(q, B)$ be the matrix with entries θ_{jk}. Using the flow of V_k, $k=1,\ldots,N$, to move μ_k to $\hat{\mu}_k$ and $y_2(1,\mu_k,q)$ to $-b\xi_{\sigma_k}^{-1}(\hat{\mu}_k)$, $i=1,\ldots,N$, one reaches $(\hat{q},\hat{B})$, where

$$\hat{q}=q-2\frac{d^2}{dx^2}\log|\det\theta(q,B)|,$$

$$\hat{a}=a-\sum_{k=1}^{N} b^2\prod_{l\neq k}\left(\frac{\hat{\mu}_k-\mu_l}{\hat{\mu}_k-\hat{\mu}_l}\right)(\hat{\mu}_k-\mu_k)\frac{f_{\sigma_k}(1,\hat{\mu}_k,q,B)}{\xi_{\sigma_k}(\hat{\mu}_k)},$$

$$\hat{c}=c-\sum_{k=1}^{N} bd\prod_{l\neq k}\left(\frac{\hat{\mu}_k-\mu_l}{\hat{\mu}_k-\hat{\mu}_l}\right)(\hat{\mu}_k-\mu_k)\frac{f_{\sigma_k}(1,\hat{\mu}_k,q,B)}{\xi_{\sigma_k}(\hat{\mu}_k)},$$

$\hat{d}=d$ and $\hat{b}=b$.

Proof. Letting $(q_0,B_0)=(q,B)$ and (q_m,B_m) be the result of moving from (q_{m-1},B_{m-1}) in the flow of V_k, formula (2.24) implies for $j,k\neq m$

$$\frac{(\mu_j-\mu_m)(\hat{\mu}_k-\hat{\mu}_m)}{(\mu_j-\hat{\mu}_m)(\hat{\mu}_k-\mu_m)}\theta_{jk}(q_m,B_m)$$
$$=\theta_{jk}(q_{m-1},B_{m-1})-\frac{\theta_{jm}(q_{m-1},B_{m-1})\theta_{mk}(q_{m-1},B_{m-1})}{\theta_{mm}(q_{m-1},B_{m-1})}. \tag{2.25}$$

Consider the process of reducing θ to upper triangular form by Gauss elimination. Letting $\theta_{jk}(q_m,B_m)=\theta_{jk}(m)$ and

$$c_{jk}(m)=\frac{(\mu_j-\mu_m)(\hat{\mu}_k-\hat{\mu}_m)}{(\mu_j-\hat{\mu}_m)(\hat{\mu}_k-\mu_m)}=a_j(m)b_k(m),$$

after the entries below the diagonal in the first column have been cancelled, (2.25) implies

$$\det\theta=\det\begin{pmatrix}\theta_{11}(0) & & \cdots & \theta_{1N}(0)\\ 0 & c_{22}(1)\theta_{22}(1) & \cdots & c_{2N}(1)\theta_{2N}(1)\\ \vdots & \vdots & & \vdots\\ 0 & c_{N2}(1)\theta_{N2}(1) & \cdots & c_{NN}(1)\theta_{NN}(1)\end{pmatrix}$$
$$=\prod_{j=2}^{N}a_j(1)\prod_{k=2}^{N}b_k(1)\det\begin{pmatrix}\theta_{11}(0) & & \cdots & \theta_{1N}(0)\\ 0 & \theta_{22}(1) & \cdots & \theta_{2N}(1)\\ \vdots & \vdots & & \vdots\\ 0 & \theta_{N2}(1) & \cdots & \theta_{NN}(1)\end{pmatrix}.$$

Thus, proceeding in the same manner,

$$\det\theta=\left(\prod_{m=1}^{N-1}\prod_{j=m+1}^{N}a_j(m)\prod_{k=m+1}^{N}b_k(m)\right)\prod_{k=1}^{N}\theta_{kk}(k-1).$$

Since theorem 2.3 implies

$$\hat{q}=q-\sum_{k=1}^{N}2\frac{d^2}{dx^2}\log|\theta_{kk}(q_{k-1},B_{k-1})|,$$

we have proven the formula for $\hat{q}$ in theorem 2.4.

To prove the formula for $\hat{B}$ in theorem 2.4, it clearly suffices to prove the formula for $\hat{a}$ and we will use induction on N for that. In the case $N=1$ the formula reduces to (2.20). Assuming the formula for $N = M-1$ and applying it to (q_1, B_1),

$$\hat{a} = a_1 - \sum_{k=2}^{M} b^2 \prod_{\substack{l=2\\ l\neq k}}^{M} \left(\frac{\hat{\mu}_k - \mu_l}{\hat{\mu}_k - \hat{\mu}_l}\right)(\hat{\mu}_k - \mu_k)\frac{f_{\sigma_k}(1, \hat{\mu}_k, q_1, B_1)}{\xi_{\sigma_k}(\hat{\mu}_k)}.$$

However, (2.22) implies for $k = 2, \ldots, M$

$$f_{\sigma_k}(1, \hat{\mu}_k, q_1, B_1) = \frac{\hat{\mu}_k - \mu_1}{\hat{\mu}_k - \hat{\mu}_1} f_{\sigma_k}(1, \hat{\mu}_k, q_0, B_0) + \frac{\hat{\mu}_1 - \mu_1}{\hat{\mu}_1 - \hat{\mu}_k} f_{\sigma_1}(1, \hat{\mu}_1, q_0, B_0).$$

Thus, using (2.20) to replace a_1 by a_0, to complete the proof of theorem 2.4, it suffices to show

$$1 + \sum_{k=2}^{M} \frac{\hat{\mu}_k - \mu_k}{\hat{\mu}_1 - \hat{\mu}_k} \prod_{l=2}^{M} \left(\frac{\hat{\mu}_k - \mu_l}{\hat{\mu}_k - \hat{\mu}_l}\right) = \prod_{l=2}^{M} \frac{\hat{\mu}_1 - \mu_l}{\hat{\mu}_1 - \hat{\mu}_l}. \tag{2.26}$$

Formula (2.26) is just the statement that the sum of the residues of

$$\frac{1}{z - \hat{\mu}_1} \prod_{l=2}^{M} \frac{z - \mu_l}{z - \hat{\mu}_l} = F(z)$$

at $z = \hat{\mu}_1, \ldots, \hat{\mu}_M$ must be 1, which follows from $\lim_{|z|\to\infty} zF(z) = 1$. □

Remark 2.5. In theorem 2.4 we assume $\hat{\mu}_k \neq \mu_k$, $k = 1, \ldots, N$, since otherwise θ will be undefined. However, in theorem 2.3 the corresponding assumption, $\mu_n \neq \mu_n(t)$, is not made. While strictly speaking $[f_*(\mu_n, q), y_2(\mu_n, q)]$ is undefined, it should be clear from the computation in the proof of lemma 2.1 that in theorem 2.3 one should define

$$[f_*(\mu_n, q), y_2(\mu_n, q)](x) = 1 + \frac{\xi_*(\mu_n) + b y_2^{-1}(1, \mu_n)}{a \dfrac{\partial y_2}{\partial \mu}(1, \mu_n) + b \dfrac{\partial y_2'}{\partial \mu}(1, \mu_n)} \int_0^x y_2^2(x, \mu_n)\, dx.$$

It is possible to extend theorem 2.4 to general $\hat{\mu}_k$, $k = 1, \ldots, M$, in a similar fashion, but, since we will not have further use for $\hat{q}$, we will leave the extended formula for $\hat{q}$ to the reader. To extend the formulae for $\hat{B}$ in theorems 2.3 and 2.4 to general $\hat{\mu}_k$, one has only to observe that

$$\lim_{\hat{\mu}_k \to \mu_k} (\hat{\mu}_k - \mu_k) \frac{f_*(1, \hat{\mu}_k)}{\xi_*(\hat{\mu}_k)} = \frac{y_2(1, \mu_k) + b\xi_*^{-1}(\mu_k)}{a \dfrac{\partial y_2}{\partial \mu}(1, \mu_k) + b \dfrac{\partial y_2'}{\partial \mu}(1, \mu_k)}.$$

3. *The manifold M*

In this section we give M the structure of a real analytic manifold. The strategy for doing this is quite simple. In § 1 we showed that the mapping Φ was an analytic homeomorphism. Elementary considerations show $\Phi(M)$ is contained in an explicit analytic submanifold N of S. Using the flows of § 2, we can show $\Phi(M \cap \{d = d_0\}) = N \cap \{d = d_0\}$ when $M \cap \{d = d_0\} \neq \varnothing$. In § 5 we will see that the range of d on M is $\mathbf{R}$, and we will conclude $\Phi(M) = N$. Here we only show that there is a d_∞ so that $\Phi(M) = N \cap \{d < d_\infty\}$. Hence M inherits its real analytic structure from $N \cap \{d < d_\infty\}$.

The manifold N is the subset of S defined by the relations

$$\lambda_{2n-1} \leq \mu_n \leqq \lambda_{2n}, \qquad \kappa_n^2 + b\Delta(\mu_n)\kappa_n + b^2 = 0, \qquad n = 1, 2, \ldots, \tag{3.1}$$

$$b = b_0, \qquad \frac{2d}{b} + r = \frac{2(a_0 + d_0)}{b} + \int_0^1 q_0 \, dx = I_0, \tag{3.2}$$

where $(q_0, B_0) \in M$. The function r is the coordinate function on S from (1.13). We have already seen that

$$\Delta(\mu_n(q, B)) = \frac{-b}{y_2(1, \mu_n(q, B), q)} - \frac{y_2(1, \mu_n(q, B), q)}{b}$$

(see (2.12)), so that (3.1) holds on $\Phi(M)$. Moreover, combining theorems 1.3 and 1.4, we see that (3.2) holds on $\Phi(M)$ also. Thus $\Phi(M) \subset N$.

That N is an analytic submanifold of S is a consequence of the properties of Δ stated in the Introduction. These imply that

(i) $|\Delta(\lambda_k)| = 2$ for all k,

(ii) $|\Delta(\mu)| > 2$ for $\lambda_{2n-1} < \mu < \lambda_{2n}$,

(iii) if $\lambda_{2n-1} < \lambda_{2n}$, then $\partial\Delta/\partial\lambda \neq 0$ at λ_{2n} or λ_{2n-1}.

Hence for each n (3.1) defines a real analytic curve—topologically a circle in the (μ_n, k_n)-plane if $\lambda_{2n-1} < \lambda_{2n}$. When $\lambda_{2n-1} = \lambda_{2n}$ this curve degenerates to a point and (μ_n, k_n) is constant on N. By (1.6) the number of indices n for which $\lambda_{2n} = \lambda_{2n-1}$ must be finite, and we delete them in what follows. We may give N locally as an explicit submanifold in S by solving

$$\kappa_n^2 + b\Delta(\mu_n)\kappa_n + b^2 = 0$$

for k_n or μ_n as is appropriate for each admissible index n.

Since the topology on S is induced by the topology on $l^2_{\mathbf{R}}$ via the coordinate map $s \to \tilde{s}$, any point $s_0 \in N$ has a neighbourhood U in S such that on U

$$|\mu_n - (n - \tfrac{1}{2})^2\pi^2 - r(s_0)| < 1 \qquad \text{and} \qquad |\kappa_n < b$$

for $n \geq n(s_0)$. Hence, since N will be an analytic submanifold if it is given locally as the graph of an analytic function, and the preceding remarks show we can ignore any finite set of indices in proving this analyticity, it suffices to prove the real analyticity on $V = \{\{\mu_n\}_{n>n_1} : \{b, d, \mu_1, \kappa_1, \ldots\} \in U$ for some $\{b, d, \mu_1, \kappa_1, \ldots, \mu_{n_1}, \kappa_{n_1}\kappa_{n_1+1}, \ldots\}\}$ of

$$F: \{\mu_n\}_{n>n_1} \to \{\kappa_n\}_{n>n_1},$$

where

$$\kappa_n = (b/2)\Delta(\mu_n)(\sqrt{1 - 4(\Delta(\mu_n))^{-2}} - 1)$$

and $n_1 > n(s_0)$ is chosen large enough so that on U one has $\lambda_{2n-1} < \mu_n < \lambda_{2n}$ for $n \geq n_1$. Since each component of F is clearly analytic on $V_1 = \{\{\mu_n\}_{n>n_1} : \{\text{Re}\{\mu_n\}\}_{n>n_1} \in V, |\text{Im}\{\mu_n\}| < 1$ and $\{\text{Im}\{\mu_n\}\}_{n>n_1} \in l^2\}$, it will suffice to show that the image of V_1 under F is bounded, i.e. that $\{\{\tilde{\kappa}_n\}_{n>n_1}: \{\kappa_n\}_{n>n_1} \in F(V_1)\}$ is bounded in l^2, where $\tilde{\kappa}_n = n^2(\kappa_n + (-1)^n/n)$; cf. theorem A.3 of [6].

Showing F is locally bounded requires a sharper estimate on $\Delta(\mu_n)$ than we have used up to now. Since, given $(q_0, B_0) \in M$,

$$\Delta(\mu) = a_0 y_1(1, \mu, q_0) + b y_1'(1, \mu, q_0) + c_0 y_2(1, \mu, q_0) + d_0 y_2'(1, \mu, q_0)$$

and $|\sqrt{\mu_n} - (n - \frac{1}{2})\pi| < A/n$ on V_1, one sees easily from theorem 1.1 that on V_1

$$|\Delta(\mu_n) - b y_1'(1, \mu_n, q_0)| < B/n.$$

To avoid the $O(1)$ error term in the basic estimate for y_1', we use

$$y_1'(1, \mu_n, q_0) = \frac{y_2' y_1 - 1}{y_2}\bigg|_{(1,\mu_n,q_0)} = \frac{-\sin^2(\mu_n^{1/2}) + (\cos(\mu_n^{1/2}))O(\mu_n^{-1/2}) + O(\mu_n^{-1})}{\mu_n^{-1/2}\sin\mu_n^{1/2} + O(\mu_n^{-1})}$$

Thus $|\Delta(\mu_n) - b\pi(n - \frac{1}{2})(-1)^n| \le C/n$ on V_1 and

$$\left|\kappa_n(\mu_n) + \frac{(-1)^n}{\pi(n - \frac{1}{2})}\right| \le \frac{D}{n^3}.$$

Thus, since D is constant on V_1, we see $\{(\tilde{\kappa}_{n_1}(\mu_{n_1}), \ldots): (\mu_{n_1}, \ldots) \in V_1\}$ is bounded in the l^2-norm. Thus we conlude that N is a real analytic submanifold of S.

Given $(q_0, B_0) \in M$, one sees from theorem 2.2 that by using the flows from the vector fields $V_1, V_2, \ldots$ successively, one can reach a subset of $M \cap \{d = d_0\}$ whose image under Φ contains all points of $N \cap \{d = d_0\}$ such that for n sufficiently large

$$(\mu_n, \kappa_n) = (\mu_n(q_0, B_0), y_2(1, \mu_n(q_0, B_0), q_0)). \tag{3.3}$$

By (3.2) on $N \cap \{d = d_0\}$

$$r = I_0 - 2d_0/b_0,$$

i.e. on $N \cap \{d = d_0\}$

$$\mu_n = (n - \tfrac{1}{2})^2\pi^2 + I_0 - 2d_0/b_0 + l^2(n),$$

and we see that the points of $N \cap \{d = d_0\}$ satisfying (3.3) for all but finitely many n are dense in $N \cap \{d = d_0\}$. Since Φ^{-1} is continuous and $M \cap \{d = d_0\}$—as the intersection of closed sets—is closed in $L^2_{\mathbf{R}}[0, 1] \times \mathrm{SL}(2, \mathbf{R})$, we have proven:

LEMMA 3.1. $\Phi(M \cap \{d = d_0\}) = N \cap \{d = d_0\}$.

To complete the argument outlined in the first paragraph of this section, we only need the following:

LEMMA 3.2. *The range of d on M is $(-\infty, d_\infty)$ for some $d_\infty \le \infty$.*

Proof. In this lemma we make our first use of the $*$-involution. Since this involution fixes M and interchanges a and d (cf. lemma 4.1), a and d have the same range on M. Thus we only need consider the range of a on M. Given $(q, B) \in M$, we have from theorem 2.3

$$a(t) = a - b^2(\mu_n(t) - \mu_n)\frac{y_2(1, \mu_n(t), q) + b\xi_*^{-1}(\mu_n(t))}{y_2(1, \mu_n(t), q) + b y_2'(1, \mu_n(t), q)}$$

as one moves (q, B) under the flow of V_n. In particular, if we move to the point

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

with $\mu_n = \mu_n(q, B)$ and $\kappa_n = b^2(y^2(1, \mu_n, q))^{-1}$, we have (see remark 2.5)

$$a_n = a - b^2 \left.\frac{y_2 - b^2 y_2^{-1}}{a\frac{\partial y_2}{\partial \mu} + b\frac{\partial y_2'}{\partial \mu}}\right|_{(1,\mu_n,q)} = a + b\frac{y_2^2(1, \mu_n, q) - b^2}{\int_0^1 y_2^2(x, \mu_n, q)\, dx}$$

by (1.2). Since

$$y_2^2(1, \mu_n, q) = \sin^2 \pi(n - \tfrac{1}{2})x / \pi^2(n - \tfrac{1}{2})^2 + O(1/n^3),$$

we have

$$a_n = a - 2b^2\pi^2 n^2 + O(n). \tag{3.4}$$

Thus $d(M) = a(M) \supset (-\infty, a]$.

Since (q, B) was an arbitrary point of M, to complete the proof we only need show that d cannot assume its least upper bound on M. Suppose it did assume this maximum value at (q_0, B_0). Then we could choose a λ_0 such that $a_0 y_2(1, \lambda_0, q_0) + b y_2'(1, \lambda_0, q_0) \neq 0$. By theorem 2.1 $d \neq 0$ at (q_0, B_0) on the integral curve of $V(q, B, \lambda_0)$ through (q_0, B_0). Since this curve lies in M, the proof is complete. □

4. *Even points*

In this section we will describe the points of M which are fixed by the involution

$$q(x) \to q^*(x) = q(1-x),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

on M. The fixed points of this involution play the role of even potentials for Dirichlet and periodic boundary conditions and, following [**4**], we call them even points. These points are important when one considers the submanifold of M, M_B, obtained by fixing the boundary condition B.

We begin with two simple lemmas.

LEMMA 4.1. *The $*$-involution maps M onto M.*

Proof. Suppose $-y'' + q(x)y = \lambda y$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}.$$

Letting $z = y(1-x)$, we have $-z'' + q^*(x)z = \lambda z$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z(0) \\ -z'(0) \end{pmatrix} = \begin{pmatrix} z(1) \\ -z'(1) \end{pmatrix}.$$

Thus

$$\begin{aligned}\begin{pmatrix} z(0) \\ z'(0) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z(1) \\ z'(1) \end{pmatrix} \\ &= \begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} z(1) \\ z'(1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^*\begin{pmatrix} z(1) \\ z'(1) \end{pmatrix}\end{aligned}$$

since $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. Thus $(q, B) \in M \Rightarrow (q^*, B^*) \in M$ and, since $*$ is an involution, the proof is complete. □

LEMMA 4.2. *The following identities hold*:

(i) $y_1(1, \lambda, q^*) = y_2'(1, \lambda, q)$,

(ii) $y_2(1, \lambda, q^*) = y_2(1, \lambda, q)$,

(iii) $y_1'(1, \lambda, q^*) = y_1'(1, \lambda, q)$,

(iv) $y_2^2(1, \lambda, q) + b\Delta(\lambda)y_2(1, \lambda, q) + b^2$

$$= (ay_2(1, \lambda, q) + by_2'(1, \lambda, q))(a^*y_2(1, \lambda, q^*) + by_2'(1, \lambda, q^*))$$
$$= y_2^2(1, \lambda, q^*) + b\Delta(\lambda)y_2(1, \lambda, q^*) + b^2.$$

We have already used (iv) in the special case $\lambda = \mu_n(a, b, q)$.

Proof. To verify (i)–(iii), we note that since $y_i(1-x, \lambda, q^*)$, $i = 1, 2$, solves $-y'' + qy = \lambda y$, we must have

$$\begin{pmatrix} y_1(1-x, \lambda, q^*) \\ y_2(1-x, \lambda, q^*) \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} y_1(x, \lambda, q) \\ y_2(x, \lambda, q) \end{pmatrix} \tag{4.1}$$

for some constants α_i, $i = 1, \ldots, 4$. Evaluating (4.1) and its derivative at $x = 1$, we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} y_1(1, \lambda, q) \\ y_2(1, \lambda, q) \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} y_1'(1, \lambda, q) \\ y_2'(1, \lambda, q) \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} y_1 & -y_1' \\ y_2 & -y_2' \end{pmatrix}^{-1} = \begin{pmatrix} y_2'(1, \lambda, q) & -y_1'(1, \lambda, q) \\ y_2(1, \lambda, q) & -y_1(1, \lambda, q) \end{pmatrix}$$

by the Wronskian identity. Substituting for $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ in (4.1) and evaluating the resulting expression and its derivative at $x = 0$ gives (i)–(iii).

To verify (iv), we compute as follows:

$$\begin{aligned}
\Delta(\lambda) &= [ay_1 + by_1' + cy_2 + dy_2'](1, \lambda, q) \\
&= \left[ay_1 + \frac{b(y_1y_2' - 1)}{y_2} + cy_2 + dy_2' \right](1, \lambda, q) \\
&= \left[\frac{bay_1y_2 + b^2y_1y_2' + bcy_2^2 + bdy_2y_2' - b^2}{by_2} \right](1, \lambda, q) \\
&= \left[\frac{bay_1y_2 + b^2y_1y_2' + ady_2^2 + bdy_2y_2'}{by_2} - \frac{b}{y_2} - \frac{y_2}{b} \right](1, \lambda, q) \\
&= \left[\frac{(ay_2 + by_2')(dy_2 + by_1)}{by_2} - \frac{b}{y_2} - \frac{y_2}{b} \right](1, \lambda, q).
\end{aligned}$$

Now (iv) follows by multiplying by by_2 and collecting terms. □

The identity (iv) is important here because it is related to the basic formula

$$\hat{a} = a - b^2(\tilde{\mu}_j - \mu_j) \left[\frac{y_2 + b\xi_*}{ay_2 + by_2'} \right](1, \tilde{\mu}_j, q) \tag{4.2}$$

of § 2. By (iv), $[y_2+b\xi_*](1, \tilde{\mu}_j, q)=0$ only if $\tilde{\mu}_j$ is a root of $[ay_2+by_2'](1, \lambda, q)=0$ or of $[dy_2+by_1](1, \lambda, q)=0$. We let the roots of $[dy_2+by_1](1, \lambda, q)=0$ be $\nu_1, \nu_2, \ldots$. Since the ν are just the μ for (q^*, B^*), they have all the properties of the μ established in § 1. In the next lemma we think of $y_2+b\xi_*$ as a function on

$$\{(\mu, \xi): \xi^2-\Delta(\mu)\xi+1=0\},$$

but we denote the two branches as $\xi_\pm(\mu)$ as before.

LEMMA 4.3. *If $\nu_n=\mu_n$ and $\lambda_{2n-1}\neq\lambda_{2n}$, then $\lambda_{2n-1}<\mu_n<\lambda_{2n}$ and one branch of $y_2+b\xi_*$ has a zero of order 2 in μ at $\mu=\mu_n$. $y_2+b\xi_*$ has no other zeros and hence is of one sign for $\lambda_{2n-1}\le\mu\le\lambda_{2n}$. Moreover, $y_2+b\xi_*$ is of one sign for $\lambda_{2n-1}\le\mu\le\lambda_{2n}$ only if $\nu_n=\mu_n$.*

Proof. By theorem 1.3, ay_2+by_2' and dy_2+by_1 have simple zeros at μ_n and ν_n for $n=1,2,\ldots$ respectively and no other zeros. The first step in this proof is showing that, if $\mu_n=\nu_n=\lambda_{2n}$ or $\mu_n=\nu_n=\lambda_{2n-1}$, then $\lambda_{2n}=\lambda_{2n-1}$. If either of these occurs, it follows that $y_2(1, \mu_n)=\pm b$, $y_2'(1, \mu_n)=\mp a$, $y_1(1, \mu_n)=\mp d$ and thus $y_1'(1, \mu_n)=\pm c$, since $y_1y_2'-y_1'y_2=1$. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1(1, \mu_n) & y_2(1, \mu_n) \\ y_1'(1, \mu_n) & y_2'(1, \mu_n) \end{pmatrix}=\begin{pmatrix} \mp 1 & 0 \\ 0 & \mp 1 \end{pmatrix}.$$

Thus the generalized periodic eigenvalue at $\lambda=\mu_n$ has multiplicity 2 and we conclude $\lambda_{2n-1}=\lambda_{2n}$.

Since $[ay_2+by_2'](1, \lambda)$ and $[dy_2+by_1](1, \lambda)$ both have the asymptotic form

$$b\cos\sqrt{\lambda}+O(1/\sqrt{\lambda}),$$

it follows that their derivatives at μ_n and ν_n respectively have the same sign. If $\nu_n=\mu_n$, it follows from lemma 4.2(iv) that $[(y_2+b\xi_+)(y_2+b\xi_-)](1, \mu)$ is non-negative on $[\lambda_{2n-1}, \lambda_{2n}]$ and that it has a double zero at $\mu=\mu_n$ and no other zeros in $[\lambda_{2n-1}, \lambda_{2n}]$. Since $\xi_+\neq\xi_-$ on $(\lambda_{2n-1}, \lambda_{2n})$, it follows that, if $\lambda_{2n-1}<\lambda_{2n}$, one of the factors $y_2+b\xi_\pm$ must have a double zero at $\mu=\mu_n$ and the other factor is non-zero on $[\lambda_{2n-1}, \lambda_{2n}]$. On the other hand, if $\nu_n\neq\mu_n$, then lemma 4.2(iv) implies $[(y_2+b\xi_+)(y_2+b\xi_-)](1, \mu)$ is negative on the interval between ν_n and μ_n. The lemma follows immediately. □

Now we are ready to study the fixed points of the $*$-involution. The point of contact with the preceding discussion is the following lemma:

LEMMA 4.4. *The point (q, B) is fixed by the involution if and only if $\nu_n(q, B)=\mu_n(q, B)$ for all n.*

Proof. By lemma 4.2(i) and (ii) for each n

$$\nu_n(q, B)=\mu_n(q^*, B^*).$$

Thus the necessity of the condition is obvious.

Suppose $\mu_n=\nu_n$, $n=1,2,\ldots$. Since by theorem 1.3

$$\mu_n=(n-\tfrac{1}{2})^2\pi^2+2\frac{a}{b}+\int_0^1 q\,dx+l^2(n),$$

$$\nu_n=(n-\tfrac{1}{2})^2\pi^2+2\frac{d}{b}+\int_0^1 q^*\,dx+l^2(n),$$

we conclude $a=d$. Since $y_2(1,\mu_n,q)=y_2(1,\mu_n,q^*)$ for all n by lemma 4.2(i), we see $\Phi(q,B)=\Phi(q^*,B^*)$ and hence $(q,B)=(q^*,B^*)$. □

Lemmas 4.2, 4.3 and 4.4 make it possible to assign an index to the fixed points of the $*$-involution. If (q,B) is fixed, then

$$F_j(\mu,\xi)=(\mu-\mu_j)\left(\frac{y_2(1,\mu,q)+b\xi}{ay_2(1,\mu,q)+by_2'(1,\mu,q)}\right)$$

is of one sign on the jth gap, i.e. on

$$\{(\mu,\xi):\lambda_{2j-1}\le\mu\le\lambda_{2j},\,0=\xi^2-\Delta(\mu)\xi+1\}.$$

The index of (q,B) is the set $I(q,B)$ of j for which $\lambda_{2j-1}<\lambda_j$ and F_j is non-positive on the jth gap. This set is always finite: since $y_2(1,\lambda,q)\to 0$ as $\lambda\to\infty$, one sees from lemma 4.2(iv) that the sign of F_j on the jth gap for j large will be the sign of $\xi_\pm(\mu_j)(a\partial y_2/\partial\mu(1,\mu_j,q)+b\partial y_2/\partial\mu(1,\mu_j,q))$. From (1.2) and the asymptotics of $\Delta(\lambda)$ and $y_2(1,\lambda)$ as $\lambda\to\infty$, one sees that this expression is positive for j large.

The main result of this section is the following converse to the preceding remark:

THEOREM 4.5. *For each finite subset I of $\mathbf{Z}_+\backslash\{j:\lambda_{2j-1}=\lambda_{2j}\}$ there is a unique fixed point of $*$ on M of index* **I**.

Our strategy in proving theorem 4.5 will be to exploit the fact that the fixed points are precisely the critical points of the function a on M. Using the mapping Φ of § 1 to identify M with $N\cap\{d<d_\infty\}$, we may consider a as an analytic function on $N\cap\{d<d_\infty\}$. Given a finite subset I of $\mathbf{Z}_+\backslash\{j:\lambda_{2j-1}=\lambda_{2j}\}$, it is natural to split $N\cap\{d<d_\infty\}$ into the product of the finite-dimensional torus

$$N_I=\{(\mu_{j_1},\kappa_{j_1},\ldots,\mu_{j_l},\kappa_{j_l}):j_i\in I\}$$

and the infinite-dimensional analytic manifold

$$N_{I^c}=\{(b,d,\mu_{j_1},\kappa_{j_1},\ldots,\mu_{j_l},\kappa_{j_l},\ldots:j_i\in I^c,d<d_\infty\}.$$

The points of N_I and N_{I^c} are subject to (3.1) as well as, in the case of N_{I^c}, (3.2) and the asymptotic conditions defining S. Clearly $N\cap\{d<d_\infty\}=N_I\times N_{I^c}$, and we denote points of N as pairs (r,s), $r\in N_I$, $s\in N_{I^c}$.

LEMMA 4.6. *There is an analytic function $r(s)$ from N_{I^c} to N_I such that*

$$a(r(s),s)=\min_{N_I}a(r,s).$$

Moreover, $a(r,s)>a(r(s),s)$ for $r\ne r(s)$.

Proof. Since $f(r)=a(r,s)$ is a continuous function on a finite-dimensional torus, it assumes its minimum at a point we call $r(s)$. Then, using the formula from theorem 2.4 with $(q,B)=(q(s),B(s))=\Phi^{-1}(r(s),s)$, the values of $a(r,s)$, $r\in N_I$, are given by

$$a=a(r(s),s)-\sum_{j\in I}b^2(\sigma_j-\mu_j)\prod_{\substack{i\in I\\ j\ne i}}\left(\frac{\sigma_j-\mu_i}{\sigma_j-\mu_i}\right)\left[\frac{y_2+b\xi_*}{ay_2+by_2'}\right](1,\sigma_j),\qquad(4.3)$$

where $(\sigma_j,\xi_*(\sigma_j))$ ranges over $C_j=\{(\mu,\xi)=\xi^2-\Delta(\mu)+1=0,\ \lambda_{2j-1}\le\mu\le\lambda_{2j}\}$ for $j\in I$, and we use remark 2.5 when $\sigma_j=\mu_j$. Since $y_2+b\xi$ must be non-positive on C_j, we see from lemma 4.3 that $\nu_j(q(s),B(s))=\mu_j(q(s),B(s))$, $j\in I$. Moreover, since lemma 4.3 also implies that

$$(\mu_j(q(s), B(s)), -b^{-1}y_2(1, \mu_j(q(s), B(s)), q(s)))$$

is the only zero of $y_2+b\xi$ on C_j, it follows from (4.3) that $a(r(s), s) < a(r, s)$ for $r \neq r(s)$.

To see that $r(s)$ is real analytic, we first note that given $s_0 \in N_{I^c}$, $\lambda_{2j-1} < \mu_j(q(s_0), B(s_0)) < \lambda_{2j}$, $j \in I$, and hence $\tilde{\mu}_j$, $j \in I$, defined in (1.13) are admissible coordinates on N_I near $r(s_0)$. From (4.3)

$$\frac{\partial^2 a}{\partial \tilde{\mu}_i \partial \tilde{\mu}_j}(r(s_0), s_0) = \begin{cases} 0, & i \neq j, \\ -b^2 \left[\dfrac{(\partial^2/\partial\mu^2)(y_2 + b\xi_*)}{a\, \partial y_2/\partial\mu + b\, \partial y_2'/\partial\mu} \right](1, \mu_j(q(s_0), B(s_0))), & i = j. \end{cases}$$

Moreover, since the zeros of $y_2 + b\xi_*$ at

$$(\mu_j(q(s_0), B(s_0)), -b^{-1}y_2(1, \mu_i(q(s_0), B(s_0)), q(s_0))), \qquad j \in I,$$

are exactly of order two, $(\partial^2 a/\partial\tilde{\mu}_j^2)(r(s_0), s_0) > 0$, $j \in I$. Thus we may use the implicit function theorem to solve

$$\frac{\partial a}{\partial \tilde{\mu}_j}(r, s) = 0, \qquad j \in I,$$

with base point $(r(s_0), s_0)$. This yields an analytic function $\tilde{r}(s)$ defined near s_0 such that $(\partial a/\partial\tilde{\mu}_j)(\tilde{r}(s), s) = 0$, $j \in I$, and $(\partial^2 a/\partial\tilde{\mu}_j\partial\tilde{\mu}_i)(\tilde{r}(s), s)$ is positive definite. Setting $(\tilde{q}(s), \tilde{B}(s)) = \Phi^{-1}(\tilde{r}(s), s)$, it follows from lemma 4.2(iv) that $\nu_j(\tilde{q}(s), \tilde{B}(s)) = \mu_j(\tilde{q}(s), \tilde{B}(s))$, $j \in I$, and hence $F_j(\mu, \xi)$ is of one sign on C_j, $j \in I$. The positivity of the Hessian of a at $(F(s), s)$ then implies $\tilde{r}_j$ is non-positive. Thus (4.3) with $(r(s), s)$ replaced by $(\tilde{r}(s), s)$ implies $a(\tilde{r}(s), s) \leq a(r, s)$, $r \in N_I$, and we conclude $\tilde{r}(s) = r(s)$ for s near s_0. □

The uniqueness statement in theorem 4.5 is a consequence of the following:

PROPOSITION 4.7. *Suppose $(q_0, B_0) = \Phi^{-1}(r_0, s_0)$ a fixed point for the $*$-involution of index* **I**. *Then $r_0 = r(s_0)$ and*

$$a(r(s_0), s_0) > \min\{a(r(s), s), d(r(s), s)\}$$

for $s \neq s_0$.

Proof. If (q_0, B_0) is of index **I**, it follows immediately from lemma 4.6 that $r_0 = r(s_0)$.

Let $a' = a(r(s'), s')$, $d' = d(r(s'), s')$ and $a_0 = a(r(s_0), s_0) = d(r(s_0), s_0)$, and suppose $\min\{a', d'\} \geq a_0$. Since the $*$-involution fixes $y_2(1, \mu)$ and $\Delta(\mu)$, and hence $a(r(s)^*, s^*) = \min_{N_I} a(r, s^*)$, the involution must take $(r(s), s)$ to $(r(s^*), s^*)$ for some $s^* \in N_{I^c}$ by lemma 4.6. Hence we can assume without loss of generality that $a' \leq d'$. If $a' > a_0$, then, as in the proof of lemma 3.2, we can use the flow of V_j for j sufficiently large, and hence in I^c to move a' down below a_0. Thus we have s'' such that $a(r(s'), s'') < a_0$ and hence $a(r(s''), s'') < a_0$, but $d(r(s''), s'') = d'$. Since $N_{I^c} \cap \{d = d'\}$ is connected and a is continuous, it follows that there is an s''' such that $a(r(s'''), s''') = a_0$ and $d(r(s'''), s''') = d'$. Finally, using the involution, we get s_1 such that $a(r(s_1), s_1) = d' \geq a_0$ and $d(r(s_1), s_1) = a_0 = d(r(s_0), s_0)$.

Let $(q', B') = \Phi^{-1}(r(s_0), s_1)$ and set

$$(\tilde{\mu}_j, \xi_*(\tilde{\mu}_j)) = (\mu_j(q_1, B_1), -b^{-1}y_2(1, \mu_j(q_1, B_1), q_1))$$

for $j \in I^c$. By theorem 2.4 and the continuity of Φ^{-1}

$$a(r(s_0), s_1) = \lim_{N\to\infty} a_0 - b^2 \sum_{j\in S_N} (\tilde{\mu}_j - \mu_j) \prod_{\substack{i\in S_N \\ i\neq j}} \left(\frac{\tilde{\mu}_j - \mu_i}{\tilde{\mu}_j - \tilde{\mu}_i}\right)\left[\frac{y_2 + b\xi_*}{a_0 y_2 + b y_2'}\right](1, \tilde{\mu}_i), \quad (4.4)$$

where the μ_i and y_2 are computed at (q_0, B_0) and $S_N = I^c \cap \{j \leq N\}$. Thus $a(r(s_0), s_1) \leq a_0$ and $d' = a(r(s_1), s_1) \leq a(r(s_0), s_1)$. Since $d' \geq a_0$, we conclude $d' = a_0$. This means that we can repeat the preceding argument with $(r(s'), s')$ in place of $(r(s_1), s_1)$, i.e. (4.4) holds with $a(r(s_0), s_1)$ replaced by $a(r(s_0), s')$ and $(\tilde{\mu}_j, \xi_*(\tilde{\mu}_j))$, $j \in I^c$, defined as before with s_1 replaced by s'. Thus, since $a' \geq a_0$ by assumption, we conclude $a' = a_0$ and $(\tilde{\mu}_j, \xi_*(\tilde{\mu}_j)) = (\mu_j(q_0, B_0), -b^{-1}y_2(1, \mu_j(q_0, B_0), q_0))$ for $j \in I^c$. Since $d' = d(r(s'), s') = a_0 = d(r(s_0), s_0)$, we have $s' = s_0$. □

Proposition 4.7 implies that to find a fixed point of the involution with index I, we only need find an $s_0 \in I$, where $\sup_{N_{I^c}} a(r(s), s)$ is assumed. However, since N_{I^c} is not compact, we have not been able to do this directly. Instead we will exploit a connection between maxima of a and minima of

$$\int_0^1 (k+q)^2\, dx$$

on $M \cap \{a = d\}$.

LEMMA 4.8. *For any real constant k, the derivative of* $\frac{1}{4}\int_0^1 (k+q)^2\, dx$ *on the flow of the vector field* $(\dot{q}, \dot{B}) = V(q, B, \lambda)$ *in theorem* 2.1 *is given by*

$$\left[\frac{1}{4}\int_0^1 (k+q)^2\, dx\right]^{\boldsymbol{\cdot}} = [(d-a)((\lambda+k)y_2 + y_1) + (c + (\lambda+k)b)(y_1 - y_2')](1, \lambda, q). \quad (4.5)$$

Proof. We have

$$\left[\frac{1}{4}\int_0^1 (k+q)^2\, dx\right]^{\boldsymbol{\cdot}} = \frac{1}{2}\int_0^1 (k+q)\dot{q}\, dx.$$

Since $(2/b)(a+b) + \int_0^1 q\, dx$ is constant on M (see (3.2)) and the vector field is tangent to M, we have (see (2.7))

$$\frac{k}{2}\int_0^1 \dot{q}\, dx = -\frac{k}{b}(\dot{a} + \dot{d}) = [k(d-a)y_2 + kb(y_1 - y_2')](1, \lambda). \quad (4.6)$$

Using the abbreviations

$$\langle y\rangle = ay(1, \lambda) + by'(1, \lambda), \qquad [y] = cy(1, \lambda) + dy'(1, \lambda),$$

the formula for $\partial\Delta(\lambda)/\partial q(x)$ given in (2.3) becomes

$$\frac{\partial\Delta(\lambda)}{\partial q(x)} = y_1(x)(\langle y_2\rangle y_1(x) - \langle y_1\rangle y_2(x)) + y_2(x)([y_2]y_1(x) - [y_1]y_2(x)).$$

Substituting the derivative of this expression for $\frac{1}{2}\dot{q}$ and systematically replacing qy

by $y''+\lambda y$, we arrive at

$$\begin{aligned}\tfrac{1}{2}\int_0^1 q\dot{q}\,dx &= \langle y_2\rangle((y_1')^2+\lambda y_1^2)|_0^1-\langle y_1\rangle(y_1'y_2'+\lambda y_1y_2)|_0^1\\ &\quad+[y_2](y_1'y_2'+\lambda y_1y_2)|_0^1-[y_1]((y_2')^2+\lambda y_2^2)|_0^1\\ &=[(ay_2+by_2')((y_1')^2+\lambda y_1^2)+(cy_2+dy_2'-ay_1-by_1')(y_1'y_2'+\lambda y_1y_2)\\ &\quad-(cy_1+dy_1')((y_2')^2+\lambda y_2^2)](1,\lambda)-[\lambda(ay_2+by_2')-cy_1-dy_1'](1,\lambda)\\ &=[-ay_1'+\lambda by_1-cy_2'+\lambda dy_2-\lambda ay_2-\lambda by_2'+cy_1+dy_1'](1,\lambda)\\ &=[(d-a)(y_1'+\lambda y_2)+(c+\lambda b)(y_1-y_2')](1,\lambda). \end{aligned} \tag{4.7}$$

In passing to the third equality, we used $(y_1y_2'-y_2y_1')=1$ four times. Combining (4.6) and (4.7) gives (4.5). □

The following proposition completes the proof of theorem 4.5.

PROPOSITION 4.9. *Given a finite index set* **I**, *let* $M_I=\Phi^{-1}(\{(r(s),s): s\in N_{I^c}\})$, *where* $r(s)$ *is the analytic function of lemma* 4.6. *Then for k sufficiently large*

$$\lim_{M_I\cap\{a=d\}}\int_0^1 (q+k)^2\,dx$$

is assumed at a fixed point of index **I**.

Proof. First we show the minimum is assumed. Since, as noted in the proof of proposition 4.7, the involution leaves M_I invariant if $a-d$ is not identically zero on M_I, it must be of two signs on M_I. Hence, since N_{I^c} is connected, we conclude $M_I\cap\{a=d\}$ is non-empty.

Choosing a sequence (q_j, B_j) in $M_I\cap\{a=d\}$ on which

$$\tfrac{1}{4}\int_0^1 (q+k)^2\,dx$$

tends to its infimum, we can choose a subsequence such that $q_{j_i}\to q_\infty$ weakly in $L^2[0,1]$. The sequence $a_j=d_j$ is bounded, since $(2/b)(a_j+d_j)+\int_0^1 q_j\,dx$ is constant on M, and $c_j=a_j^2-1/b$, since $B_j\in \mathrm{SL}(2,\mathbf{R})$. Thus we may assume $B_{j_i}\to B_\infty$ with $a_\infty=d_\infty$.

To see that $(q_\infty, B_\infty)\in M_I\cap\{a=d\}$, we note first that theorem 1.5 of [**6**] implies, for $l=1, 2$, $y_l(x,\lambda,q_{j_i})\to y_l(x,\lambda,q_\infty)$ uniformly on bounded subsets of $[0,1]\times\mathbf{C}$. From the formulae

$$y_1'(x,\lambda,q)=-\lambda^{1/2}\sin\lambda^{1/2}x+\int_0^x \cos\lambda^{1/2}(x-t)q(t)y_1(t,\lambda,q)\,dt,$$

$$y_2'(x,\lambda,q)=\cos\lambda^{1/2}x+\int_0^x \cos\lambda^{1/2}(x-t)q(t)y_2(t,\lambda,q)\,dt,$$

one sees that $y_l'(1,\lambda,q_{j_i})\to y_l'(1,\lambda,q_\infty)$ for $\lambda\in\mathbf{C}$. Thus $\Delta(\lambda,q_\infty,B_\infty)=\Delta(\lambda)$ and $(q_\infty,B_\infty)\in M$. Finally, since

$$y_2(1,\mu,q_{j_i})+b\xi_*(\mu)\to y_2(1,\mu,q_\infty)+b\xi_*(\mu)$$

on C_j for $j\in I$, the characterization of $r(s)$ in lemma 4.6 shows $(q_\infty,B_\infty)\in M_I$.

Since $\int_0^1 (q+k)^2\,dx$ can only jump down on weakly convergent sequences,

$$\int_0^1 (k+q_\infty)^2\,dx = \min_{M_I\cap\{a=d\}} \int_0^1 (k+q)^2\,dx,$$

and the derivative of

$$\int_0^1 (k+q_\infty)^2\,dx$$

along any differentiable curve in $M_I \cap \{a=d\}$ must vanish when the curve passes through (q_∞, B_∞). If $M_I \cap \{a=d\}$ is a smooth submanifold of M, the preceding condition is equivalent to: the derivative of

$$\int_0^1 (k+q_\infty)^2\,dx$$

along any differentiable curve in M, tangent to M_I and $\{a=d\}$ at (q_∞, B_∞), vanishes when the curve passes through (q_∞, B_∞). The second form is much more convenient for us, so we will show that $\{a=d\}$ and M_I intersect transversally and hence $M_I \cap \{a=d\}$ is smooth.

Given $(q, B) \in M_I \cap \{a=d\}$, theorems 1.7 and 2.2 imply that $\{V_{n_j}(q, B): j\in I\}$ is a basis for the tangent space at (q, B) to $\Phi^{-1}(N_I \times \{s\})$, where s is the component of $\Phi(q, B)$ in N_{I^c}. Thus, since $V_\lambda(q, B)$ is in the tangent space to M at (q, B), there are unique $c_j \in \mathbf{R}$ such that $V_\lambda - \sum_{j\in I} c_j V_{n_j}$ is in the tangent space to M_I at (q, B). Let $\gamma(t)$ be a smooth curve in M with $\gamma(0)=(q, B)$ and $\gamma'(0) = V_\lambda - \sum_{j\in I} c_j V_{\mu_j}$. Then, since $\mu_j(q, B) = \nu_j(q, B)$, $j \in I$, for $(q, B) \in M_I$,

$$\frac{d}{dt}[d-a](\gamma(t))\bigg|_{t=0} = [b(a+d)y_2 + b^2(y_2'+y_1)](1, \lambda, q) \equiv h(\lambda)$$

and, since $h(\lambda) = 2b^2 \cos \lambda^{1/2} + O(\lambda^{-1/2})$, it follows that for suitable λ

$$\frac{d}{dt}[d-a](\gamma(t)) \neq 0.$$

Thus $\{d=a\}$ and M_I intersect transversally at (q, B).

Now continuing with the same notation, let $(q, B) = (q_\infty, B_\infty)$ and let $\gamma(t)$ be a smooth curve in M with $\gamma(0) = (q_\infty, B_\infty)$ and, for $l \in I^c$,

$$\gamma'(0) = \left[V_{\mu_l} - \frac{h(\mu_l)}{h(\lambda)}\left(V_\lambda - \sum_{j\in I} c_j V_{\mu_j}\right)\right](q_\infty, B_\infty).$$

By construction $\gamma'(0)$ is tangent to M_I and $\{a=d\}$, hence

$$\frac{d}{dt}\left(\frac{1}{4}\int_0^1 (k+q(\gamma(t)))^2\,dx\right)\bigg|_{t=0} = 0. \tag{4.8}$$

Since $d_\infty = a_\infty$, and $\nu_j(q_\infty, B_\infty) = \mu_j(q_\infty, B_\infty)$ for $j \in I$, it follows that $y_2'(1, \mu_j, q_\infty) = y_1(1, \mu_j, q_\infty)$ for $j \in I$. Thus, when one uses lemma 4.8 to compute the derivative in (4.8), it yields

$$0 = (c_\infty + b(\mu_l + k))[y_1 - y_2'](1, \mu_l, q_\infty) - (c_\infty + b(\lambda + k))[y_1 - y_2'](1, \lambda, q_\infty)\frac{h(\mu_l)}{h(\lambda)}.$$

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

However, since $h(\mu_l)=[b^2(y_1-y_2')](1,\mu_l,q_\infty)$, we have

$$0=([y_1-y_2'](1,\mu_l,q_\infty))\left((c_\infty+b(\mu_l+k))-b^2\frac{(c_\infty+b(\lambda+k))}{h(\lambda)}[y_1-y_2'](1,\lambda,q_\infty)\right). \tag{4.9}$$

Thus either $[y_1-y_2'](1,\mu_l,q_\infty)=0$ or the other factor in (4.9) vanishes identically in λ. Since $c>-1/b$ on $M\cap\{a=d\}$, we may assume that k was chosen large enough at the outset that $c_\infty+b(\lambda+k)$ is strictly positive on all the gaps. Thus, if the second factor vanishes identically, $[y_1-y_2'](1,\lambda,q_\infty)$ must vanish at every zero of $h(\lambda)$. Note that

$$h(\lambda)=b([a_\infty y_2+by_2'](1,\lambda,q_\infty)+[d_\infty y_2+by_1](1,\lambda,q_\infty))\equiv bf(\lambda)+bf^*(\lambda).$$

Since f and f^* have the same sign at λ_{2n-1} and the same but opposite sign at λ_{2n} for all n, we see that $h(\lambda)$ has a zero ξ_n in $\lambda_{2n-1}\le\lambda\le\lambda_{2n}$ for all n. If $[y_1-y_2'](1,\xi_n)=0$ as well, then $\xi_n=\mu_n(q_\infty,B_\infty)=\nu_n(q_\infty,B_\infty)$. Thus, if the second factor in (4.9) vanishes identically, we can still conclude $[y_1-y_2'](1,\mu_l,q_\infty)=0$. Since l was an arbitrary element in I^c, we get $\nu_j(q_\infty,B_\infty)=\mu_j(q_\infty,B_\infty)$ for all j, and by lemma 4.4 (q_∞,B_∞) is a fixed point of the $*$-involution.

Now we only need show that (q_∞,B_∞) has index **I**. As in the preceding paragraph, we assume k was chosen so that $c_\infty+b(\lambda+k)$ is positive on all gaps. Since $h(\mu_j,q_\infty,B_\infty)=0$ for all j, given $l\in I^c$, $V_{\mu_l}(q_\infty,B_\infty)$ is tangent to $M\cap\{a=d\}$. However, V_{μ_l} is tangent to M_I as well. To see this, note that, as in the proof of lemma 4.6, near $\Phi(q_\infty,B_\infty)$, d, r and the $\tilde{\mu}$ are admissible coordinates on $N\cap\{d<d_\infty\}$. As in the proof of lemma 4.6,

$$\frac{\partial^2 a}{\partial\tilde{\mu}_i\partial\tilde{\mu}_j}(\Phi(q_\infty,B_\infty))=0,\qquad i\neq j.$$

Thus, differentiating the equations

$$\frac{\partial a}{\partial\tilde{\mu}_j}(r(s),s)=0,\qquad j\in I,$$

with respect to $\tilde{\mu}_l$ we see

$$\left.\frac{\partial\tilde{\mu}_j(r(s),s)}{\partial\tilde{\mu}_l}\right|_{(r(s),s)=\Phi(q_\infty,B_\infty)}=0,\qquad j\in I.$$

Thus $V_{\mu_l}(q_\infty,B_\infty)$ is tangent to M_I at (q_∞,B_∞).

Note that the image of $\{V_{\mu_j}(q,B): j=1,2,\ldots\}$ under $\Phi'(q,B)$ is clearly codimension 1 in the tangent space to $N\cap\{d<d_\infty\}$ at $\Phi(q,B)$. Moreover, $V_\lambda(q,B)$ for $\lambda\neq\mu_j(q,B)$, $j=1,2,\ldots$, is not in the closed linear span of

$$\{V_{\mu_j}(q,B): j=1,2,\ldots\},$$

since $\dot{d}\neq0$ on its integral curve at (q,B). Thus, since Φ' is an isomorphism, we may conclude that the closed linear span of $\{V_\lambda(q,B):\lambda\in\mathbf{R}\}$ is the tangent space to M at (q,B).

If we set $(\dot{q}, \dot{B}) = V_\lambda(q_\infty, B_\infty, \lambda)$, it follows immediately that $\dot{a}+\dot{d}=0$ for all λ. Likewise by lemma 4.8

$$\tfrac{1}{4}\int_0^1 (k+q)^2\,dx = D$$

for all λ. Thus the gradients of $a+d$ and $\int_0^1 (k+q)^2\,dx$ as functions on M vanish at (q_∞, B_∞). Hence by the chain rule, given $\gamma(t)$ in M with $\gamma(0)=(q_\infty, B_\infty)$, the second derivatives at $t=0$ of these functions restricted to $\gamma(t)$ depend only on $\gamma'(0)$.

To use the observation of the preceding paragraph, we note that the second derivative at $t=0$ of $\frac{1}{4}\int_0^1 (k+q)^2\,dx$ along $\gamma(t)$ in $M_I \cap \{a=d\}$ with $\gamma(0)=(q_\infty, B_\infty)$ and $\gamma'(0)=V_{\mu_l}(q_\infty, B_\infty)$ must be non-negative. Hence the same is true if we take $V(t)$ to be the integral curve through (q_∞, B_∞) of $V_{\xi_l(q,B)}(q, B)$, where $\xi_l(q, B)$ is the zero of $h(\lambda, q, B)$ in $\lambda_{2l-1} \le \mu \le \lambda_{2l}$. An argument from Rouché's theorem like that used in the proof of theorem 1.3 shows $h(\lambda, q, B)$ has exactly one root counted by multiplicity in $\lambda_{2l-1} \le \mu \le \lambda_{2l}$ for each l. Thus $\xi_l(q, B)$ is a well defined analytic function on M. Since $\dot{a}-\dot{d}=0$ on γ, $\gamma(t)$ lies in $M \cap \{a=d\}$.

By theorem 4.8 for t near zero

$$\frac{d}{dt}\left(\int_0^1 (k+q(\gamma(t)))^2\,dx\right) = (c(t)+b(\xi_l(t)+k))[y_1-y_2'](1, \xi_l(t), q(t)),$$

and by (2.7)

$$\frac{d}{dt}[a+d](\gamma(t)) = -b^2[y_1-y_2'](1, \xi_l(t), q(t)).$$

Thus we must have $(d^2/dt^2)([a+d](\gamma(t)))|_{t=0} \le 0$. Again, since the gradient of $a+d$ on M vanishes at (q_∞, B_∞), this remains true if we replace $\gamma(t)$ by the integral curve of V_{μ_l} through (q_∞, B_∞). Thus we conclude $F_l(\mu, \xi, q_\infty, B_\infty)$ is non-negative on C_l, $l \in I^c$, and (q_∞, B_∞) has index **I**. □

In the course of the proof of proposition 4.9 we saw that a fixed point of the involution was necessarily a critical point of $a+d$ on M. Conversely, lemma 4.5 implies a critical point of $a+d$ must be fixed by the involution. Thus we have:

COROLLARY 4.10. *The fixed points of the involution on M coincide with the critical points of $a+d$ on M.*

5. *The range of B*

If we consider B as a function on M, then its range is simply the set of boundary conditions which give rise to the generalized periodic spectrum defining M for some potential q. Since b is constant on and $c=b^{-1}(ad-1)$, it suffices to determine the range of (a, d). Using the results of §§ 2 and 4, one can find the range of (a, d) on M_I for all **I**.

THEOREM 5.1. *Let (q_I, B_I) be the fixed point of $*$ on M of index* **I**. *Then the range of (a, d) on M_I is $\{(a, d): a+d<2a_I\} \cup (a_I, a_I)\}$. In particular, this gives the range of B on M when I is the empty set.*

In proving theorem 5.1, we will use one of the formulae of theorem 2.4 extended to the case where one moves all the μ, namely

$$\hat{a}=a-b^2\sum_{j=1}^{\infty}(\sigma_j-\mu_j)\left(\prod_{i\pm j}\frac{\sigma_j-\mu_i}{\sigma_j-\sigma_i}\right)\left[\frac{y_2+b\xi_*}{ay_2+by_2'}\right](1,\sigma_j,q). \tag{5.1}$$

We claim that (5.1) is valid whenever

$$s=(b,d,\sigma_1,-b\xi_*(\sigma_1),\sigma_2,-b\xi_*(\sigma_2),\ldots)$$

is in the range of Φ. From the continuity of Φ^{-1}, defining $(\hat{q},\hat{B})=\Phi^{-1}(s)$, it follows that $\hat{a}=\lim_{N\to\infty}a_N$, where

$$a_N=a-b^2\sum_{j=1}^{N}(\sigma_k-\mu_k)\left(\prod_{\substack{i=1\\ i\neq j}}^{N}\frac{\sigma_j-\mu_i}{\sigma_j-\sigma_i}\right)\left[\frac{y_2+b\xi_*}{ay_2+by_2'}\right](1,\sigma_j,q). \tag{5.2}$$

Thus (5.1) will hold if the sum of products on the right-hand side of (5.2) converges to the right-hand side of (5.1). This holds under the weaker assumption $|\sigma_k-\mu_k|<C$ for all k and $|\xi_*(\sigma_k)|<1$ for $k>k_0$. To see this, we note that

$$\prod_{\substack{i=1\\ i\neq j}}^{N}\frac{\sigma_j-\mu_i}{\sigma_j-\sigma_i}=\prod_{i=1}^{N}\left(1+\frac{\sigma_i-\mu_i}{\sigma_j-\sigma_i}\right)$$

and

$$\left|\frac{\sigma_i-\mu_i}{\sigma_j-\sigma_i}\right|\leq\frac{C}{\pi^2|(j-\frac{1}{2})^2-(i-\frac{1}{2})^2|-C'} \tag{5.3}$$

for some C' independent of i,j. Thus we see that the products are convergent and uniformly bounded in j. Using theoreem 1.1 and lemma 4.2(iv), we estimate the term $(ay_2+b\xi_*)(ay_2+by_2')^{-1}$ by

$$\left|\left[\frac{y_2+b\xi_*}{ay_2+by_2'}\right](1,\mu)\right|=\left|\left[\frac{dy_2+by_1}{y_2+b\xi_*^{-1}}\right](1,\mu)\right|$$
$$=\frac{b\cos\mu^{1/2}+O(\mu^{-1/2})}{(b/2)\Delta(\mu)(1+\sqrt{1-4\Delta(\mu)^{-2}})+O(\mu^{-1/2})}$$
$$=\frac{b\cos\mu^{1/2}+O(\mu^{-1/2})}{-b\mu^{1/2}\sin\mu^{1/2}+O(1)}.$$

Thus, since we have $(\sigma_j)^{1/2}=\pi(j-\frac{1}{2})+O(1/j)$,

$$\left|\left[\frac{y_2+b\xi_*}{ay_2+by_2'}\right](1,\sigma_j)\right|\leq\frac{K}{j^2}. \tag{5.4}$$

Thus the right-hand side of (5.1) is convergent and, by dominated convergence, is the limit of the sequence of the a_N.

Since proposition 4.7 implies

$$a_l=\max_{M_l}(\min\{a,d\}),$$

if $(a,d)=(a_1,d_1)$ at $(q_1,B_1)\in M_l$, we may assume, using the involution if necessary, that $a_1\leq a_l$. Choosing k sufficiently large, we may use the flow of V_k as in the proof of lemma 3.2 to move a_l down to a_1, moving (q_l,B_l) to (q_2,B_2). Then, applying the involution, we arrive at (q_2^*,B_2^*) with $(a_2^*,a_2^*)=(a_l,a_1)$. Since (q_1^*,B_1^*) has

$(a_1^*, d_1^*) = (d_1, a_1)$, and we know from lemma 3.1 that $\Phi(M \cap \{d = a_1\}) = N \cap \{d = a_1\}$, it follows that we can choose $(\sigma_k, \xi_*(\sigma_k))$, $k = 1, \ldots$, so that

$$d_1 = a_I - b^2 \sum_{j=1}^{\infty} (\sigma_j - \nu_j)\left(\prod_{i \neq j}^{\infty} \frac{\sigma_j - \nu_i}{\sigma_j - \sigma_i}\right)\left[\frac{y_2 + b\xi_*}{a_2^* y_2 + by_2'}\right](1, \sigma_j, q_2^*) \tag{5.5}$$

as in (5.1), where ν_k is the zero of $[a_2^* + by_2'](1, \lambda, q_2^*)$ in $\lambda_{2j-1} \leq \lambda \leq \lambda_{2j}$.

To arrive at the formula we need for the proof of theorem 5.1, we need to follow the operations that lead to (q_2^*, B_2^*) back to (q_I, B_I). This is not difficult. We have $y_2(1, \lambda, q_2^*) = y_2(1, \lambda, q_2)$ and, combining (2.20) and (2.21),

$$y_2(1, \lambda, q_2) = y_2(1, \lambda, q_I) + \frac{a_1 - a_I}{b^2}\left[\frac{a_I y_2 + by_2'}{\lambda - \mu_k}\right](1, \lambda, q_I), \tag{5.6}$$

where μ_k is the zero of $[a_I y_2 + by_2'](1, \lambda, q_I)$ in $\lambda_{2k-1} < \lambda < \lambda_{2k}$. Since $[a_2^* y_2 + by_2'] \times (1, \lambda, q_2^*)$ and $[a_I y_2 + by_2'](1, \lambda, q_I)$ are entire functions of order $\frac{1}{2}$ with the same asymptotics as $\lambda \to \infty$,

$$[a_2^* y_2 + by_2'](1, \lambda, q_2^*) = \left(\prod_{i=1}^{\infty} \frac{\lambda - \nu_i}{\lambda - \mu_i}\right)[a_I y_2 + by_2'](1, \lambda, q_I). \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5), we have

$$d_1 = a_I - b^2 \sum_{j=1}^{\infty}\left(\prod_{\substack{i=1 \\ j \neq i}}^{\infty} \frac{\sigma_j - \mu_i}{\sigma_j - \sigma_i}\right)\left(\left[\frac{y_2 + b\xi_*}{a_I y_2 + by_2'}\right](1, \sigma_j, q_I) + \frac{a_1 - a_I}{b^2}\frac{\sigma_j - \mu_j}{\sigma_j - \mu_k}\right). \tag{5.8}$$

Note that the infinite products in (5.8) are convergent by (5.3).

LEMMA 5.2

$$\sum_{j=1}^{\infty}\left(\frac{\sigma_j - \mu_j}{\sigma_j - \mu_k}\right)\prod_{i \neq j} \frac{\sigma_j - \mu_i}{\sigma_j - \sigma_i} = 1. \tag{5.9}$$

Proof. Since $|\sigma_j - \pi^2(j - \frac{1}{2})^2| < C$ for all j,

$$S_N = \sum_{j=1}^{N}\left(\frac{\sigma_j - \mu_j}{\sigma_j - \mu_k}\right)\prod_{i \neq j} \frac{\sigma_j - \mu_i}{\sigma_j - \sigma_i}$$

is the sum of the residues of

$$f(z) = \frac{1}{z - \mu_k}\prod_{i=1}^{\infty} \frac{z - \mu_i}{z - \sigma_i}$$

in $|z| < \pi^2 N^2$ for N sufficiently large. On $|z| = \pi^2 N^2$, estimating as in (5.3),

$$\left|\frac{z - \mu_i}{z - \sigma_i} - 1\right| = \left|\frac{\sigma_i - \mu_i}{z - \sigma_i}\right| \leq C(|\pi^2 N^2 - \pi^2(i - \tfrac{1}{2})^2| - C)^{-1},$$

where C is independent of N and i. Hence, as $N \to \infty$, $\prod_{i=1}^{\infty}(z - \mu_i)(z - \sigma_i)^{-1} \to 1$ uniformly on $|z| = \pi^2 N^2$. Hence, as $|z| = \pi^2 N^2$,

$$\frac{1}{z - \mu_k}\prod_{i=1}^{\infty} \frac{z - \mu_i}{z - \sigma_i} = \frac{1 + O(1)}{z - \mu_k}.$$

Using $\oint_{|z| = \pi^2 N^2} f(z)\, dz = 2\pi i S_N$, we conclude $\lim_{N \to \infty} S_N = 1$. To see that $\lim_{N \to \infty} S_N$ is the sum in (5.9), one may again use (5.3). □

Substituting (5.9) into (5.8) gives a formula which implies theorem 5.1:

$$a_1+d_1=2a_I-b^2\sum_{j=1}^{\infty}(\sigma_j-\mu_j)\left(\prod_{i\neq j}\frac{\sigma_j-\mu_i}{\sigma_j-\sigma_i}\right)\left[\frac{y_2+b\xi_*}{a_Iy_2+by_2'}\right](1,\sigma_j,q_1). \tag{5.10}$$

Proof of theorem 5.1. Since (q_1, B_1) is in M_I, and, as noted in the proof of proposition 4.7, M_I is $*$-invariant, it follows that $(q_1^*, B_1^*)\in M_I$. Thus, if we move σ_j to μ_j and $\xi_*(\sigma_j)$ to $\xi_*(\mu_j)$ for $j\in I$, we will not decrease a_1^*, and (5.5) will become

$$d_1\le a_I-b^2\sum_{j=1}^{\infty}(\tilde\sigma_j-\nu_j)\left(\prod_{i\neq j}\frac{\tilde\sigma_j-\nu_i}{\tilde\sigma_j-\tilde\sigma_i}\right)\left[\frac{y_2+b\xi_*}{a_2^*y_2+by_2'}\right](1,\tilde\sigma_j,q_2^*),$$

where

$$\tilde\sigma_j=\begin{cases}\sigma_j, & j\in I^c,\\ \mu_j, & j\in I.\end{cases}$$

Hence the remainder of the proof of (5.10) after (5.5) shows

$$a_1+d_1\le 2a_I-b^2\sum_{j\in I^c}(\sigma_j-\mu_j)\left(\prod_{\substack{i\neq j\\ i\in I^c}}\frac{\sigma_j-\mu_i}{\sigma_j-\sigma_i}\right)\left[\frac{y_2+b\xi_*}{a_Iy_2+by_2'}\right](1,\sigma_j,q_1). \tag{5.11}$$

Since (q_I, B_I) has index **I**,

$$F_j(\mu,\xi)=(\mu-\mu_j)\left[\frac{y_2+b\xi}{a_Iy_2+by_2'}\right](1,\mu,q_I)$$

is non-negative on C_j for $j\in I^c$. Thus (5.11) shows $a_1+d_1\le 2a_I$. Moreover, if $a_1+d_1=2a_I$, then $\sigma_j=\mu_j$ for $j\in I^c$. By theorems 1.3 and 1.4

$$\begin{aligned}\mu_j&=\pi^2(j-\tfrac12)^2+I_M-2a_I/b+l_2(j),\\ \sigma_j&=\pi^2(j-\tfrac12)^2+I_M-2a_1/b+l_2(j),\end{aligned} \tag{5.12}$$

where $I_M=2(a+d)/b+\int_0^1 q\,dx$ is constant on M. Thus, since I is finite, $a_1=a_I$.

To complete the proof, we only need show, for arbitrary $a_1\le a_I$, $\sup\,(a+a_1)=2a_I$, where the supremum is taken over $M_I\cap\{d=a_1\}$. Let (q_N, B_N) be the element of M_I with

$$\mu_j(q_N,B_N)\equiv\sigma_j=\begin{cases}\mu_j(q_I,B_I), & j\in I^c\cap\{j<N\},\\ \mu_j(q_I,B_I)+\dfrac{2(a_I-a_1)}{b}, & j\in I^c\cap\{j\ge N\},\end{cases}$$

$$y_2(1,\sigma_j,q_N)=y_2(1,\sigma_j,q_I),\qquad j\in I^c\cap\{j<N\}$$

and

$$y_2(1,\sigma_j,q_N)=-b\xi_*(\sigma_j),$$

where $|\xi_*(\sigma_j)|\le 1$ for $j\in I^c\cap\{j\ge N\}$. To see that there is an element of M, and hence an element of M_I, with these coordinates, note that by § 3 $\Phi(M)\supset N\cap\{d\equiv d_I\}$, and by (5.12) there are points in $N\cap\{d=a_1\}$ with the given data. Since $a_1\le a_I=d_I$, we see that (q_N, B_N) exists and lies in $M_I\cap\{d=a_1\}$. Thus, setting $\mu_j=\mu_j(q_I,B_I)$, $\mu_j^N=\mu_j(q_N,B_N)$ and $c=2(a_I-a_1)/b$, and applying (5.10) to

(q_N^*, B_N^*), we have for N large

$$\begin{aligned} a_N + a_1 &= 2a_I - b^2 \sum_{j=1}^{\infty} (\mu_j^N - \mu_j)\left(\prod_{i\neq j} \frac{\mu_j^N - \mu_i}{\mu_j^N - \mu^N}\right)\left[\frac{y_2 + b\xi_*}{a_I y_2 + b y_2'}\right](1, \mu_j^N, q_I) \\ &= 2a_I - b^2 \sum_{j\in I} (\mu_j^N - \mu_j)\left(\prod_{i\neq j} \frac{\mu_j^N - \mu_i}{\mu_j^N - \mu_i^N}\right)\left[\frac{y_2 + b\xi_*}{a_I y_2 + b y_2'}\right](1, \mu_j^N, q_I) \\ &\quad - b^2 c \sum_{j\geq N} \prod_{i\in I}\left(1 + \frac{\mu_i^N - \mu_i}{\mu_j^n - \mu_i^n}\right) \prod_{\substack{i\geq N\\ j\neq i}} \left(1 + \frac{c}{\mu_j - \mu_i}\right)\left[\frac{y_2 + b\xi_*}{a_I y_2 + b y_2'}\right](1, \mu_j^N, q_I). \end{aligned} \tag{5.13}$$

Since the μ_i^N, $i\in I$, lie in bounded intervals and I is finite, (5.3) and (5.4) imply that the final sum in (5.13) tends to zero as $N\to\infty$. However, since (q_I, B_I) has index **I**,

$$F_j(\mu, \xi) = (\mu - \mu_j)\left[\frac{y_2 + b\xi}{a_I y_2 + b y_2'}\right](1, \mu, q_I)$$

is non-positive on C_j for $j \in I$. Thus

$$\liminf_{N\to\infty} a_N + a_1 \geq 2a_1. \qquad \square$$

6. *The level sets of B on M*

In this section we apply the results of the preceding sections to study isospectral sets when the boundary conditions are fixed. The subset of M with boundary condition given by $B = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ will be denoted by M_B. Since $q \to q^*$ is an analytic homeomorphism of M_B onto M_B, we only need consider the case $a \geq d$ for the results given here.

Proposition 4.7 implies that, when $a = d = a_I$ and $I = \varnothing$, then M_B consists of a single point, and by theorem 5.1, if $a + d \geq 2a_\varnothing$, then $M_B = \varnothing$ unless $a = d = a_\varnothing$. The first result of this section is that these are the only cases where M_B is compact.

THEOREM 6.1. *If $a + d < 2a_\varnothing$, then M_B is not compact.*

Proof. If $a + d < 2a_\varnothing$, then by theorem 5.1 there is a point $(q_1, B_1) \in M$ with $d_1 = d$ and $a_1 > a$. The strategy for proving this theorem is to show that using the flows from V_j, j large, to move a_1 down to a, we can construct a non-compact set of points in M_B. By theorem 2.3 on the orbit of (q_1, B_1) under the flow of V_j

$$a(t) = a_1 - (\mu_j(t) - \mu_j) b^2 \left[\frac{y_2 + b\xi_*}{a_1 y_2 + b y_2'}\right](1, \mu_j(t), q_1).$$

For $|\mu - \pi^2(j - \frac{1}{2})^2| < C$ and $|\xi_*(\mu)| < 1$, (5.4) implies that

$$\left[\frac{y_2 + b\xi_*}{a_1 y_2 + b y_2'}\right](1, \mu, q_1) = O\left(\frac{1}{j^2}\right).$$

Thus, since $\mu_j(q_1, B_1) = \pi^2(j - \frac{1}{2})^2 + O(1)$, there is a j_0 such that for $j > j_0$, $|\mu - \mu_j(q_1, B_1)| < 1$ and $|\xi_*(\mu)| < 1$

$$a_1 - a < a_1 - b^2(\mu - \mu_j(q_1, B_1))\left[\frac{y_2 + b\xi_*}{a_1 y_2 + b y_2'}\right](1, \mu, q_1). \tag{6.1}$$

On the other hand,

$$\left[\frac{y_2+b\xi_*}{a_1y_2+by_2'}\right](1,\lambda_{2j},q_1)=\frac{b(-1)^j+O(1/j)}{b\cos\sqrt{\lambda_{2j}}+O(1/j)}=1+O\left(\frac{1}{j}\right) \tag{6.2}$$

by (1.6). Thus there is a $j_1>j_0$ such that for $j>j_1$

$$a_1-a>a_1-b^2(\lambda_{2j}-\mu_j(q_1,B_1)\left[\frac{y_2+b\xi_*}{a_1y_2+by_2'}\right](1,\lambda_{2j},q_1). \tag{6.3}$$

Combining (6.1) and (6.3), we see that for each $j>j_1$, M_B contains a point (q_j,B) with

$$\mu_k(q_j,B)=\mu_k(q_1,B_1)$$

for $k\neq j$ and

$$|\mu_j(q_j,B)-\mu_j(q_1,B_1)|\geq 1.$$

This implies that the image of M_B under Φ is not compact and hence M_B is not compact. □

Since lemma 4.2(iv) and lemma 4.4 imply the gradient of a on $M\cap\{d=d_0\}$ can vanish only at fixed points of the involution, and conversely one sees from lemma 4.2(iv) and formula (5.1) with $(q,B)=(q_I,B_I)$ that M_B does have a conic singularity at fixed points of the involution, we have the conclusion:

THEOREM 6.2. *M_B is an analytic submanifold of $L^2[0,1]\times(\mathrm{SL}(2,\mathbf{R})\backslash b=0)$ unless it contains a fixed point of $*$; hence it is analytic unless $(a,d)=(a_I,d_I)$ for some $I\neq\emptyset$. If $(a,d)=(a_I,a_I)$ for some $I\neq\emptyset$, then M_B has a conic singularity at (q_I,B_I).*

One property of M_B which holds without exception is connectivity.

THEOREM 6.3. *M_B is connected.*

Proof. Given any point (q_0,B_0) in M_{B_0} which is not fixed by the involution, there is a j such that $(\partial a/\partial\mu_j)(\Phi(q_0,B_0))\neq 0$ when we consider a as a function on N. Thus, using the splitting $N=N_I\times N_{I^c}$ with $I=\{j\}$ and the implicit function theorem, we get $r(s)$ such that $\Phi^{-1}(r(s),s)\in M_{B_0}$ for s in an open subset of $N_{I^c}\cap\{d=d_0\}$. Since points with

$$\mu_n=\pi^2(n-\tfrac{1}{2})^2+I_M-2d_0/b$$

and $|\kappa_n|<b$ for n beyond some point are dense in $N_{I^c}\cap\{d=d_0\}$, it follows that we can connect (q_0,B_0) to (q_1,B_0) in M_{B_0}, where

$$\mu_n(q_1,B_0)=\pi^2(n-\tfrac{1}{2})^2+I_M-2d_0/b \tag{6.4}$$

and $|y_2(1,\mu_n,q_1)|<b$ for $n>n_0$.

Let (q_2,B_2) be a point where a assumes its maximum on the subset S_0 of M satisfying (6.4). As in lemma 4.6 (see formula (4.3)), one sees that a is strictly less than its maximum on S_0 away from (q_2,B_2) and hence (q_2,B_2) is unique. Moreover, a has no other local maxima on S_0 and all its critical points on the torus S_0 must be non-degenerate and hence finite in number. Thus, if $(q_1,B_0)\neq(q_2,B_0)$, we may construct a continuous curve $\gamma(t)=(q(t),B(t))$ in S_0 with $\gamma(0)=(q_1,B_0)$ and $\gamma(1)=(q_2,B_2)$ such that a is strictly increasing on γ.

Next we will move the values of a on γ down to a_0 by using the flow from V_J, $J > n_0$ sufficiently large. Since $\{\gamma(t): 0 \le t \le 1\}$ is compact, it follows that the estimate (6.2) with (a_1, q_1) replaced by $(a(t), q(t))$ and the corresponding estimate for $\mu = \lambda_{2j-1}$ hold uniformly for $t \in [0, 1]$. Moreover, as always, for each t the function

$$F_j(\mu, \xi; t) = (\mu - \mu_j)\left[\frac{y_2 + b\xi}{a(t)y_2 + by_2'}\right](1, \mu, q(t))$$

has a unique maximum and minimum on C_j and no other critical points. Thus for J sufficiently large we have two continuous functions $\mu_\pm(t)$ with $\lambda_{2J-1} < \mu_-(t) \le \mu_+(t) < \lambda_{2J}$ such that

$$a_0 = a(t) - b^2 F_J(\mu_\pm(t), \xi(\mu_\pm(t)); t)$$

when $|\xi(\mu_\pm(t)| < 1$. Thus, picking $\mu_J(t) = \mu_*(t)$ and $\kappa_J = -b\xi(\mu_*(t))$, where $*$ $(=\pm)$ is chosen so that $\mu_*(0) = \pi^2(J-\frac{1}{2})^2 + I_M - 2d_0/b$, and $\mu_j(t) = \mu_j(\gamma(t))$ and $\kappa_j(t) = \kappa_j(\gamma(t))$ for $j \ne J$, we have a curve in M_{B_0} connecting (q_1, B_0) to either (q_{2+}, B_0) or (q_{2-}, B_0), where

$$\begin{aligned} &\mu_j(q_{2\pm}, B_0) = \mu_j(q_2, B_2), \qquad j \ne J, \\ &\mu_J(q_{2\pm}, B_0) = \mu_\pm(1), \\ &\kappa_j(q_{2\pm}, B_0) = \kappa_j(q_2, B_2) \qquad \forall j. \end{aligned} \tag{6.5}$$

To see that (q_{2+}, B_0) and (q_{2-}, B_0) can be connected in M_B, we consider a restricted to the two-dimensional torus T_0 obtained by fixing the coordinates on $M \cap \{d = d_0\}$ at their values in (6.5) for $j \ne J, J+1$. As before, a has only one local maximum on T_0 and this is non-degenerate. Moreover, any other critical value for a on T_0 must be a global minimum of a on the orbit of the critical point under the flow of V_J or V_{J+1}. Using the representation for a on T_0 from theorem 2.4 with base point (q_2, B_2) and moving on the flow of V_J or V_{J+1} to the point used in the proof of lemma 3.2, one sees that any critical value of a on T_0 below the maximum must be below a_0 for J sufficiently large. Thus the level set $a_0 = a$ on T_0 which contains (q_{2+}, B_0) and (q_{2-}, B_0) is connected. Thus (q_1, B_0) can be connected to both (q_{2+}, B_0) and (q_{2-}, B_0).

If (q, B_0) and (p, B_0) are points in M_{B_0} which are not fixed by the involution, taking $n_0 = \max\{n_0(q), n_0(p)\}$, we can assume (q_1, B_0) and (p_1, B_0) are both in S_0. Hence the preceding argument shows any two points of M_{B_0} which are not fixed by the involution can be connected in M_{B_0}. Since fixed points of the involution are never isolated in M_{B_0}, we conclude M_{B_0} is connected. □

If we let $A = \{I: a_0 + d_0 > 2a_I\}$ and $B_0 = \begin{pmatrix} a_0 & b \\ c_0 & d_0 \end{pmatrix}$, then theorem 5.1 implies the M_{B_0} is contained in

$$S_0 = \left(M - \bigcup_{I \in A} M_I\right) \cap \{d = d_0\}.$$

We will conclude our study of M_{B_0} with a theorem on the relation on the topological structure of M_{B_0} to that of S_0.

THEOREM 6.4. *Assume that $a_0+d_0\neq 2a_l$ for all l. Then the homotopy groups of M_{B_0} and S_0 are isomorphic.*

Proof. We will show that the natural homomorphism of the homotopy groups of M_B into those of S_0 is injective and surjective. The proof of injectivity will have two main steps. We will show:

(i) If $\gamma: S^m\to M_{B_0}$, then given $j>j_0(\gamma)$, γ is homotopic in M_{B_0} to $\tilde{\gamma}$, where for all $s\in S^m$ the roots of

$$0=F_j(\mu,\xi;\tilde{\gamma}(s))=(\mu-\mu_j(\tilde{\gamma}(s)))\left[\frac{y_2+\xi}{a_0y_2+by_2'}\right](1,\mu,q(\tilde{\gamma}(s)))$$

on C_j are contained in $|\xi|<1$ and $\mu=\mu_j(\tilde{\gamma}(s))$ is the greater root.

(ii) If $\Gamma: S^m\times[0,1]\to S_0$ is a homotopy of γ in (i) to a constant map, then we can replace Γ by a homotopy $\tilde{\Gamma}$ of γ to a constant such that $\tilde{\Gamma}: S^m\times[0,1]\to S_0\cap\{a\geq a_0\}$.

Given (i) and (ii), one can complete the proof of injectivity as follows. Choose $j_1>j_0(\gamma)$ sufficiently large that the range of $F_{j_1}(\mu,\xi;\tilde{\Gamma}(s,t))$ on $C_j\cap\{|\xi|<1\}$ contains $(a_0-a(\tilde{\Gamma}(s,t)))b^{-2}$ for all $(t,s)\in[0,1]\times S^m$. Then extend $\tilde{\Gamma}$ to $S^m\times[-1,1]$ by letting $\tilde{\Gamma}$ on $S^m\times[-1,0]$ be the homotopy from (i) in M_{B_0} of $\tilde{\gamma}$ with $j=j_1$ to γ. Finally, changing $\mu_{j_1}(\tilde{\Gamma}(s,t))$ to $\mu^+(s,t)$ as in the proof of theorem 6.3 gives a homotopy in M_{B_0} of $\tilde{\gamma}$ to a constant and completes the proof of injectivity.

The proof that the mapping of the homotopy groups is surjective is similar and simpler. Given $f: S^m\to S_0$, we deform f in S_0 to $\gamma: S^m\to S_0\cap\{a>a_0\}$ as in (ii) and then deform γ in $S_0\cap\{a_1>a_0\}$ to $\tilde{\gamma}$ as in (i). Then moving $\mu_{j_1}(\tilde{\gamma}(s))$ continuously up to $\mu^+(s)$, we deform $\tilde{\gamma}$ to a curve in M_{B_0}. Note that this deformation automatically remains in S_0 since $a\geq a_0$. We believe that the proofs of the simpler versions of (i) and (ii) used here to prove surjectivity will be evident from the proofs of (i) and (ii).

Proof of (i). Since M_{B_0} contains no critical points of $a+d$ on M by hypothesis, and hence no critical points of a on $M\cap\{d=d_0\}$, the gradient of a as a function on $M\cap\{d=d_0\}$ does not vanish on M_{B_0}. Moreover, since $\gamma(S^m)$ is a *compact* subset of M_{B_0}, there is a finite set I_0 such that the gradient of a as a function on N_{I_0} does not vanish on $\gamma(S^m)$.

The idea of this proof is to use the N_{I_0}-gradient flow of a to bring the deformation of γ moving $\mu_j(\gamma(s))$ into $\mu_j(\tilde{\gamma}(s))$ back into M_{B_0}. However, getting the necessary uniform estimates in j requires some additional work.

Using theorem 1.1 as in the derivation of (5.4), we have for $(\mu,\xi)\in C_j\cap\{|\xi|<1\}$

$$\begin{aligned}F_j(\mu,\xi;q,B)&=(\mu-\mu_j(q,B))\left[\frac{y_2+b\xi}{ay_2+by_2'}\right](1,\mu,q)\\&=(\mu-\mu_j(q,B))\left(\frac{b\cos\mu^{1/2}+O(\mu^{-1/2})}{-b\mu^{1/2}\sin\mu^{1/2}+O(1)}\right),\end{aligned}\tag{6.6}$$

where both O terms are uniform on bounded sets in $(\|q\|,\|B\|)$. Moreover, if $\dot{}$ denotes the derivative along a flow moving q and a but not d, $\Delta(\lambda,q,B)$ or $\mu_j(q,B)$,

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

theorems 1.1, 1.2 and lemma 4.2(iv) give

$$|\dot{F}_j(\mu, \xi; q, B)| = \left|(\mu - \mu_j(q, B))\left[\frac{d(\partial y_2/\partial q, \dot{q}) + b(\partial y_1/\partial q, \dot{q})}{y_2 + b\xi^{-1}} - \frac{dy_2 + by_1}{(y_2 + b\xi^{-1})^2}\left(\frac{\partial y_2}{\partial q}, \dot{q}\right)\right](1, \mu, q)\right|$$

$$\leq \frac{M(\|q\|, \|B\|)}{j}\left|\frac{\mu - \mu_j(q, B)}{y_2(1, \mu, q) + b\xi^{-1}(\mu)}\right| \|\dot{q}\| \tag{6.7}$$

for $(\mu, \xi) \in C_j \cap \{|\xi| < 1\}$, $j > j_0(\|q\|, \|B\|)$. In (6.6) we set

$$\mu = \mu_\delta = \delta\pi(j - \tfrac{1}{2}) + \mu_j(q, B)$$

with $|\delta| \leq 1$. Using theorem 1.3, this gives for $|\xi_\delta| < 1$

$$F_j(\mu_\delta, \xi_\delta; q, B) = \delta \tan \delta/2 + O(1/j), \tag{6.8}$$

uniformly on bounded sets in $(\|q\|, \|B\|, 1/b)$ and $|\delta| \leq 1$. In (6.7) we consider $(\mu, \xi) \in D_j = C_j \cap \{|\xi| < 1\} \cap \{|\mu - \mu_j(q_j, B)| \leq \pi(j - \frac{1}{2})\}$ and conclude

$$|\dot{F}_j(\mu, \xi; q, B)| \leq \frac{C(\|q\|, \|B\|, 1/b)}{j}\|\dot{q}\| \tag{6.9}$$

for $j > j_1(\|q\|, \|B\|)$. The estimates (6.8) and (6.9) will suffice for this proof.

Since F_j has just one maximum and one minimum on C_j for all (j, B, q), it is clear from (6.8) that for any $\delta \in (0, 1)$ and $j \geq j_2(\|q\|, \|B\|, 1/b, 1/\delta)$ the minimum of F_j on C_j must occur in

$$I_\delta(j) = \{(\mu, \xi) \in C_j : |\mu - \mu_j(q, B)| \leq \delta\pi(j - \tfrac{1}{2})\},$$

and the range of F_j on $I_\delta(j)$ is contained in $[-\delta, \delta]$. However, (6.8) also implies that for $j \geq j_3(\|q\|, \|B\|, 1/b, 1/\delta)$ the range of F_j on $I_j(\delta)$ includes $[0, \delta/2 \tan \delta/2]$. Hence, taking $0 < \varepsilon < \frac{1}{2}\tan\frac{1}{2}$, we can move along the flow of V_j, $j > \max\{j_i(\|q\|, \|B\|, 1/b, 2/\varepsilon),\ i = 2, 3\}$, in the direction of increasing μ until $a = a(q, B) - \varepsilon b^2$ at $\mu = \mu_j^+(q, B)$ and be sure that:

(i) $\mu_j^+(q, B)$ is greater than the minimum of $F_j(\mu, \xi; \varepsilon, B)$ on C_j,

(ii) the values of a in this process remain in $|a - a(q, B)| < 4\varepsilon b^2$ and (μ, ξ) remains in D_j. Since the flow is transverse to the level surface $a = a(q, B) - \varepsilon b^2$, this gives us smooth functions $\mu(t, q, B)$ and $t(q, B)$ such that $\mu_j(q, B) = \mu(0, q, B)$ and $\mu(t(q, B), q, B) = \mu_j^+(q, B)$. We set $\mu_j(\tau, q, B) = \mu(t(g, B)\tau, g, B)$ for

$$j > \max\left\{j_i\left(\|q\|, \|B\|, \frac{1}{b}, \frac{2}{\varepsilon}\right), i = 2, 3\right\} = j_4\left(\|q\|, \|B\|, \frac{1}{b}, \frac{2}{\varepsilon}\right).$$

Since the N_{I_0}-gradient of a does not vanish on $\gamma(S^m)$, we may choose a neighbourhood U in N_{I_0} of the projection of $\gamma(S^m)$ onto N_{I_0} such that the N_{I_0}-gradient of a is bounded away from zero on U. Moreover, by (6.9) there is a $j_S(\gamma, U)$ such that the N_{I_0}-gradient of a is bounded away from zero uniformly on

$$\{(r, u, v) \in N_{I_0} \times N_{\{j\}} \times N_{(I_0 \cup \{j\})^c} : r \in U, u \in D_j(\gamma(s)), v = v(\gamma(s)), s \in S^m\},$$

uniformly in $j > j_S(\gamma, U)$. Hence we can choose an $\varepsilon > 0$ such that for $j > j_S(\gamma, U)$ the N_{I_0}-gradient flow out of $E_j = \{(r, u, v) \in N_{I_0} \times N_{\{j\}} \times N_{(I_0 \cup \{j\})^c} : r = r(\gamma(s)),$

$u \in D_j(\gamma(s))$, $v = v(\gamma(s))\}$ moves a through $[a(e)-8\varepsilon b^2, a(e)+8\varepsilon b^2]$ for all $e \in E_j$ before r leaves U. (We define the N_{I_0}-gradient flow of a as the flow of the vector field $\sum_{j\in I_0} \dot{a}_j V_j$, where $\dot{a}_j$ denotes the derivative of a on the flow of V_j in (2.7).)

Now define

$$(\alpha, \beta) = (\max_G \|q(\Phi^{-1}(r, u))\|, \max \|B(\Phi^{-1}(r, u))\|),$$

where

$$G = \{(r, u) \in N_{I_0} \times N_{I_0^c} : r \in N_{I_0}, u = u(\gamma(s)), s \in S^m\}.$$

We let $j_0 > j_6(\gamma) = \max\{j_4(\alpha, \beta, 1/b, 2/\varepsilon), j_5(\gamma, U)\}$. For such a j_0 we have a deformation $H(t, g)$ of G given in our coordinates on N by

$$\mu_j = \begin{cases} \mu_j(\tau, g), & j = j_0, \\ \mu_j(g), & j \neq j_0, \end{cases}$$

$$\kappa_j = \begin{cases} -b\xi(\mu_j(\tau, g)), |\xi| < 1, & j = j_0, \\ \kappa_j(g), & j \neq j_0, \end{cases}$$

such that for $(\tau, g) \in [0, 1] \times G$:

(i) $|a(H(\tau, g)) - a(g)| < 2\varepsilon b^2$,

(ii) $a(H(1, g)) = a(g) - \varepsilon b^2$,

(iii) $\mu_{j_0}(1, g)$ is greater than the value of μ at the minimum of $F_j(\mu, \xi; g)$ on C_j.

Moreover, letting ϕ_r be the N_{I_0}-gradient flow of a,

$$a_0 = a(H(\tau, \phi_r(\gamma(s)))) = a(\phi_r(H(\tau, \gamma(s))))$$

is uniquely solvable for $r(\tau, s)$ with $\phi_{r(\tau,s)} \in G$ for $(\tau, s) \in [0, 1] \times S^m$. Now $F(s, \tau) = H(\tau, \phi_{r(s,\tau)}(\gamma(s)))$ gives the homotopy of γ required for step (i) with the $j_0(\gamma)$ in (i) given by $j_6(\gamma)$.

Proof of (ii). In proving (5.10) we used the assumption $(q, B) \in M_I$ only to insure that $\min\{a, d\} \leq a_I$. Hence (5.10) gives a representation of $a + d$ at a general point (q, B) of M in terms of the coordinates of (q, B) when $d \leq a_I$. Since $F_j(\mu, \xi; q_I, B_I)$ is positive on C_j for $j \in I^c$ and negative on C_j for $j \in I$, one sees as in (5.13) that for $d_0 \leq a_I$

$$\min_{N_I} \sup_{N_{I^c} \cap \{d = d_0\}} a \geq 2a_I - d_0. \tag{6.10}$$

Since Γ maps $S^m \times [0, 1]$ into a compact subset of $M \cap \{d = d_0\}$, there is a k such that $\Gamma(S^m \times [0, 1]) \cap M_{\{j\}} = \varnothing$ for $j > k$. Let T denote the set of subsets I of $\{1, \ldots, k\}$ such that $a_0 + d_0 < 2a_I$. Since $\Gamma(S^m \times [0, 1])$ is compact, one sees that the tails in (5.10) are uniformly small on $\Gamma(S^m \times [0, 1])$. Hence the proof of (6.10) implies that there is a $l > k$ such that for all $I \in T$ and $(s, t) \in S^m \times [0, 1]$

$$\min_{r \in N_I} \max_{E_I(r,s,t)} a > 2a_I - d_0 > a_0, \tag{6.11}$$

where

$$E_I(r, s, t) = \{(u, v, w) \in N_I \times N_{I^c \cap \{j \leq l\}} \times N_{\{j > l\}} : u = r, w = w(\Gamma(s, t))\}.$$

In what follows we will work on the compact tori, parametrized by $S^m \times [0, 1]$,

$$N(s, t) = \{(u, v) \in N_{\{j \le l\}} \times N_{\{j > l\}}: v = v(\Gamma(s, t))\}.$$

Formula (6.11) will be the basis for the rest of this proof. We are going to imitate the proofs of Morse theory and use flows $\phi_r(s, t)$ on $N(s, t)$ closely related to the $N_{\{j \le l\}}$-gradient flow of a to deform Γ into the set where $a > a_0$.

Given any finite subset I of $\mathbf{Z}_+$, we can repeat the construction of $r(s)$ in lemma 4.6 to get an analytic function $h(s)$ on N_{I^c} such that $a(h(s), s)$ is the strict maximum of $a(h, s)$ for $h \in N_I$. Then we set

$$S_I = \{\Phi^{-1}(h(s), s): s \in N_{I^c}\},$$

in analogy with M_I. Finally, for $I \subset \{1, \ldots, l\}$ and $(s, t) \in S^m \times [0, 1]$ we set

$$M(s, t) = \Phi^{-1}(N(s, t)),$$
$$M_I(s, t) = M_I \cap \Phi^{-1}(N(s, t)),$$
$$S_I(s, t) = S_I \cap \Phi^{-1}(N(s, t)).$$

Now (6.11) can be rephrased as: for all $I \in T$ and $(s, t) \in S^m \times [0, 1]$

$$a(S_{I^c \cap \{j \le l\}}(s, t)) > a_0. \tag{6.12}$$

Moreover, by hypothesis, for all $(s, t) \in S^m \times [0, 1]$

$$\Gamma(s, t) \notin M_I(s, t), \tag{6.13}$$

when $I = \{j\}$, $k < j \le l$, and when $I \subset \{1, \ldots, l\}$ and $a_0 + d_0 > 2a_I$.

As defined earlier, the $N_{\{j \le l\}}$-gradient field of a is given by

$$V(q, B) = \sum_{j=1}^{l} \dot{a}_j(q, B) V_j(q, B),$$

where $\dot{a}_j$ denotes $\dot{a}$ on the flow of V_j. The flow of this vector field leaves $M(s, t)$ invariant and $\dot{a} \ge 0$ on the flow, vanishing only at zeros of V. If (q, B) is a zero of V on $M(s, t)$, then for some set $J \subset \{1, \ldots, l\}$

$$(q, B) = M_J(s, t) \cap S_{J'}(s, t), \tag{6.14}$$

where $J' = \{1, \ldots, l\} \cap J^c$. By (6.13), $\Gamma(s, t)$ can be a zero of V only if J in (6.14) belongs to T - note that $M_I(s, t) \cap M_{I'}(s, t) = M_{I \cup I'}(s, t)$ - and hence $a(\Gamma(s, t)) > a_0$ by (6.12). However, this does not imply that the flow of V will eventually make $a > a_0$. In what follows on $M(s, t)$ we change V to $V(s, t)$, depending continuously on (s, t), with the following properties:

(a) $V(s, t)$ has the same zeros as V on $N(s, t)$ and a is non-decreasing on $V(s, t)$.

(b) $V(s, t)$ is tangent to $M_I(s, t)$ for all $I \subset \{1, \ldots, l\}$.

(c) If $(q, B) = M_J(s, t) \cap S_{J'}(s, t)$ is a zero of V, then $M_J(s, t)$ contains the stable manifold of (q, B) under the flow of $V(s, t)$.

(d) $V(s, 1)$ is independent of s.

Since $S^m \times [0, 1]$ is compact and all dependence on (s, t) is continuous, it follows from (6.12), (6.13), (6.14) and (a)-(d) that, letting $\phi_r(q, B_j s, t)$ denote the flow of $V(s, t)$, there is an R such that for all $(s, t) \in S^m \times [0, 1]$

$$a(\phi_R(\Gamma(s, t); s, t)) > a_0. \tag{6.15}$$

Given (6.15), we can build $\tilde{\Gamma}$ by defining

$$\Gamma_1(s,t)=\begin{cases}\Gamma(s,0), & 0\le t\le\frac{1}{2},\\ \Gamma(s,2t-1), & \frac{1}{2}\le t\le 1,\end{cases}$$

$$r(t)=\begin{cases}2tR, & 0\le t\le\frac{1}{2},\\ R, & \frac{1}{2}\le t\le 1,\end{cases}$$

$$v(t)=\begin{cases}0, & 0\le t\le\frac{1}{2},\\ 2t-1, & \frac{1}{2}\le t\le 1,\end{cases}$$

$$\tilde{\Gamma}(s,t)=\phi_{r(t)}(\Gamma_1(s,t);\, s, v(t)).$$

Hence this proof will be complete once we construct $V(s,t)$.

The construction of $V(s,t)$ is based on facts already used in proving lemma 4.6: if $(q,B)\in M_I\cap S_{I'}$, $I\cap I'=\varnothing$, then $(\partial^2 a/\partial\tilde{\mu}_i\partial\tilde{\mu}_j(\Phi(q,B)))$, $i,j\in I\cup I'$, is a diagonal matrix with $\partial^2 a/\partial\tilde{\mu}_i^2>0$ for $i\in I$ and $\partial^2 a/\partial\tilde{\mu}_i^2<0$ for $i\in I'$ – we allow I or I' to be empty. As in lemma 4.6, an implicit function argument implies that locally one has an analytic function $\tilde{r}$: $N_{(I\cup I')^c}\to N_{I\cup I'}$ so that $\Phi(M_I\cap S_I)$ is given by $(\tilde{r}(u),u)$. If we choose $r(u)$ so that

$$\max_{v\in N_{I'}} a(h(u,v),v,u)$$

is assumed at $(h(u,v),v)=r(u)$, where $\Phi(M_I)$ is given by $(h(w),w)$, it follows that $\Phi^{-1}(r(u),u)\in M_I\cap S_{I'}$, and the argument used to prove proposition 4.7 shows $r(u)$ is unique. Hence $r(u)=\tilde{r}(u)$ and is analytic. Now setting $R_{I,I'}=M_I\cap S_{I'}$, we have

$$R_{I,I'}=\Phi^{-1}(\{r(u),u\}\colon u\in N_{(I\cup I')^c}\}),$$

generalizing the constructions of M_J and $S_{J'}$. Hence for $I, I'\subset\{1,\dots,l\}$, setting

$$R_{I,I'}(s,t)=R_{I,I'}\cap M(s,t)=M_I(s,t)\cap S_{I'}(s,t),$$

we see that $\Phi(R_{I,I'}(s,t))$ is an analytic subset of $N(s,t)$, given globally by r: $N_{(I\cup I|)^c\cap\{j\le l\}}\to N_{I\cup I'}$, where r also depends continuously on (s,t). In the case $I\cup I'=\{1,\dots,l\}$, if $(q,B)=R_{I,I'}(s,t)$, the fact that $(\partial^2 a/\partial\tilde{\mu}_i\partial\tilde{\mu}_j(\Phi(q,B)))$, $1\le i$, $j\le l$, is diagonal implies that the tangent spaces satisfy

$$\begin{aligned}T_{(q,B)}(M_{\{j\}}(s,t))&=\operatorname{span}\{V_i(q,B)\colon i\ne j,\, i\le l\}\qquad\text{for } j\in I,\\ T_{(q,B)}(S_{\{j\}}(s,t))&=\operatorname{span}\{V_i(q,B)\colon i\ne j,\, i\le l\}\qquad\text{for } j\in I'.\end{aligned}\tag{6.16}$$

Moreover, $\partial^2 a/\partial\tilde{\mu}_i^2>0$, $i\in I$, and $\partial^2 a/\partial\tilde{\mu}_i^2<0$, $i\in I'$, imply that the $N_{\{j\le l\}}$-gradient flow of a on $M(s,t)$ has a hyperbolic fixed point at (q,B), with stable manifold tangent to $M_I(s,t)$ at (q,B) and unstable manifold tangent to $S_{I'}(s,t)$ at (q,B).

We let V denote the $N_{\{j\le l\}}$-gradient field of a as before, and begin by constructing $V(s,t)$ near the zeros of V. As noted earlier, the zeros of V on $M(s,t)$ are the union of the distinct (since we ignore degenerate gaps) points $R_{J,J'}(s,t)$, where J, J' range over disjoint sets satisfying $J\cup J'=\{1,\dots,l\}$. In view of (6.16), we can introduce coordinates $n_1,\dots,n_l$ near $R_{J,J'}(s,t)$, continuously in (s,t), so that $n_j=0$ on $M_{\{j\}}(s,t)$, $j\in J$, and $n_j=0$ on $S_{\{j\}}(s,t)$, $j\in J'$, and the Jacobian $(\partial n_j/\partial\tilde{\mu}_k)$ will be diagonal at $R_{J,J'}(s,t)$. Let V_{n_j} denote the vector field tangent to the curves $\gamma(t,n_0)$

given by $n_k = n_k^0$, $k \neq j$, $n_j = n_j^0 + t$. Near $R_{J,J'}(s, t)$ we set

$$X_J(s,t) = \sum_{j=1}^{l} c_j(s,t) n_j V_{n_j},$$

where the coefficients $c_j(s, t)$ are chosen so that $X_j(s, t) - V$ vanishes to second order at $R_{J,J'}(s, t)$. The vector field $X_J(s, t)$ is tangent to all the manifolds $M_I(s, t)$ passing through $R_{J,J'}(s, t)$, and $\dot{a}$ will be strictly positive on the flow of $X_J(s, t)$ on a punctured neighbourhood of $R_{J,J'}(s, t)$. Hence we can choose a cut-off ϕ on $N(s, t)$ depending continuously on (s, t) such that $\phi = 1$ near $R_{J,J'}(s, t)$, $\phi(s, 1)$ is independent of s, and $\dot{a}$ will be non-negative on the flow of

$$W_J(s,t) = \phi(s,t) X_J(s,t)$$

on all of $N(s, t)$.

The vector fields $W_J(s, t)$ have the properties we require near the points $R_{J,J^c \cap \{j \le l\}}(s, t)$. Now we need a vector field tangent to all the $M_J(s, t)$, $J \subset \{1, \dots, l\}$, such that $\dot{a} > 0$ on this vector field outside a small neighbourhood of the $R_{J,J' \cap \{j \le l\}}(s, t)$. To build this, we consider the set $\mathscr{C}$ of all subsets of $\{1, \dots, l\}$ and let $\mathscr{C}_j$ denote the subset of $\mathscr{C}$ with $\#\{I\} = j$. Given $I \in \mathscr{C}_n$, we set

$$U_I(s,t) = \sum_{j \in I^c \cap \{j \le l\}} \dot{a}_j V_j + \sum_{j \in I} b_j(s,t) V_j$$

on $M_I(s, t)$, where the b_j are uniquely determined by the requirement that $V_I(s, t)$ is tangent to $M_I(s, t)$. We are going to take $V(s, t) = U_I(s, t)$ on $M_I(s, t)$ outside a small neighbourhood of

$$K_n(s,t) = \left(\bigcup_{j>n} \bigcup_{I \in \mathscr{C}_j} M_I(s,t) \right) \cup \left(\bigcup_{J \subset \{1, \dots, l\}} R_{J,J^c \cap \{j \le l\}}(s,t) \right).$$

Note that $V(s, t)$ is well defined on $M_I(s, t)$ since $M_I(s, t) \cap M_J(s, t) = M_{I \cup J}(s, t)$ and $\#\{I \cup J\} > n$ unless $I \supset J$. Moreover, on $M_I(s, t)$, $\dot{a} = \sum_{i \in I^c \cap \{j \le l\}} (\dot{a}_i)^2$ and this is strictly positive on $M_I(s, t)$ off $\bigcup_{J \subset \{1,\dots,l\}} R_{J,J^c \cap \{j \le l\}}(s, t)$. We extend $U_I(s, t)$ continuously in (s, t) to a neighbourhood of $M_I(s, t)$ in $N(s, t)$ so that it is tangent to $M_{\{j\}}(s, t)$, $j \in I$, and then set

$$Y_I(s,t) = \phi U_I(s,t),$$

where $\phi = 1$ on $M_I(s, t)$ off a sufficiently small neighbourhood of $K_n(s, t)$, $\phi = 0$ on a neighbourhood of $K_{n-1}(s, t) \backslash M_I(s, t)$, and ϕ also vanishes outside a sufficiently small neighbourhood of $M_I(s, t)$. Choosing the 'sufficiently small' neighbourhoods here first for $n = l - 1$, then for $n = l - 2$ and so on, we can, for each $n > 0$, make

$$Z_n(s,t) = \sum_{j=n}^{l-1} \sum_{I \in \mathscr{C}_j} Y_I(s,t) + \sum_{J \subset \{1,\dots,l\}} W_J(s,t)$$

tangent to $M_I(s, t)$ for all I and make $\dot{a} \ge 0$ everywhere and $\dot{a} > 0$ on

$$\left(\bigcup_{j=n}^{l-1} \bigcup_{I \in \mathscr{C}_j} M_I(s,t) \right) \Big\backslash \left(\bigcup_{J \subset \{1,\dots,l\}} R_{J,J^c \cap \{j \le l\}}(s,t) \right)$$

on the flow of $Z_n(s, t)$. Moreover, we can choose the cut-offs ϕ to depend continuously on (s, t) and be independent of s when $t = 1$.

Now we conclude by setting

$$V(s, t) = Z_1(s, t) + \phi V,$$

where ϕ is a cut-off vanishing on a neighbourhood of $K_0(s, t)$ with $\phi = 1$ off a sufficiently small neighbourhood of $K_0(s, t)$, and we have a vector field with the desired properties. □

Remark 6.5. In the proof of the injectivity of the homomorphism of the homotopy groups of M_{B_0} into those of S_0, it is not necessary that the deformation of Γ into $\tilde{\Gamma}$ in step (ii) be done in S_0. However, it is essential that the corresponding deformation in the proof of the surjectivity of the homomorphism remain in S_0. Hence it is important to note that the deformation in the preceding proof can be done in S_0. To see this, recall that, given (q, B) in M, there is a j_0 such that for $j > j_0$ the coordinate κ_j satisfies $|\kappa_j| > b$ at the minimum of a on the orbit of (q, B) in the flow of V_j. Since the minimum of a on this orbit never occurs when $\mu_j = \lambda_{2j}$ or $\mu_j = \lambda_{2j-1}$ (cf. lemma 4.3), it follows that, for $j > j_0$, $|\kappa_j| > b$ on $M_{\{j\}}$. Since $\Gamma(S^m \times [0, 1])$ (or $f(S^m)$ when one is proving surjectivity) is compact, it follows that there is a $j_1 \geq j_0$ such that $|\kappa_j(\Gamma(S^m \times [0, 1]))| < b$ for $j > j_1$. Choosing k in the preceding proof larger than j_1, one sees that

$$N(s, t) \cap M_J = \phi$$

for all $(s, t) \in S^m \times [0, 1]$ unless $J \subset \{1, \ldots, l\}$. Since the flows used in the proof preserve M_J for $J \subset \{1, \ldots, l\}$, it follows that the deformation remains in S_0.

Appendix A. Boundary conditions of the form

$$\begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}.$$

1. *Asymptotics.* We will first discuss the asymptotic behaviour of the eigenvalues λ_m, $n = 0, 1, \ldots$, for these problems. For $a \neq 1$ the techniques in § 1 can be used to obtain

$$\lambda_{2n} = n^2\pi^2 + 2n\pi k + k^2 + \frac{2c}{a + a^{-1}} + \int_0^1 q(t)\, dt + l^2(n), \qquad n \geq 0,$$

$$\lambda_{2n-1} = n^2\pi^2 - 2n\pi k + k^2 + \frac{2c}{a + a^{-1}} + \int_0^1 q(t)\, dt + l^2(n), \qquad n \geq 1,$$

where $k = \cos^{-1}(2/(a + 1/a))$. This means that the nth gap length is

$$\lambda_{2n} - \lambda_{2n-1} = 4k\pi n + l^2(n), \qquad n \geq 1,$$

and the nth band length

$$\lambda_{2n-1} - \lambda_{2n-2} = 2(\pi - 2k)\pi n - \pi^2 + 2\pi k + l^2(n), \qquad n \geq 1.$$

We note that if $k \in (0, \pi/4)$, the bands grow faster than the gaps, and if $k \in (\pi/4, \pi/2)$, the gaps grow faster than the bands.

The case $a = 1$ is different and has more resemblance to the case of periodic boundary conditions. To find the asymptotics of the eigenvalues, one can here use the refined estimates of y_1, y_2, and y_2' discussed immediately following theorem 1.3

in [6]. We find that for $\lambda\in\mathbf{R}^+$

$$y_1(1,\lambda,q)=\cos\sqrt{\lambda}+\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}}\int_0^1 q(t)\,dt+\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}}\int_0^1\cos(\sqrt{\lambda}2t)q(t)\,dt$$
$$-\frac{\cos\sqrt{\lambda}}{2\sqrt{\lambda}}\int_0^1\sin(\sqrt{\lambda}2t)q(t)\,dt-\frac{\cos\sqrt{\lambda}}{8\lambda}\left(\int_0^1 q(t)\,dt\right)^2+R_1(\lambda),\tag{A.1(i)}$$

$$y_2(1,\lambda,q)=\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}-\frac{\cos\sqrt{\lambda}}{2\lambda}\int_0^1 q(t)\,dt+R_2(\lambda),\tag{A.1(ii)}$$

$$y_2'(1,\lambda,q)=\cos\sqrt{\lambda}+\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}}\int_0^1 q(t)\,dt-\frac{\sin\sqrt{\lambda}}{2\sqrt{\lambda}}\int_0^1\cos(\sqrt{\lambda}2t)\,q(t)\,dt$$
$$+\frac{\cos\sqrt{\lambda}}{2\sqrt{\lambda}}\int_0^1\sin(\sqrt{\lambda}2t)q(t)\,dt-\frac{\cos\sqrt{\lambda}}{8\lambda}\left(\int_0^1 q(t)\,dt\right)^2+R_3(\lambda),\tag{A.1(iii)}$$

where R_1, R_2, and R_3 are $O(1/\lambda)$. Using Rouché estimates on y_1, y_2 and y_2', we find that for $\lambda=\lambda_{2n-1}$ or λ_{2n}, $\sqrt{\lambda}=n\pi+r_n$ with $r_n=O(1/n)$. For a sequence $\{\lambda_n\}_1^\infty$ with such asymptotics we have

$$\{R_i(\lambda_n)\}_1^\infty\in l_2^2(\mathbf{Z}^+),\qquad i=1,2,3,$$

where $l_k^2(\mathbf{Z}^+)$ is the set of all sequences $\{a_n\}_1^\infty$ with $\{n^k a_n\}_1^\infty\in l^2(\mathbf{Z}^+)$. This is seen by observing that in this case the remainder term consists of sums of Fourier coefficients of q multiplied by $O(1/\lambda_n)$ plus terms which are $O(1/\lambda_n^{3/2})$.

We recall that the eigenvalues λ_{2n-1} and λ_{2n} are roots of the equation

$$F(\lambda)=ay_1(1,\lambda)+cy_2(1,\lambda)+(1/a)y_2'(1,\lambda)=2(-1)^n,$$

which in the case $a=1$ reduces to

$$y_1(1,\lambda)+cy_2(1,\lambda)+y_2'(1,\lambda)=2(-1)^n.$$

Now using the asymptotics for y_1, y_2 and y_2' stated above, we see that the terms involving the Fourier coefficients of q in (A.1(i)) and (A.1(iii)) cancel in the equation for the eigenvalues precisely when $a=1$. Setting $\sqrt{\lambda}=n\pi+r_n$, where $\lambda=\lambda_{2n-1}$ or λ_{2n}, we derive the equation

$$2\cos r_n=2-2\frac{\sin r_n}{2(n\pi+r_n)}\int_0^1 q(t)\,dt+2\frac{\cos r_n}{8(n\pi+r_n)^2}\left(\int_0^1 q(t)\,dt\right)^2$$
$$-c\frac{\sin r_n}{n\pi+r_n}+c\frac{\cos r_n}{2(n\pi+r_n)^2}\int_0^1 q(t)\,dt+l_2^2(n).\tag{A.2}$$

We now divide this equation by $2\cos r_n$ and solve the second-order equation in r_n that we obtain by deleting terms in $l_2^2(\mathbf{Z}^+)$. For $c\geq 0$ this gives us the asymptotics

$$\lambda_{2n}=n^2\pi^2+2c+\int_0^1 q\,dt+l^2(n),\qquad n\geq 0,$$

$$\lambda_{2n-1}=n^2\pi^2+\int_0^1 q\,dt+l^2(n),\qquad n\geq 1,$$

and for $c<0$

$$\lambda_{2n}=n^2\pi^2+\int_0^1 q\,dt+l^2(n), \qquad n\geq 0,$$

$$\lambda_{2n-1}=n^2\pi^2+2c+\int_0^1 q\,dt+l^2(n), \qquad n\geq 1.$$

The nth gap length is therefore

$$\lambda_{2n}-\lambda_{2n-1}=2|c|+l^2(n), \qquad n\geq 1,$$

and the nth band length

$$\lambda_{2n-1}-\lambda_{2n-2}=2n\pi-\pi^2-2|c|+l^2(n), \qquad n\geq 1.$$

2. *Isospectral flows.* The theorems and arguments of this section are completely analogous to those of § 2. We will therefore just state our results.

We fix $p\in L^2_{\mathbf{R}}[0,1]$ and define the isospectral set

$$L(p)=\{q\in L^2_{\mathbf{R}}[0,1]: \lambda_n(q)=\lambda_n(p), n\geq 1\}.$$

To study the geometric nature of $L(p)$ we first construct vector fields on $L(p)$:

THEOREM. *Let $p\in L^2_{\mathbf{R}}[0,1]$. Then for each $\lambda\in\mathbf{R}$ the vector field*

$$Z_\lambda(q)=2\left(\frac{\partial\Delta(\lambda,q)}{\partial q(x)}\right)'$$

is tangent to $L(p)$. That is, a solution of the differential equation

$$\dot q(x,t)=[Z_\lambda(q)](x,t)$$

with initial data in $L(p)$ stays in $L(p)$ for all time.

As in § 1, we can map the isospectral set $L(p)$ into a space S consisting of sequences of the form

$$(\mu_1,\kappa_1,\mu_2,\kappa_2,\ldots),$$

where

$$\mu_n=n^2\pi^2+\int_0^1 p\,dt+l^2(n), \qquad \kappa_n=l^2_1(n).$$

From theorems 3.5 and 3.6 of [**6**] we know that the map

$$\phi: L^2_{\mathbf{R}}[0,1]\to S$$

given by

$$\phi=(\mu_1(q),\log(-1)y_2'(1,\mu_1(q),q),\mu_2(q),$$
$$\log(-1)^2y_2'(1,\mu_2(q),q),\ldots,\mu_n(q),\log(-1)^ny_2'(1,\mu_n(q),q),\ldots)$$

is an analytic homeomorphism of $\{q\in L^2_{\mathbf{R}}[0,1): \int_0^1 q\,dt=\int_0^1 p\,dt\}$ and the sequence space S with the l^2-coordinatization given in [**6**]. Using the definition of Δ,

$$\Delta(\mu_n)=ay_1(1,\mu_n)+(1/a)y_2'(1,\mu_n),$$

and the Wronskian identity

$$y_1(1,\mu_n)=1/y_2'(1,\mu_n),$$

https://doi.org/10.1017/S0143385700009470 Published online by Cambridge University Press

we derive the equation

$$(y_2'(1, \mu_n))^2 - a\Delta(\mu_n)y_2'(1, \mu_n) + a^2 = 0,$$

which we can write in terms of the κ_n as

$$(e^{\kappa_n})^2 - a\Delta(\mu_n)(-1)^n e^{\kappa_n} + a^2 = 0.$$

Thus $\phi(L(p))$ is contained in a product of real analytic curves which are topologically circles.

Let

$$p_n(q) = (\mu_n(q), \kappa_n(q)), \qquad n \geq 1,$$

and let $\sigma_n(q) \in \{+, -\}$ denote the sign of the radical in the equation

$$ay_1(1, \mu_n(q), q) = \Delta(\mu_n)/2 \pm \tfrac{1}{2}(\Delta^2(\mu_n) - 4)^{1/2}.$$

Now let Z_n denote the vector fields Z_λ with $\lambda = \mu_n(q(\cdot, t))$. For these vector fields the integral curves have a very simple description in terms of the p_n as is seen by the following result.

THEOREM. *Under the flow of Z_n the points $p_m(q(\cdot, t))$, $m \neq n$, remain fixed on their circles, while the point $p_n(q(\cdot, t))$ moves clockwise around its circle. It moves in such a way that*

$$d\mu_n(q(\cdot, t))/dt = \sigma_n(q(\cdot, t)) \cdot (\Delta^2(\mu_n) - 4)^{1/2}.$$

We can in fact integrate the vector fields Z_n explicitly. Thus we will be able to see directly how a potential changes as we move along an integral curve of Z_n. The explicit integration of Z_n is given by the following theorem, analogous to theorem 2.3.

THEOREM. *Let $q_0 \in L(p)$ and let $\mu_n(t)$ denote the unique solution of*

$$\frac{d}{dt}\mu_n(t) = \pm(\Delta^2(\mu_n(t)) - 4)^{1/2}$$

for which the point $p_n(t) = (\mu_n(t), \kappa_n(t))$ starts at $p_n(q_0)$ and moves clockwise around its circle without pausing. Here the sign in front of the radical is chosen to be $\sigma_n(q_0)$ for $t = 0$ and then to change when the radical vanishes. Then the integral curve of Z_n passing through q_0 is given by

$$q(x, t) = q_0(x) - 2\frac{d^2}{dx^2}\log\left[f_{\sigma_n(t)}(\mu_n(t), q_0), y_2(\mu_n, q_0)\right],$$

where $f_\pm(x, \lambda, q)$ are the solutions of $-u'' + qu = \lambda u$ associated with eigenvectors of $BF(\lambda)$ as in § 2.

3. *Geometry of* $L(p)$. We are now in a position to describe $L(p)$ as a submanifold of $L^2_{\mathbf{R}}[0, 1]$. As in § 3, we will proceed by determining the image of $L(p)$ in the sequence space S under the coordinate map ϕ.

As noted earlier, the coordinates $(\mu_1, \kappa_1, \mu_2, \kappa_2, \ldots)$ of a point in $L(p)$ satisfy the equations

$$e^{2\kappa_n} + (-1)^{n+1}a\Delta(\mu_n)\, e^{\kappa_n} + a^2 = 0, \qquad n \geq 1. \tag{A.3}$$

Let N be the subset of S defined by these equations. We claim that N is a real analytic submanifold of S. From the properties of Δ it follows that for each n equation (A.3) defines a real analytic curve which topologically is a circle in the (μ_n, κ_n)-plane unless $\lambda_{2n-1} = \lambda_{2n}$, in which case the circle degenerates to a point. From the asymptotics of the λ it follows that this degeneracy can only occur a finite number of times. We may give N locally as an explicit submanifold of S by solving (A.3) for μ_n or κ_n as appropriate for each n for which the corresponding circle is non-degenerate. In the case $a \neq 1$ the asymptotics of

$$\Delta(\mu_n) = ay_1(1, \mu_n) + \frac{1}{a} y_2'(1, \mu_n) = (-1)^n\left(a + \frac{1}{a}\right) + O\left(\frac{1}{n}\right)$$

imply that there is an n_0 such that for $n > n_0$

$$\kappa_n = \log\left[\frac{(-1)^n a\Delta(\mu_n)}{2} - \left(\frac{a^2\Delta^2(\mu_n)}{4} - a^2\right)^{1/2}\right]$$

on the intersection of N with a bounded subset of S. Thus N is qualitatively the same as $N \cap \{d = d_0\}$ in § 3 and, since the proof that N is real analytic exactly follows the proof given there, we will not repeat it here.

In the case $a = 1$ but $c \neq 0$ the situation is different. We will see that in this case we can solve for μ_n as a function of κ_n in equation (A.3) for n large. This will enable us to prove that N is a real analytic submanifold of S. More precisely our strategy is as follows: given $(\hat{\mu}, \hat{\kappa}) \in N$,

$$(\hat{\mu}, \hat{\kappa}) = (\hat{\mu}_1, \hat{\kappa}_1, \hat{\mu}_2, \hat{\kappa}_2, \ldots),$$

we need to show that the intersection of N with a neighbourhood of $(\hat{\mu}, \hat{\kappa})$ in S consists of points in S with $\mu_n = \mu_n(\kappa_n)$ for $n \geq n_0$, where each component of the function $\psi(\kappa) = (\mu_{n_0}(\kappa_{n_0}), \mu_{n_0+1}(\kappa_{n_0+1}), \ldots)$ is analytic on the ball $\|\kappa - \hat{\kappa}\| < \varepsilon$ in *complex-valued* l_1^2, and the l^2-norm of the sequence

$$\left\{\mu_n(\kappa) - n^2\pi^2 - \int_0^1 p\,dt\right\}_{n=n_0}^{\infty}$$

is uniformly bounded on that ball.

As in § 3, we can then conclude that Ψ is real analytic. Since N for $n \geq n_0$ is the graph of Ψ and for $n \leq n_0$ we can solve for μ_n or κ_n locally using real analytic functions, N is a real analytic submanifold of S. We will now carry out this plan in detail.

Since $a = 1$, we can write (A.3) as

$$\Delta(\mu_n) = 2(-1)^n \cosh \kappa_n. \tag{A.3'}$$

On $\{\|\kappa - \hat{\kappa}\| < \varepsilon\}$, $|\kappa_n| < 2\varepsilon/n$ for $n > n_1(\hat{\kappa}, \varepsilon)$. From this it follows, using Rouché's theorem to compare the roots of $(-1)^n\Delta(\lambda)(2\cosh\kappa_n)^{-1} - 1$ and the roots of $(-1)^n \cos\sqrt{\lambda} - 1$, that there is a constant A and an $n_2(\hat{\kappa}, \varepsilon)$ such that for all κ in $\{\|\kappa - \hat{\kappa}\| < \varepsilon\}$, (A.3') has exactly two roots $\mu_n(\kappa_n)$ such that

$$\sqrt{\mu_n} = n\pi + \beta_n$$

with $|\beta_n| < A/n$, when $n \geq n_2(\hat{\kappa}, \varepsilon)$.

Using the asymptotic expansions from (A.1) and analysing the error terms as before, we can write (A.3′) in a form analogous to (A.2):

$$2\cosh\kappa_n = 2\cos\beta_n + 2\frac{\sin\beta_n}{2(n\pi+\beta_n)}\int_0^1 p\,dt + c\frac{\sin\beta_n}{n\pi+\beta_n}$$
$$-2\frac{\cos\beta}{8(n\pi+\beta_n)^2}\left(\int_0^1 p\,dt\right)^2 - c\frac{\cos\beta_n}{(n\pi+\beta_n)^2}\int_0^1 p\,dt + r_n, \qquad \text{(A.4)}$$

where the remainders r_n are bounded by a fixed sequence in l_2^2 for all sequences $\{\beta_n\}$ satisfying $|\beta_n|<A/n$. Dividing by $2\cos\beta_n$ and denoting another sequence of remainders bounded by a fixed sequence in l_2^2 by $\{r_n\}$,

$$\frac{\cosh\kappa_n}{\cos\beta_n} = 1 + \frac{\beta_n}{2n\pi}\int_0^1 p\,dt + \frac{c}{2}\frac{\beta_n}{n\pi} - \frac{1}{8(n\pi)^2}\left(\int_0^1 p\,dt\right)^2 - \frac{c}{4(n\pi)^2}\int_0^1 q\,dt + r_n. \qquad \text{(A.5)}$$

From the Taylor expansion of $\cos\beta_n$ we have

$$\frac{\cosh\kappa_n}{\cos\beta_n} = \cosh\kappa_n + \frac{\beta_n^2}{2}\cosh\kappa_n + r_n\cosh\kappa_n.$$

Substituting this expansion into (A.5) and treating the resulting equation as a quadratic equation for β_n, we obtain

$$\beta_n = \frac{1}{2\cosh\kappa_n}\left(\frac{\int_0^1 p\,dt + c}{n\pi} \pm \frac{1}{n\pi}[c^2 + 8\cosh\kappa_n(1-\cosh\kappa_n)n^2\pi^2 + b_n]^{1/2}\right),$$

where $\{|b_n|\}$ is bounded by a fixed l^2-sequence for all sequences $\{(\kappa_n, \beta_n)\}$ with $|\kappa_n|$ bounded and $|\beta_n| = A/n$. For ε small enough relative to c it follows that, assuming (A.3′) and $|\beta_n|\le A/n$,

$$\beta_n = \frac{1}{2n\pi}\left(\int_0^1 p\,dt + c \pm c + r_\pm(n)\right), \qquad \text{(A.6)}$$

where the error terms $r_\pm(n)$ have l^2-norm uniformly bounded on $\{\|\kappa-\hat\kappa\|<\varepsilon\}$ and there is an $n_3(\hat\kappa, \varepsilon)$ such that $|r_\pm(n)|<\varepsilon$ for $n>n_3(\hat\kappa,\varepsilon)$ uniformly on $\{\|\kappa-\hat\kappa\|<\varepsilon\}$. We know that for $n>n_2(\hat\kappa,\varepsilon)$, (A.3′) does have exactly two solutions $\mu_n(k)$ with the corresponding β satisfying $|\beta_n|<A/n$. Thus, since we can choose a neighbourhood $\mathcal{O}$ of $(\hat\mu, \hat\kappa)$ in S small enough that for any (μ, κ) in $\mathcal{O}$ we have $|\mu_n - n^2\pi^2 - \int_0^1 p\,dt| < \varepsilon$ for $n > n_4(\hat\mu, \varepsilon)$, and we may assume $\varepsilon<|c|$, we have nearly completed the proof that N is an analytic submanifold. It only remains to show that for ε sufficiently small and n sufficiently large:

(i) (A.3′) could not have two roots satisfying (A.6) with the same choice of sign so that (A.3′) does have roots satisfying (A.6) for both choices of sign.

(ii) The roots of (A.3′) in satisfying (A.6) with the minus sign are analytic functions of κ_n for

$$\{\|\kappa-\hat\kappa\|<\varepsilon\}.$$

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Fortunately both (i) and (ii) follow from an estimate on $\partial\Delta/\partial\lambda$ proven by the same method as (A.2) and (A.4):

$$\frac{\partial\Delta}{\partial\lambda}(\mu_n)=\frac{(-1)^n}{2\pi^2n^2}\left(2\pi\beta_n-\int_0^1 p\,dt-c+b_n\right),$$

where $|b_n|$ is bounded by a fixed sequence in l^2 for $|\beta_n|<A/n$.

In conclusion, let us indicate how we can give $L(p)$ a real analytic structure. For the details of a completely analogous argument see § 3. Using the flows of the vector fields Z_n, we can show that $\phi(L(p))$ contains all points of N whose coordinates agree with those of p beyond some index n, for any finite n. Since these sets are dense in N, $L(p)$ is closed in $L^2_{\mathbb{R}}[0,1]$ and ϕ is a homeomorphism, it follows that $\phi(L(p))=N$. Since ϕ is a real analytic homeomorphism, $L(p)$ inherits the real analytic structure of N.

Appendix B. Properties of $\Delta(\lambda)$

In this appendix we give proofs for the properties of $\Delta(\lambda)$ cited in the Introduction. Since these are standard results in Floquet theory, it is quite possible they have appeared elsewhere, but we were unable to locate a reference.

The key facts are:

(i) $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$ (cf. theorem 1.1).

(ii) All roots of $\Delta(\lambda)=2$ and $\Delta(\lambda)=-2$ are real, since they are eigenvalues of self-adjoint boundary value problems.

(iii) Zeros of $\Delta^2(\lambda)-4$ have multiplicity at most 2.

To prove (iii), it is convenient to use the formula

$$\Delta'(\lambda)=-\int_0^1\frac{\partial\Delta(\lambda)}{\partial q(x)}\,dx=-[ay_2+by_2'](1,\lambda)\int_0^1 f_+(x,\lambda)f_-(x,\lambda)\,dx, \tag{B.1}$$

which one obtains by combining theorem 1.2 and formula (2.5). If λ_0 is a zero of $\Delta^2(\lambda)-4$, it follows from (B.1) and the definition of $f_\pm$ that if $[ay_2+by_2'](1,\lambda_0)\neq 0$, then $f_+(x,\lambda_0)=f_-(x,\lambda_0)$ and $\Delta'(\lambda_0)\neq 0$. If λ_0 is a zero of $\Delta(\lambda)=\pm 2$ of order more than 2, then both roots of $\xi^2-\Delta(\lambda)\xi+1=0$ satisfy

$$\xi(\lambda)=\pm 1+O(\lambda-\lambda_0).$$

Thus, since the roots of $[ay_2+by_2'](1,\lambda)=0$ must be simple (cf. (1.2)) and $\partial\Delta(\lambda)/\partial q(x)\in L^2[0,1]$ for all λ, it follows that

$$\lim_{\lambda\to\lambda_0}\int_0^1 f_+(x,\lambda)f_-(x,\lambda)\,dx=\int_0^1 g^2(x)\,dx,$$

where

$$g(x)=y_1(x,\lambda_0)+\left(\lim_{\lambda\to\lambda_0}\left[\frac{\xi(\lambda)-ay_1-by_1'}{ay_2+by_2'}\right](1,\lambda)\right)y_2(x,\lambda_0).$$

However, by (B.1) this implies $\Delta'(\lambda)$ has a simple zero at $\lambda=\lambda_0$. Thus we conclude (iii) holds.

Combining (i) and (ii), Hadamard's theorem gives the product representations

$$\Delta(\lambda)\pm 2=a_\pm\lambda^{k_\pm}\prod_{j=1}^{\infty}\left(1-\frac{\lambda}{a_j^\pm}\right),$$

where the $a_j^\pm$ are real and increasing with j. Hence

$$\frac{\Delta'(\lambda)}{\Delta(\lambda)\pm 2}=\frac{k_\pm}{\lambda}+\sum_{j=1}^{\infty}\frac{1}{\lambda-a_j^\pm}, \tag{B.2}$$

$$\left(\frac{\Delta'(\lambda)}{\Delta(\lambda)\pm 2}\right)'=\frac{-k_\pm}{\lambda^2}-\sum_{j=1}^{\infty}\frac{1}{(\lambda-a_j)^2}<0. \tag{B.3}$$

Thus, by (B.3), $\Delta'(\lambda)(\Delta(\lambda)\pm 2)^{-1}$ is strictly decreasing in λ. Then, letting $\{\delta_j\}_{j=1}^{\infty}$ denote the real zeros of $\Delta'(\lambda)$ and recalling that $\Delta(\lambda)\to+\infty$ as $\lambda\to-\infty$ (cf. theorem 1.1), we see from (B.2) that

$$\lambda_0<\lambda_1\le\delta_1\le\lambda_2<\lambda_3\le\delta_2\le\lambda_4\cdots.$$

Moreover, $\lambda_{2j-1}<\delta_j<\lambda_{2j}$ unless $\lambda_{2j-1}=\lambda_{2j}$. This establishes the result on the critical values of $\Delta(\lambda)$ cited in the Introduction.

To show that the multiplicity of the λ_j as roots of $\Delta^2(\lambda)-4$ equals their multiplicity as eigenvalues, we note that if λ_0 is a zero of order 2, then $[ay_2+by_2'](1,\lambda)=0$ by the analysis in the preceding paragraph, and $[ay_1+by_1'](1,\lambda)=\pm 1$, since $BF(\lambda_0)$ must have eigenvalues equal to ± 1. The identity (cf. lemma 4.2(iv))

$$[y_2^2+b\Delta(\lambda)y_2+b^2](1,\lambda)=[(ay_2+by_2')(dy_2+b_1)](1,\lambda) \tag{B.4}$$

then implies $y_2(1,\lambda_0)=\mp b$, which in turn implies that the right-hand side of (B.4) vanishes to second order at $\lambda=\lambda_0$ and hence $[dy_2+by_1](1,\lambda_0)=0$. Thus one has $y_2(1,\lambda_0)=\mp b$, $y_2'(1,\lambda_0)=\pm a$, $y_1(1,\lambda_0)=\pm d$ and $y_2'(1,\lambda_0)=\mp c$, which implies $BF(\lambda_0)=\pm I$ and λ_0 is an eigenvalue of multiplicity 2.

Conversely, if λ_0 is a simple zero of $\Delta^2(\lambda)-4$, then either $[ay_2+by_2'](1,\lambda_0)\neq 0$, which implies $BF(\lambda_0)\neq\pm I$, or, if $[ay_2+by_2'](1,\lambda_0)=0$, then (B.4) implies $y_2(1,\lambda_0)=\mp b$ and $[dy_2+by_1](1,\lambda_0)\neq 0$. Hence $y_1(1,\lambda_0)\neq\pm d$. However, this shows

$$[cy_1+dy_1'](1,\lambda_0)=\left[\frac{-y_1}{b}\pm\frac{d}{b}\right](1,\lambda_0)$$

is non-zero, and we again conclude $BF(\lambda_0)\neq\pm I$. Thus λ_0 is a simple eigenvalue.

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