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Domains of Injective Holomorphy

This paper is dedicated to Professor Wolfgang Luh on the occasion of his retirement.

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Abstract. A domain Ω is called a domain of injective holomorphy if there exists an injective holomorphic function $f\colon \Omega \to \mathbb{C}$ that is non-extendable. We give examples of domains that are domains of injective holomorphy and others that are not. In particular, every regular domain ($\overline{\Omega}^o = \Omega$) is a domain of injective holomorphy, and every simply connected domain is a domain of injective holomorphy as well.

1 Introduction

A holomorphic function f defined on a domain Ω in the complex plane $\mathbb{C}$ is called (holomorphically) extendable over $\mathbb{C}$ if there exist two discs D_1 and D_2, where D_2 intersects both Ω and its complement, D_1 is contained in D_2 and in Ω, and there exists a holomorphic function F on D_2, so that $F = f$ on D_1. The function f is then non-extendable (over $\mathbb{C}$) if its Riemann surface (over $\mathbb{C}$) has only one sheet and its projection to $\mathbb{C}$ is exactly Ω. Equivalently, f is non-extendable if and only if for every $p \in \Omega$ the radius of convergence $R_p(f)$ of the Taylor series of f with center p is exactly $R_p(f) = \operatorname{dist}(p, \partial\Omega)$ ([5]). It is well known that every domain $\Omega \subset \mathbb{C}$ supports a non-extendable holomorphic function f; that is, every domain $\Omega \subset \mathbb{C}$ is a domain of holomorphy [7]. Moreover, such non-extendable functions may have a certain boundary regularity ([3–5]). In this paper, we consider a similar question, but f should be injective in Ω.

Definition A domain $\Omega \subset \mathbb{C}$ is called a *domain of injective holomorphy* if there exists an injective holomorphic function $f\colon \Omega \to \mathbb{C}$ that is non-extendable; that is, $R_p(f) = \operatorname{dist}(p, \partial\Omega)$ for every $p \in \Omega$. In order to cover the case where $\Omega = \mathbb{C}$, we shall agree that $\operatorname{dist}(p, \varnothing) = +\infty$.

It is not true that every domain in $\mathbb{C}$ is a domain of injective holomorphy. For instance, if $\Omega' \subset \mathbb{C}$ is any domain and $a, b \in \Omega'$, $a \neq b$, we shall see that $\Omega = \Omega' \setminus \{a, b\}$ is not a domain of injective holomorphy. This can be generalized for domains, where we remove a closed set of analytic capacity zero containing at least two points. See Proposition 2.1.

There are also plenty of domains that *are* domains of injective holomorphy. A central result in this paper, which is a generic result, implies that every domain $\Omega \subset \mathbb{C}$

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that is *regular* ($\overline{\Omega}^o = \Omega$) is a domain of injective holomorphy. There are also examples of domains of injective holomorphy that are not regular, *i.e.*, $\Omega = \{z : |z| < 1\} \setminus \{z \in \mathbb{R} : z \in [0, 1)\}$.

We shall show that every simply connected domain is a domain of injective holomorphy. We also give a complete answer in the case of finitely connected domains in $\mathbb{C}$. Finally we give an example of a domain in $\mathbb{C}$ that is not regular, is of infinite connectivity, and is a domain of injective holomorphy.

2 First Examples

The complex plane $\mathbb{C}$ is obviously a domain of injective holomorphy, because it supports the function $f(z) = z$. The domain $\mathbb{C} \setminus \{0\}$ supports the function $f(z) = 1/z$; therefore, it is a domain of injective holomorphy.

The unit disc $\mathbb{D}$ is a domain of holomorphy. In [6] we find a holomorphic function $g(z) = \sum_{n=2}^{\infty} a_n z^n \in H(\mathbb{D})$, that is non-extendable; that is, g has the unit circle as natural boundary. Moreover, $a_n \in \{0, 1\}$. Thus, integrating three times, we can find a non-extendable function $w(z) = \sum_{n=2}^{\infty} b_n z^n \in H(\mathbb{D})$ with $\sum_{n=2}^{\infty} n|b_n| < +\infty$. We consider $c > \sum_{n=2}^{\infty} n|b_n|$. Then the function $f(z) = cz + w(z)$ is injective and non-extendable; that is, $\mathbb{D}$ is a domain of injective holomorphy.

The domain $\mathbb{D} \setminus \{0\}$ is also a domain of injective holomorphy. To see this, let $f(z) = c/z + \sum_{n=2}^{\infty} b_n z^n$, where $\sum_{n=2}^{\infty} b_n z^n$ is the series we considered in the previous paragraph, when treating the case of the unit disc $\mathbb{D}$ and $c > \sum_{n=2}^{\infty} n|b_n|$. We shall show that f is injective in $\mathbb{D} \setminus \{0\}$. Suppose we have $z, w \in \mathbb{D} \setminus \{0\}$, $z \neq w$ such that $f(z) = f(w)$. Then

$$c\Big(\frac{1}{z} - \frac{1}{w}\Big) = \sum_{n=2}^{\infty} b_n (w^n - z^n).$$

It follows that

$$(w - z)c = (w - z)wz \sum_{n=2}^{\infty} b_n (w^{n-1} + w^{n-2}z + \cdots + z^{n-1}).$$

Since $|z|, |w| \leq 1$, and $z \neq w$,

$$c = |c| \leq \sum_{n=2}^{\infty} |b_n| \big(|w|^{n-1} + |w|^{n-2}|z| + \cdots + |z|^{n-1}\big) \leq \sum_{n=2}^{\infty} |b_n| n < c,$$

that is absurd. We arrive at a contradiction, and so f is injective in $\mathbb{D} \setminus \{0\}$. It is easy to see that f is non-extendable. Thus, $\mathbb{D} \setminus \{0\}$ is a domain of injective holomorphy.

Let Ω' be any domain and $a, b \in \Omega'$, $a \neq b$. Then $\Omega = \Omega' \setminus \{a, b\}$ is not a domain of injective holomorphy. Suppose that $f\colon \Omega \to \mathbb{C}$ is an injective holomorphic function that is non-extendable; we shall arrive at a contradiction. The points a, b are isolated singularities for f. If a or b is an essential singularity, then f is not injective. If a or b is a removable singularity, then f is extendable. Thus, both a and b are poles. But then, because f is an open mapping, a neighbourhood of ∞ is covered twice, one time around a and a second time around b. Therefore f is not injective, and we

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have arrived at a contradiction. We conclude that $\Omega = \Omega' \setminus \{a, b\}$ is not a domain of injective holomorphy. The last example takes the following more general form.

Proposition 2.1 *Let Ω' be any domain and let $E \subset \Omega'$ be a non-empty closed subset of zero analytic capacity that contains at least two points. Then $\Omega = \Omega' \setminus E$ is not a domain of injective holomorphy.*

Proof Let $f \colon \Omega \to \overline{\mathbb{C}}$ be injective and meromorphic; we shall show that f is meromorphically extendable.

Since any continuum is of positive analytic capacity, it follows that E is a totally disconnected subset of Ω', and so there is an open subset $U' \subset \Omega'$ such that $\overline{U}'$ is a compact subset of Ω', $U' \cap E \neq \varnothing$, and $\partial U' \cap E = \varnothing$. Thus, $K = U' \cap E$ is a non-empty compact subset of U'. Set $U = U' \setminus K$. Then U is an open subset of Ω and $\Omega \setminus \overline{U}$ is a non-empty open subset of Ω and so $f(\Omega \setminus \overline{U})$ contains a disc $\{z \in \mathbb{C} : |z - c| < \delta\}$ for some $c \in \mathbb{C}$. The function $g = 1/(f - c)$ is holomorphic and bounded in $U = U' \setminus K$, since $f(\Omega \setminus \overline{U}) \cap f(U) = \varnothing$, due to the injectivity of f on Ω. Since g is bounded and holomorphic on $U' \setminus K$ and $K \subset U'$ is of analytic capacity zero, it follows that g extends to a (bounded) holomorphic function on U'. Thus f extends meromorphically to U'. In other words, f extends to the set K, which lies outside Ω. Hence, f is extendable and Ω is not a domain of injective meromorphy.

Suppose now that E contains two distinct points a and b, and $f \colon \Omega \to \mathbb{C}$ is injective and holomorphic. We have already seen that f extends meromorphically to Ω'. If f is finite at one of the points a or b, then f extends holomorphically to that point, and so Ω is not a domain of injective holomorphy. The only other possibility is that both a and b are poles. This will lead to a contradiction. Indeed, let D_a and D_b be disjoint discs in Ω' centered at a and b respectively. There is an open neighbourhood V of ∞ contained in $f(D_a) \cap f(D_b)$. Thus $f(D_a \setminus E)$ and $f(D_b \setminus E)$ both contain open dense subsets of V. Hence, $f(D_a \setminus E) \cap f(D_b \setminus E)$ contains an open and dense subset of V and, in particular, is non-empty. This is a contradiction, since f is injective on $\Omega \setminus E$ and the sets $(D_a \setminus E)$ and $(D_b \setminus E)$ are disjoint. Thus, f extends holomorphically from Ω, and so Ω is not a domain of injective holomorphy. ■

Prime examples of Ω's as in Proposition 2.1 abound in potential theory. Any closed set E of harmonic capacity zero is of analytic capacity zero. Suppose Ω' is any domain in $\mathbb{C}$ and $u \colon \Omega \to [-\infty, +\infty)$ is any continuous subharmonic function on Ω'. We call $u^{-1}(-\infty)$ the pole set of u. If the pole set of u is not empty, then $\Omega = \Omega' \setminus u^{-1}(-\infty)$ is not a domain of injective meromorphy, and if the pole set contains at least two points, then Ω is not a domain of injective holomorphy.

Remark An obvious modification of the above proof shows that if $a \in \Omega'$ and $\Omega = \Omega' \setminus \{a\}$, then each injective meromorphic function on Ω extends to an injective meromorphic function on Ω'.

Example 2.2 Let Ω be a domain in $\mathbb{C}$. We assume that no component of $\mathbb{C} \setminus \Omega$ is a singleton. Let z_o be a point of Ω. Then Ω is a domain of injective holomorphy if and only if $\Omega' = \Omega \setminus \{z_o\}$ is a domain of injective holomorphy.

Proof Suppose first that Ω is a domain of injective holomorphy. If $\Omega = \mathbb{C}$, we have already pointed out that $1/(z - z_o)$ is an injective holomorphic function on the punctured plane that is non-extendable. Suppose $\Omega \neq \mathbb{C}$ and let f be a function that is injective, holomorphic, and non-extendable in Ω. Then

$$g(z) = \frac{1}{f(z) - f(z_o)}$$

is injective, holomorphic, and non-extendable in Ω'. Indeed, suppose g is extendable. Since g is not extendable to z_o, it follows that g is extendable through $\partial\Omega$. There are open discs $D_1 \subset D_2$ such that $D_1 \subset \Omega$, $D_2 \cap \partial\Omega \neq \varnothing$, and there is a function G holomorphic in D_2 such that $G = g$ on D_1. Let W be the component of $D_2 \cap \Omega$ containing D_1. Choose a point $w \in D_2 \cap \partial W$ such that $G(w) \neq 0$. Now choose an open disc $U \subset D_2$ centered at w such that G has no zeros in U and choose an open disc $V \subset U \cap W$. Then $1/G + f(z_o)$ is holomorphic in U and agrees with f on V. Thus f is extendable from Ω, which contradicts the choice of f. Therefore, g is non-extendable from Ω' as claimed.

Suppose conversely, that f is a function holomorphic and injective in Ω', which is non-extendable. Then z_o is a simple pole of f. Let b be a point of $\mathbb{C} \setminus f(\Omega')$ and set $g(z) = 1/(f(z) - b)$. Then g is holomorphic and injective in Ω, and we claim that g is non-extendable. Indeed, suppose g is extendable. Then there are open discs $D_1 \subset D_2$ such that $D_1 \subset \Omega$, $D_2 \cap \partial\Omega \neq \varnothing$, and there is a function G holomorphic in D_2 such that $G = g$ on D_1. Let W be the component of D_2 containing D_1. Choose a point $w \in D_2 \cap \partial W$ such that $G(w) \neq 0$. Now choose an open disc $U \subset D_2$ centered at w such that G has no zeros in U and choose an open disc $V \subset U \cap W$. Then $1/G + b$ is holomorphic in U and agrees with f on V. Since $w \neq z_o$, it follows that f is extendable from Ω', which contradicts the choice of f. Therefore g is non-extendable as claimed. ∎

3 A Generic Result

Let $\Omega \subset \mathbb{C}$ be a bounded domain that is regular; that is $\overline{\Omega}^o = \Omega$. This is equivalent to saying that, for every $w \in \partial\Omega$, there exists a sequence $w_n \in \overline{\Omega}^c$, $n = 1, 2, \dots$ converging to w. The space $H(\Omega)$ of functions holomorphic on Ω is endowed with the topology of uniform convergence on compacta. The space $A(\Omega)$ consists of all holomorphic functions $f \in H(\Omega)$ that extend continuously to $\overline{\Omega}$. This space is endowed with the supremum norm $\|f\|_\infty = \sup_{z\in\Omega} |f(z)|$. The space $A^\infty(\Omega)$ consists of all holomorphic functions $f \in H(\Omega)$ such that all derivatives $f^{(\ell)}, \ell = 0, 1, 2, \dots$ extend continuously to $\overline{\Omega}$. We continue to denote these extensions by $f^{(\ell)}$. If $f, g, \in A^\infty(\Omega)$, then

$$\operatorname{dist}(f, g) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{\|f^{(j)} - g^{(j)}\|_\infty}{1 + \|f^{(j)} - g^{(j)}\|_\infty}.$$

Let Y denote the set of restrictions to Ω of rational functions y that are holomorphic and injective on some neighbourhood V_y (depending on y) of the closure of Ω.

Let X be one of the following spaces:

(i) the closure of Y in $H(\Omega)$ endowed with the relative topology from $H(\Omega)$;
(ii) the closure of Y in $A(\Omega)$ endowed with the relative topology from $A(\Omega)$;
(iii) the closure of Y in $A^{\infty}(\Omega)$ endowed with the relative topology from $A^{\infty}(\Omega)$.

Note that, by Hurwitz's Theorem, all functions in X are either injective or constant. Hence, the non-extendable elements of X are injective.

Theorem 3.1 *Let X be one of the above mentioned spaces, where $\Omega \subset \mathbb{C}$ is a bounded regular domain. Then the set U of non-extendable elements of X is G_δ and dense in X; in particular, it is non-void, and so Ω is a domain of injective holomorphy.*

Proof Let $P = \Omega \cap (\mathbb{Q} + i\mathbb{Q})$. For $p \in P$ and $f \in X$ we denote by $R_p(f)$ the radius of convergence of the Taylor series of f with center p. Then, as in [5, Prop. 2.3], we have $U = \cap_{p \in P}\{f \in X : R_p(f) = \operatorname{dist}(p, \partial\Omega)\}$. Again, as in [5], we can easily see that U is a G_δ subset of X. We shall show that U contains a *dense* G_δ subset of X.

By Baire's Theorem, it suffices to fix a $p \in P$ and prove that the set

$$A = \{f \in X : R_p(f) = \operatorname{dist}(p, \partial\Omega)\}$$

contains a dense G_δ subset of X. Therefore, we fix $p \in P$ and let $w \in \partial\Omega$ be such that $|p-w| = \operatorname{dist}(p, \partial\Omega)$. By assumption, there exists a sequence $w_m \in \overline{\Omega}^c$, $m = 1, 2, \dots$ converging to w.

We consider the sets $E(n, k, m) = \{g \in X : |S_n(g, p)(w_m)| > k\}$, where

$$S_n(g, p)(z) = \sum_{\ell=0}^{n} \frac{g^{(\ell)}(p)}{\ell!}(z - p)^{\ell}.$$

Then it is easy to see that

$$A \supset \bigcap_{k,m} \bigcup_{n=0}^{\infty} E(n, k, m).$$

Further, by the Cauchy estimates each $E(n, k, m)$ is open in X, and so this intersection is indeed a G_δ.

The density of this G_δ in X will follow once we show that $\bigcup_{n=0}^{\infty} E(n, k, m)$ is dense in X, where k and m are fixed.

Let $\varphi \in Y$ and let $\overline{\Omega} \subset V \subset \overline{V} \subset V_1$, where V, V_1 are bounded and open and φ is holomorphic and injective in V_1. We may also assume that $w_m \notin V_1$ for the above fixed m. For q a positive integer strictly greater than all multiplicities of the (finitely many) poles of φ and for $\delta > 0$, set

$$\varphi_\delta(z) \equiv \varphi(z) + \frac{\delta}{(z - w_m)^q}.$$

As $\delta \to 0^+$, we also have that $\varphi_\delta \to \varphi$ uniformly on compact subsets of V_1. It follows from Rouché's Theorem that, for small $\delta > 0$, φ_δ is injective on $\overline{V}$ (and on V). Therefore, for $\delta > 0$ small enough, $\varphi_\delta \in Y \subset A^{\infty}(\Omega) \subset A(\Omega) \subset H(\Omega)$. Moreover,

$\varphi_\delta \to \varphi$, as $\delta \to 0^+$, in the topologies of $A^\infty(\Omega)$, $A(\Omega)$ and $H(\Omega)$. Hence, for $\delta > 0$, we have that $\varphi_\delta \in X$ and $\varphi_\delta \to \varphi$ in X as $\delta \to 0^+$.

Fix a neighbourhood of φ in X. From the preceding paragraph, it follows that we may choose $\delta > 0$ so that φ_δ is in this neighbourhood.

From the choice of q, we see that w_m is a pole of greatest multiplicity for φ_δ. If w_m is among the closest poles of φ_δ to the center p, then $\lim_n S_n(\varphi_\delta, p)(w_m) = \infty$, according to a result of Dienes [1]. Then we can choose n big enough so that $|S_n(\varphi_\delta, p)(w_m)| > k$.

If w_m is outside the closed disc of convergence with center p, then it is well known that $S_n(\varphi_\delta, p)(w_m), n = 0, 1, 2, \dots,$ is an unbounded sequence. Indeed, suppose this sequence is bounded, then the sequence

$$S_n(\varphi_\delta, p)(w_m) - S_{n-1}(\varphi_\delta, p)(w_m) = \frac{\varphi_\delta^{(n)}(p)}{n!}(w_m - p)^n, n = 0, 1, 2, \dots,$$

is also bounded. It follows from Abel's Lemma that $R_{\varphi_\delta}(p) \geq |w_m - p|$. But w_m lies outside the closed disc of convergence, that is absurd. Thus, $S_n(\varphi_\delta, p)(w_m), n = 0, 1, 2, \dots,$ is an unbounded sequence. Thus, again we can find $n \in \mathbb{N}$, so that $|S_n(\varphi_\delta, p)(w_m)| > k$.

This proves that $\bigcup_{n=0}^\infty E(n, k, m)$ is dense in X. It follows that U is a dense G_δ subset of the complete metrizable space X. Since $X \neq \varnothing$ (it contains the function $f(z) = z$), it follows that U is a non-void subset of X. Every element of X is either injective in Ω or constant. The constants are extendable, so they cannot belong to U. Hence, every element of U is injective. Since $U \neq \varnothing$ and the elements of U are injective and non-extendable, it follows that Ω is a domain of injective holomorphy. This completes the proof. ∎

We remark that obvious modifications allow one to prove an analogue of Theorem 3.1 for regular domains of the Riemann sphere. Moreover, an attentive examination of the proof of Theorem 3.1 allows one to prove a local version with few modifications of the proof. We shall state such a local version without proof.

An horocycle of a domain Ω is a pair (w, D), where w is a boundary point of Ω and D is an open disc in Ω having w on its boundary. We say that (w, D) is an horocycle of the domain Ω at the boundary point w. If f is a function holomorphic in Ω, (w, D) is a horocycle of Ω, V is an open disc containing the point w as well as the disc D, and F is a function holomorphic in V such that $F = f$ on D, then we say that the function F extends the function f across the horocycle (w, D). Moreover, we shall say that the function f is extendable across the horocycle (w, D) if there is such a V and such an F. Let us say that a boundary point w of Ω is a regular boundary point if it is also a boundary point of the exterior of Ω. If (w, D) is a horocycle at a regular boundary point w of Ω and w is the only boundary point of Ω on the boundary of D, we say that (w, D) is a regular horocycle of Ω.

Theorem 3.2 *Let X be one of the above mentioned spaces, where $\Omega \subset \mathbb{C}$ is a bounded domain and (w, D) a regular horocycle of Ω. Then the set A of elements of X that are not extendable across (w, D) is G_δ and dense in X. In particular, A is non-empty.*

For an arbitrary domain Ω in $\mathbb{C}$, let us define the regular horocyclic boundary of Ω to be the collection of all regular horocycles (w, D) of Ω. Let us say that a function f holomorphic in Ω is nowhere extendable across the regular horocyclic boundary if f is extendable across no regular horocycle (w, D) of Ω.

Corollary 3.3 *Let X be one of the above mentioned spaces, where $\Omega \subset \mathbb{C}$ is an arbitrary bounded domain in $\mathbb{C}$. Then the set U of elements of X that are nowhere extendable across the regular horocyclic boundary is G_δ and dense in X; in particular, it is non-void.*

Proof We can construct a countable family (w_j, D_j) of regular horocycles that is sufficiently "dense". That is, if a function f holomorphic in Ω is extendable across none of the regular horocycles (w_j, D_j), then it is extendable across no regular horocycle. By Theorem 3.2, for each j, the set A_j of elements of X that are not extendable across (w_j, D_j) is G_δ and dense in X. The corollary now follows from Baire's theorem. ■

For a regular domain Ω, the regular horocyclic boundary is sufficiently "dense" in the boundary $\partial\Omega$ so that Theorem 3.1 follows from Corollary 3.3. Thus, Theorem 3.1 is a consequence of Theorem 3.2. But, of course, the proof of Theorem 3.2 (that we have omitted) incorporates the main ingredients of the proof of Theorem 3.1.

4 Further Examples of Domains of Injective Holomorphy

It follows from Theorem 3.1 that every Jordan domain is a domain of injective holomorphy. The same holds for every domain that is bounded by a finite number of disjoint Jordan curves. Moreover, we can have unbounded examples as well.

Proposition 4.1 *Let $\Omega \subset \mathbb{C}$ be a regular domain. Then Ω is a domain of injective holomorphy.*

Proof If Ω is bounded, the result follows by Theorem 3.1.

If $\Omega = \mathbb{C}$, the result holds because of the function $f(z) = z$.

Suppose Ω is unbounded and $\Omega \neq \mathbb{C}$. As $\overline{\Omega}^o = \Omega$, it follows that $\overline{\Omega} \neq \mathbb{C}$ and therefore, there exists $w \in \mathbb{C}$ and $\delta > 0$ so that $\delta < \operatorname{dist}(w, \Omega)$. We set $\varphi(z) = 1/(z - w)$. Then $\varphi(\Omega)$ is bounded.

Consider first the case where $\mathbb{C} \setminus \Omega$ is unbounded. Then $\varphi(\Omega)$ is regular. So, according to Theorem 3.1, there exists $g \in H(\varphi(\Omega))$ injective and non-extendable. We consider the function $g \circ \varphi \in H(\Omega)$, which is injective. If $g \circ \varphi$ were extendable through a boundary point $Q \in \partial\Omega, Q \neq \infty$, then one could see that g would be extendable through the boundary point $\varphi(Q) = 1/(Q - w) \neq 0$ in $\partial(\varphi(\Omega))$, that would give a contradiction. This completes the proof in case $\mathbb{C} \setminus \Omega$ is unbounded.

If $\mathbb{C} \setminus \Omega$ is bounded, then $\varphi(\Omega) = \omega \setminus \{0\}$, where ω is a bounded regular domain. Thus, ω is a domain of injective holomorphy and no component of $\overline{\mathbb{C}} \setminus \omega$ is a singleton. It follows from Example 2.2, that $\varphi(\Omega) = \omega \setminus \{0\}$ is a domain of injective holomorphy as well. Let g be an injective holomorphic function on $\varphi(\Omega)$, which is non-extendable from $\varphi(\Omega)$. Then $g \circ \varphi$ is an injective holomorphic function on Ω, which is non-extendable. This completes the proof. ■

We shall see that there are many non-regular bounded domains that are domains of injective holomorphy, but for this we must look more closely at the boundary. To this end, it will be convenient to introduce the notion of a "dense" set of horocycles that applies to an arbitrary domain $\Omega \subset \mathbb{C}$, which is non-degenerate, that is, not empty nor all of $\mathbb{C}$. A holomorphic function f in Ω is holomorphically extendable over $\mathbb{C}$ if there are two open discs U and V with $U \subset \Omega \cap V$ and $V \cap \partial\Omega \neq \varnothing$, and there is a function F holomorphic in V such that $F = f$ in U. In this situation, let us say that the function f is extended through (or over) the pair of discs (U, V). There exists a countable family (U_j, V_j) of such pairs of discs that is sufficient for the above definition. That is, a function f holomorphic in Ω is holomorphically extendable over $\mathbb{C}$ if and only if f is extendable over the pair (U_j, V_j) for some j. That is, f is extendable if and only if for some j there is a function F holomorphic in V_j such that $F = f$ in U_j. Moreover, we may construct such a sequence of pairs having the property that for any $\epsilon > 0$, the family of pairs (U_j, V_j) for which the diameter of V_j is greater than ϵ is locally finite. Thus, we may consider that the sequence of pairs (U_j, V_j) approaches the boundary. Let us say that a boundary point p of Ω is *horocyclically* accessible if p lies on the boundary of an open disc D contained in Ω. We say that D is an horocycle at p, or we also say that (p, D) is an horocycle in Ω. Now, for each pair (U_j, V_j), we may choose an horocycle (p_j, D_j) such that D_j is in the component of $V_j \cap \Omega$ containing U_j. This family of horocycles (p_j, D_j) is sufficiently "dense" for our purposes. That is, if a holomorphic function F on V_j extends f from U_j, then F extends f from D_j. We say that F extends f over the horocycle (p_j, D_j). Thus, a holomorphic function f in Ω extends holomorphically over $\mathbb{C}$ if and only if it extends over one of the horocycles (p_j, D_j). Without bothering to formally introduce an appropriate topology, let us agree to call the sequence (p_j, D_j) a sequence of horocycles that approaches the boundary of Ω in a dense manner.

Theorem 4.2 *Let $\Omega \neq \mathbb{C}$ be a simply connected domain. There is a bounded injective holomorphic function on Ω that is non-extendable. Thus Ω is a domain of injective holomorphy.*

Proof Let $\varphi\colon \Omega \to \mathbb{D}$ be a conformal mapping of Ω onto the unit disc $\mathbb{D}$. If $w_1, w_2, \dots$ is an interpolating sequence in $\mathbb{D}$, we shall say that the sequence $z_n = \varphi^{-1}(w_n)$, $n = 1, 2, \dots$, is an interpolating sequence in Ω. We shall construct an interpolating sequence $\{z_n\}$ that approaches the boundary $\partial\Omega$ in a sufficiently dense manner. We shall interpolate along the sequence $\{w_n\}$ by a bounded function b so as to assign the values 0 and i to the sequence $\{b(w_n)\}$. Choose a real number $c > 0$ such that $\Re b + c > 0$ and set

$$h(w) = \int_0^w (b(\omega) + c)\,d\omega$$

for $w \in \mathbb{D}$. Then $h' = b + c$, so $\Re h' = \Re b + c > 0$, and so h is injective in $\mathbb{D}$. Thus, $f = h \circ \varphi$ is injective in Ω and $f'(z_n) = h'(w_n)\varphi'(z_n) = (b(w_n) + c)\varphi'(z_n)$. We shall choose the sequence $\{z_n\}$ so as to approach a sufficiently dense set of points $p \in \partial\Omega$, and we may assume that for each such p the corresponding subsequence of $\arg \varphi'(z_n)$ has a limit modulo 2π. The corresponding subsequence of $\arg f'(z_n)$ will take the two values $\arg \varphi'(z_{n(j)})$ and $\arg(i{+}c){+}\arg \varphi'(z_{n(k)})$ for different subsequences $z_{n(j)}$ and $z_{n(k)}$. If f' were extendable across p with $f'(p) \neq 0$, then the difference of

$\arg\varphi'(z_{n(j)})$ and $\arg(i+c)+\arg\varphi'(z_{n(k)})$ should converge to zero modulo 2π, as $z_{n(j)}$ and $z_{n(k)}$ converge to p. However, this converges to $\arg(i+c)$ modulo 2π, that is absurd. But even if $f'(p)$ were 0, there would be a nearby point, with the same extension, where $f'(q)\neq 0$. At such a nearby point, we would have the preceding contradiction. Thus, f' is non-extendable across p and consequently, the same is true of f. If the set of such p is sufficiently dense, then f will be nowhere extendable.

There remains the choice of an appropriate interpolating sequence $\{z_n\}$ in Ω. A function f in Ω is non-extendable if it is non-extendable through each point p of the boundary $\partial\Omega$. Of course, there are uncountably many boundary points, and the situation seems to be even more complicated, since a given boundary point can be approached from "different sides" and f should be non-extendable through p for all of these approaches. The number of different approaches to a given boundary point p could be uncountable. For example, consider the simply connected domain

$$\Omega=\big(\mathbb{C}\setminus[0,+\infty)\big)\setminus\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{2^n}[0,n^{-1}\exp(im\pi 2^{-n})],$$

where $[0,b]$ denotes the segment from 0 to a point b. Then, from the point of view of holomorphic continuation, there are uncountably many different approaches to the boundary point 0.

Fortunately, we need not consider all boundary points, nor need we consider all approaches to those boundary points that we do consider. Indeed, it is sufficient to consider horocyclic approach. Let us say that a holomorphic function f on Ω is extendable through the horocycle D at p belonging to the boundary of Ω, if there are open discs U and V with $U\subset D$, $U\subset V$, $p\in V$ and a holomorphic function F on V such that $F=f$ on $D\cap U$. A holomorphic function f on Ω is non-extendable if and only if, for each horocyclically accessible boundary point p and each horocycle D at p, f is non-extendable through D at p.

Let (p_j,D_j), $j=1,2,\dots$, be a sequence of horocycles that approaches the boundary in a dense manner in the sense defined above. Let α_j be the radius of D_j ending at p_j. Now, let $\{z_n\}$ be an interpolating sequence in Ω constructed by selecting distinct points successively on the radii:

$$\alpha_1;\ \alpha_1,\alpha_2;\ \alpha_1,\alpha_2,\alpha_3;\ \dots.$$

Now, as in the beginning of the proof, let b be a bounded holomorphic function in Ω taking the values 0 and i along the interpolating sequence $\{z_n\}$. Moreover, we shall be careful to assure that along each α_j both values 0 and i are assumed infinitely often. We may easily arrange so that for every $i=1,2,\dots$, for the indices n such that z_n belongs to α_i, the corresponding subsequence $\arg\phi'(z_n)$ belonging to $[0,2\pi)$ has a limit. Then f', constructed in the first paragraph of this proof, is non-extendable through each (p_j,D_j), $j=1,2,\dots$, unless perhaps $f'(p_{kj})=0$. But if $f'(p_j)$ were 0, there would be nearby points with the same extension where the derivative is not 0, and for such points we would have the non-extendability, which is a contradiction. Because of the way the horocycles (p_j,D_j), $j=1,2,\dots$, were chosen, the function f' is non-extendable over (U_j,V_j); $j=1,2,\dots$. From the way in that the sequence

(U_j, V_j) was defined, it follows that f' is non-extendable. The first paragraph of this proof has now been justified, and the proof is complete. ■

Corollary 4.3 *Let $\Omega \subset \overline{\mathbb{C}}$ be a simply connected domain containing ∞ and different from $\overline{\mathbb{C}}$. We set $\Omega' = \Omega \cap \mathbb{C}$. There is an injective holomorphic function on Ω', which is non-extendable and bounded and hence holomorphically extendable to the point ∞.*

This follows from Theorem 4.2, if we compose with a transformation of the form $w = 1/(z - z_o)$, where $z_o \notin \Omega$.

Let us say that an open set Ω in $\mathbb{C}$ is of injective holomorphy if it supports an injective holomorphic function whose restriction to every component of Ω is non-extendable. If an open set Ω is of injective holomorphy, then obviously every component of Ω is a domain of injective holomorphy. The converse does not hold, as we can see by taking as Ω the union of two disjoint punctured discs. However, a corollary of Theorem 4.2 is that every simply connected open set Ω included in $\mathbb{C}$ and different from $\mathbb{C}$ supports a bounded injective holomorphic function whose restriction to every component of Ω is non-extendable; in particular, Ω is of injective holomorphy.

Indeed if O_n are the components of the simply connected open set Ω, then Theorem 4.2 easily implies the existence of functions f_n on O_n injective, holomorphic, non-extendable, and satisfying $1 - 1/n < |f_n(z)| < 1 - 1/(n + 1)$. Then the functions f_n build a function f on Ω, that is injective, bounded, and holomorphic and whose restriction to each component of Ω is non extendable.

We could also define an open set Ω to be a set of injective *meromorphy* if it supports an injective meromorphic function whose restriction to every component of Ω is non-extendable. Again, if an open set Ω is of injective meromorphy, then obviously every component of Ω is a domain of injective meromorphy. However, we do not know whether the converse is true.

We remark that with the above construction of a sufficiently dense countable family of horocycles, it is easy to prove the well-known fact that every domain Ω of $\mathbb{C}$ is a domain of holomorphy over $\mathbb{C}$. We may assume $\Omega \neq \mathbb{C}$. For a sequence $\{z_n\}$ of distinct points in Ω that has no limit points in Ω and is frequently in each of these countably many horocycles, there exists, by the Weierstrass theorem, a holomorphic function f in Ω whose zeros are precisely the points of $\{z_n\}$. Such a function is non-extendable, because any extension would be identically zero, that is absurd. Replacing the Weierstrass theorem by the Mittag–Leffler theorem (that is more elementary), we have a meromorphic function in Ω whose poles are precisely the points of $\{z_n\}$. Since poles are always isolated, it follows that this function is meromorphically non-extendable over $\mathbb{C}$. Thus, every domain of $\mathbb{C}$ is also a domain of meromorphy over $\mathbb{C}$. In fact, it is known that domains of holomorphy and domains of meromorphy over $\mathbb{C}^n$ coincide ([2]). However, the proof of Proposition 2.1 shows that domains of *injective* holomorphy are not the same as domains of injective meromorphy. More precisely, $\mathbb{D} \setminus \{0\}$, where $\mathbb{D}$ denotes the open unit disc, is not a domain of injective meromorphy, but it is a domain of injective holomorphy.

Combining previous results we have the following theorem.

Theorem 4.4 *Let Ω be a domain in $\mathbb{C}$ having finitely many complementary components in $\overline{\mathbb{C}}$. If at least two of these are singletons in $\mathbb{C}$, then Ω is not a domain of injective holomorphy. Otherwise, Ω is a domain of injective holomorphy.*

Proof Let $K_1, K_2, \ldots, K_n$ be the complementary components of Ω in the Riemann sphere $\overline{\mathbb{C}}$, where K_n is the component containing ∞. Suppose two components are singletons in $\mathbb{C}$. Then, by Proposition 2.1, Ω is not a domain of injective holomorphy.

Now suppose at most one of the components is a singleton in $\mathbb{C}$. We claim that Ω is a domain of injective holomorphy.

If $n = 1$, then $\Omega = \overline{\mathbb{C}} \setminus K_1$ is simply connected, and so by Theorem 4.2, Ω is a domain of injective holomorphy.

If $n > 1$, consider the case that there is indeed a complementary component in $\overline{\mathbb{C}}$ that is a singleton in $\mathbb{C}$, say $K_1 = \{z_o\}$. Then, by Example 2.2, Ω is a domain of injective holomorphy if and only if $\Omega \cup \{z_o\}$ is a domain of injective holomorphy.

We may thus assume that no $K_1, K_2, \ldots, K_{n-1}$ is a singleton. We claim that we may also assume that K_n is not a singleton. Indeed, suppose K_n is a singleton, that is $K_n = \{\infty\}$, and suppose we knew that if a domain W in $\mathbb{C}$ has only finitely many complementary components in $\overline{\mathbb{C}}$ and none of them are singletons, then W is a domain of injective holomorphy. Set

$$\Omega_\infty = \overline{\mathbb{C}} \setminus \bigcup_{j=1}^{n-1} K_j,$$

choose a point $a \in K_1$, and define $\varphi(z) = 1/(z-a)$. Set $W = \varphi(\Omega_\infty)$. By hypothesis, W is a domain of injective holomorphy, and by Example 2.2, there is an injective holomorphic function g on $W_o = W \setminus \{0\}$, which is non-extendable. Thus, $g \circ \varphi$ is an injective holomorphic function on Ω, which is non-extendable.

We thus suppose that no K_j is a singleton. The proof is now by induction on n and the case $n = 1$ has already been established.

Suppose $n = 2$. Set $\Omega_1 = \overline{\mathbb{C}} \setminus K_1$. By Corollary 4.3 there is a bounded injective holomorphic function f_1 on Ω_1 that is non-extendable. Set $\Omega_2 = \overline{\mathbb{C}} \setminus f_1(K_2)$. Again, by Corollary 4.3, there is an injective holomorphic function f_2 on Ω_2 that is non-extendable. We claim that the holomorphic injective function $f = f_2 \circ f_1$ on Ω is non-extendable. Indeed, suppose, to obtain a contradiction, that there are two open discs $U \subset V$, with $U \subset \Omega$ and $V \not\subset \Omega$ and a function F holomorphic on V such that $F = f$ on U. Let W be the component of $V \cap \Omega$ containing U and choose a point $p \in \partial W \cap \partial\Omega$. Either $p \in K_1$ or $p \in K_2$.

Suppose first that $p \in K_1$. We may choose a disc $D \subset V$ centered at p so small that

$$\overline{f_1(D \cap \Omega)} \cap f_1(K_2) = \varnothing.$$

Hence,

$$\overline{(f_2 \circ f_1)(D \cap \Omega)} \cap \left[\overline{\mathbb{C}} \setminus f_2\{\overline{\mathbb{C}} \setminus f_1(K_2)\}\right] = \varnothing.$$

Thus, $F(p) \in f_2\{\overline{\mathbb{C}} \setminus f_1(K_2)\} \equiv f_2(\Omega_2)$. We may define $F_1 = f_2^{-1} \circ F$ near p. Then F_1 is an extension of f_1, which contradicts the choice of f_1.

If, on the other hand, $p \in K_2$, choose a disc D_2 centered at $f_1(p)$ so small that $f_1^{-1}(D_2) \subset V$ and choose a disc $D_1 \subset D_2 \cap f_1(W)$. On D_2, the holomorphic function $F_2 = F \circ f_1^{-1}$ is well defined and it coincides with f_2 on D_1. Thus f_2 is extendable, which contradicts the choice of f_2. This completes the proof that f is non-extendable in Ω and establishes the theorem for $n = 2$.

Suppose the theorem is valid if Ω has $n-1$ complementary components in $\overline{\mathbb{C}}$, none of that is a singleton, and suppose Ω has n complementary components, none of that is a singleton.

Set $\Omega_1 = \overline{\mathbb{C}} \setminus K_1$. By Corollary 4.3 there is an injective holomorphic function f_1 on Ω_1 that is non-extendable. Set

$$\Omega_2 = \overline{\mathbb{C}} \setminus f_1(\cup_{j=2}^n K_j).$$

By the inductive hypothesis, there is an injective holomorphic function f_2 on Ω_2 that is non-extendable. As for the case $n = 2$, one can show that the holomorphic injective function $f = f_2 \circ f_1$ on Ω is non-extendable. ∎

We remark that the situation for meromorphy is more elegant. Let us say that a finitely-connected domain in $\mathbb{C}$ is non-degenerate if no complementary component is a singleton. Then a finitely connected domain in $\mathbb{C}$ is a domain of injective meromorphy if and only if it is non-degenerate.

Corollary 4.5 *Let $\Omega \subset \mathbb{C}$ be a doubly connected domain. Then Ω is a domain of injective holomorphy.*

Recall that a continuum (compact connected set) is said to be *degenerate* if it is a singleton.

Example 4.6 Let Ω be a domain whose complementary components are bounded, isolated, and non-degenerate. Then Ω is a domain of injective holomorphy.

Proof We may assume that Ω contains the origin. We shall construct a sequence f_n of conformal mappings of Ω such that $f_n(0) = 0$ and $f'(0) = 1$, for each n.

First, we arrange the complementary components of Ω in a sequence K_n. Set $\Omega_n = \overline{\mathbb{C}} \setminus \cup_{j=1}^n K_j$. Let f_0 be the identity mapping $f_0(z) = z$. Let h_1 be an injective holomorphic function on $\overline{\mathbb{C}} \setminus K_1$ that is non-extendable and such that $h_1(0) = 0$ and $h_1'(0) = 1$. Set $f_1 = h_1$. Suppose, for $j = 1, \dots, n$, the functions h_j and f_j have been defined such that h_j is an injective holomorphic function on $\overline{\mathbb{C}} \setminus f_{j-1}(K_j)$ that is non-extendable,

$$f_j = h_j \circ f_{j-1}, \quad f_j(0) = h_j(0) = 0, \quad \text{and} \quad f_j'(0) = h_j'(0) = 1.$$

We let h_{n+1} be an injective holomorphic function on $\overline{\mathbb{C}} \setminus f_n(K_{n+1})$ such that $h_{n+1}(0) = 0$ and $h_{n+1}'(0) = 1$ and we set $f_{n+1} = h_{n+1} \circ f_n$. Then the sequences h_n and f_n are defined by induction. For each n, the function f_n is an injective holomorphic function on Ω_n that is non-extendable.

By the Koebe distortion theorem, the family $\{f_n\}$ is normal on $\Omega = \mathbb{C} \setminus \cup_j K_j$, and so there is a subsequence of the sequence $\{f_n\}$ that converges to a function f on Ω that is either identically infinite or holomorphic. Since $f(0) = 0$, the function f is holomorphic. Since each f_n is injective and holomorphic, the limit function f is either injective or constant, but since $f'(0) = 1$, the function f is injective.

We shall show that f is non-extendable. Suppose, for the sake of contradiction, that f extends to a point $p \in \partial\Omega$. Then $p \in \partial K_n$ for some n. For $m > n$ we may write

$$f_m = h_m \circ h_{m-1} \circ \cdots \circ h_{n+1} \circ f_n = g_n \circ f_n,$$

where g_n is injective holomorphic on $\mathbb{C} \setminus \bigcup_{j=n+1}^{m} f_n(K_j)$. Recall that a subsequence of $\{f_m\}$ converges to f. By the same argument invoking the Koebe distortion theorem, the corresponding subsequence of g_m has a subsequence that converges to a function g, that is injective holomorphic on

$$W_n = \mathbb{C} \setminus \Big\{ f_n(\infty) \cup \bigcup_{j=n+1}^{\infty} f_n(K_j) \Big\}.$$

Hence, there is a sequence of indices n_k such that f_{n_k} converges to f on Ω and g_{n_k} converges to g on W_n. We may write

$$f = \lim_{k\to\infty} f_{n_k} = \lim_{k\to\infty} h_{n_k} \circ \cdots \circ h_{n+1} \circ f_n = \lim_{k\to\infty} g_{n_k} = g \circ f_n.$$

Let σ_n be a Jordan curve in Ω that separates K_n from the other K_m. Let U_n be the topological annulus bounded by σ_n and ∂K_n. Now $f_n(U_n)$ is a topological annulus on $\overline{\mathbb{C}}$ whose boundary components are the Jordan curve $f_n(\sigma_n)$ and ∂Q_n, where Q_n is the component of $\overline{\mathbb{C}} \setminus f_n(U_n)$ that is disjoint from $f_n(\sigma_n)$. Now g is holomorphic and injective in a neighborhood W of Q_n. There is a neighborhood G of K_n such that $f(G \cap \Omega) \subset g(W)$. Choose a disc D centered at p lying in G such that f extends to D. Then, in $D \cap \Omega$, we have $f_n = g^{-1} \circ f$. But f extends to D, and g^{-1} is defined on $f(D)$. Therefore, f_n extends to D, which is the desired contradiction. ■

An instance of a domain as in Example 4.6 would be the multiply slit plane

$$\mathbb{C} \setminus \bigcup_{-\infty}^{+\infty} [2n, 2n+1].$$

Let Ω be a domain of $\mathbb{C}$ and let K_j be a sequence of bounded non-degenerate complementary components of Ω that are isolated in the sense that for each j there is an open neighborhood G_j of K_j such that

$$(\mathbb{C} \setminus \Omega) \cap G_j = K_j.$$

Then the proof of Example 4.6 shows that there exists an injective holomorphic function on Ω that extends to no boundary point of Ω lying on any K_j. It follows that, if for each open disc D each non-empty component of $D \cap \Omega$ meets some K_j, then Ω is a domain of injective holomorphy. For example, let E be the Cantor set and let a_j be a sequence in $\mathbb{R} \setminus E$ that in each complementary interval (bounded or not) accumulates precisely to the (finite) end points. Let

$$K_j = \{a_j\} \times [0,1] \quad \text{and} \quad K = \Big(E \cup \bigcup_{j=1}^{\infty} \{a_j\} \Big) \times [0,1].$$

Then $\Omega = \mathbb{C} \setminus K$ is a domain of injective holomorphy whose complement K has uncountably many non-degenerate components.

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