Multi-scale (time and mass) dynamics of space objects
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Back-tracing space debris using proper elements

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Abstract. Normal form methods allow one to compute quasi-invariants of a Hamiltonian system, which are referred to as proper elements. The computation of the proper elements turns out to be useful to associate dynamical properties that lead to identify families of space debris, as it was done in the past for families of asteroids. In particular, through proper elements we are able to group fragments generated by the same break-up event and we possibly associate them to a parent body. A qualitative analysis of the results is given by the computation of the Pearson correlation coefficient and the probability of the Kolmogorov-Smirnov statistical test.

Keywords. Proper elements, Normal form, Space debris, Geopotential, Sun and Moon attractions, Solar radiation pressure, Statistical data analysis.

1. Introduction

The dynamics around the Earth can be classified taking into account the forces involved, which depend on the distance from the Earth's surface. To this end, we recall the three main regions above the Earth surface: Low-Earth-Orbits (LEO), Medium-Earth-Orbits (MEO) and Geosynchronous-Earth-Orbits (GEO). In all these regions, the motion is mainly governed by the gravitational field of the Earth, but other forces might influence the long term evolution of a spatial object, according to its altitude. Such forces include, in particular, the non-spherical shape of the Earth, the attraction of the Moon and the Sun, the solar radiation pressure. Once the Hamiltonian model is given, we implement a Lie series normalization procedure to compute quasi-invariants of the motion, namely the so-called proper elements, which are associated to semi-major axis, eccentricity and inclination. Using simulated data after a break-up event (a collision or an explosion), we analyzed the connection of the computation of the proper elements with the dynamics observed immediately after the catastrophic event. The results are corroborated by a statistical data analysis based on the check of the Kolmogorov-Smirnov test and the computation of the Pearson correlation coefficient. We refer to Celletti, Pucacco, & Vartolomei (2021a) for full details on the procedure and to Celletti, Pucacco, & Vartolomei (2021b) for an application to concrete cases.

2. The Model

A model of dynamics for space debris has been developed taking into account four main forces that act on a satellite or a space debris, namely the gravitational potential of the Earth, the attraction of the Moon and Sun, and the Solar radiation pressure. The entire model has been described in Celletti *et al.* (2017a) starting from the Cartesian equations of motion and using the expansion in the orbital elements. The Hamiltonian function has the following form

$$\mathscr{H} = \mathscr{H}_{Kep} + \mathscr{H}_E + \mathscr{H}_M + \mathscr{H}_S + \mathscr{H}_{SRP}\ ,$$

where $\mathscr{H}_{Kep}$ is given by

$$\mathscr{H}_{Kep} = -\frac{GM_E}{2a},$$

where G is the gravitational constant, M_E is the mass of the Earth and a denotes the semi-major axis.

Following Kaula (1966), the Hamiltonian part corresponding to the Earth's perturbation can be written as an expansion in orbital elements of the space object. In the quasi-inertial reference frame, the Hamiltonian function can be written as

$$\begin{aligned}\mathscr{H}_E &= GM_E\frac{R_E^2}{a^3}J_2\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)\frac{1}{(1-e^2)^{3/2}}\\ &+ GM_E\frac{R_E^3}{a^4}J_3\left(\frac{15}{8}\sin^3 i - \frac{3}{2}\sin i\right) e\sin\omega\frac{1}{(1-e^2)^{5/2}}.\end{aligned}$$

The perturbation of the space object due to the Moon and Sun attractions can be written as an expansion in orbital elements of the 3rd body perturber and the debris, using the following formula (see Kaula (1962)):

$$\begin{aligned}\mathscr{H}_M = &-Gm_M\sum_{l\geq 2}\sum_{m=0}^{l}\sum_{p=0}^{l}\sum_{s=0}^{l}\sum_{q=0}^{l}\sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}(-1)^{m+s}(-1)^{[m/2]}\frac{\varepsilon_m\varepsilon_s}{2a_M}\frac{(l-s)!}{(l+m)!}\left(\frac{a}{a_M}\right)^l\\ &F_{lmp}(i)F_{lsq}(i_M)H_{lpj}(e)G_{lqr}(e_M)\{(-1)^{t(m+s-1)+1}U_l^{m,-s}\cos(\phi_{lmpj}+\phi'_{lsqr}-y_s\pi)\\ &+(-1)^{t(m+s)}U_l^{m,-s}\cos(\phi_{lmpj}-\phi'_{lsqr}-y_s\pi)\}\ ,\end{aligned}$$

where m_M is the mass of the Moon, $y_s = 0$, if (s mod 2)=0, $y_s = \frac{1}{2}$, if (s mod 2)= 1, $t = (l-1)$ mod 2, and

$$\varepsilon_m = \begin{cases}1, & m=0\\ 2, & m\in\mathbb{Z}\backslash\{0\}\end{cases}$$

$$\phi_{lmpj} = (l-2p)\omega + (l-2p+j)M + m\Omega$$

$$\phi'_{lsqr} = (l-2q)\omega_M + (l-2q+r)M_M + s\left(\Omega_M - \frac{\pi}{2}\right)\ .$$

The functions $F_{lmp}(i)$, $F_{lsq}(i_M)$ and $G_{lqr}(e_M)$ have been introduced in Kaula (1962), Celletti *et al.* (2017b); $H_{lpj}(e)$ are the Hansen coefficients, while the terms $U_l^{m,s}$ are given by

$$U_l^{m,s} = \sum_{r=\max(0,-(m+s))}^{\min(l-s,l-m)}(-1)^{l-m-r}\binom{l+m}{m+s+r}\binom{l-m}{r}\cos^{m+s+2r}\left(\frac{\varepsilon}{2}\right)\sin^{-m-s+2(l-r)}\left(\frac{\varepsilon}{2}\right),$$

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where $\varepsilon = 23^o 26' 21.406''$ is the Earth's obliquity. In applications, the expansion of the Moon's Hamiltonian will be truncated to $l = 2$ and averaged over the mean anomalies of the object M and of the Moon M_M.

The Hamiltonian due to the Sun depends on the orbital elements of the Sun and the debris. The expansion of $\mathscr{H}_S$ is given below and, again, we will consider the expansion to $l = 2$, averaging over the mean anomalies of the object and perturber body:

$$\mathscr{H}_S = -Gm_S \sum_{l\geq 2} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{h=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{a^l}{a_S^{l+1}} \varepsilon_m \frac{(l-m)!}{(l+m)!} F_{lmp}(i) F_{lmh}(i_S) H_{lpq}(e) \\ G_{lhj}(e_S) \cos(\phi_{lmphqj}),$$

where m_S is the mass of the Sun and

$$\phi_{lmphqj} = (l-2p)\omega + (l-2p+q)M - (l-2h)\omega_S - (l-2h+j)M_S + m(\Omega - \Omega_S).$$

The contribution to the Hamiltonian due to Solar radiation pressure is given below:

$$\mathscr{H}_{SRP} = C_r P_r \frac{A}{m} a_S^2 \sum_{l=1}^{1} \sum_{s=0}^{l} \sum_{p=0}^{l} \sum_{h=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{a^l}{a_S^{l+1}} \varepsilon_s \frac{(l-s)!}{(l+s)!} F_{lsp}(i) F_{lsh}(i_S) H_{lpq}(e) \\ G_{lhj}(e_S) \cos(\phi_{lsphqj}) \ .$$

where C_r, P_r are, respectively, the reflectivity coefficient and the radiation pressure for an object located at $a_S = 1AU$, while $\frac{A}{m}$ denotes the area-to-mass ratio. This function is averaged over the mean anomaly of the object M, but it depends on the mean anomaly of the sun M_S.

3. Normal Forms

We construct a normal form, which consists of a procedure implementing iteratively canonical changes of coordinates, in such a way that the Hamiltonian function is transformed into a given form. In particular, we will require that the Hamiltonian is integrable up to a remainder term. Each time we implement the iterative procedure, we decrease the size of the norm of the remainder term; however, it is well known that such iteration is in general not converging Poincare (1892); besides, the complexity of the computation usually grows when increasing the normalization steps.

Proper elements are computed through normal form theory Efthymiopoulos (2011), which we shortly summarize as follows. Consider the following Hamiltonian function

$$\mathscr{H}(\underline{I}, \underline{\varphi}) = \mathscr{H}_0(\underline{I}) + \varepsilon \mathscr{H}_1(\underline{I}, \underline{\varphi}) \ , \tag{3.1}$$

where $(\underline{I}, \underline{\varphi})$ denote action-angle variables with $(\underline{I}, \underline{\varphi}) \in B \times \mathbb{T}^n$, where n is the number of degrees of freedom and $B \subset \mathbb{R}^n$ denotes an open set. The function $\mathscr{H}_0(\underline{I})$ appearing in (3.1) is called the integrable part, $\varepsilon \in \mathbb{R}$ is a small parameter, $\mathscr{H}_1(\underline{I}, \underline{\varphi})$ is called the perturbing function.

As we mentioned before, we implement a change of coordinates which transforms the Hamiltonian to remove the perturbation to orders of ε^2. Usually, such normalization procedure can only be iterated for some steps, after which it starts to diverge Poincare (1892).

Let the function $\mathscr{H}_1$ be expanded in Fourier series as

$$\mathscr{H}_1(\underline{I}, \underline{\varphi}) = \sum_{\underline{k} \in K} b_{\underline{k}}(\underline{I}) \exp(i\underline{k} \cdot \underline{\varphi}) \ ,$$

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where $K \subseteq \mathbb{Z}^n$ and $b_{\underline{k}}$ denote functions with real coefficients. We call χ the generating function associated to the canonical transformation from the original coordinates $(\underline{I}, \underline{\varphi})$ to the new coordinates $(\underline{I}', \underline{\varphi}')$:

$$\underline{I} = S_\chi^\varepsilon \underline{I}' \ , \qquad \underline{\varphi} = S_\chi^\varepsilon \underline{\varphi}' \ ,$$

where S_χ^ε acts on a function $\mathscr{F}$ as

$$S_\chi^\varepsilon \mathscr{F} := \mathscr{F} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \{\{...\{\mathscr{F}, \chi\}, \dots\}, \chi\}$$

with $\{\cdot,\cdot\}$ representing the Poisson bracket operator. To compute S_χ^ε, we assume that the new Hamiltonian $\mathscr{H}^{(1)} = S_\chi^\varepsilon \mathscr{H}$ takes the form

$$\mathscr{H}^{(1)}(\underline{I}', \underline{\varphi}') = Z_1(\underline{I}') + \varepsilon^2 \mathscr{H}_2(\underline{I}', \underline{\varphi}') \ , \tag{3.2}$$

where $Z_1 = \mathscr{H}_0 + \varepsilon \overline{\mathscr{H}}_1$ is the new integrable part (the bar denotes the average over the angles), while $\mathscr{H}_2$ is the new remainder function which is of order ε^2. Inserting the transformation of coordinates in (3.1), the new Hamiltonian takes the desired form (3.2), if χ satisfies the following normal form equation:

$$\mathscr{H}_1(\underline{I}', \underline{\varphi}') + \{\mathscr{H}_0(\underline{I}'), \chi(\underline{I}', \underline{\varphi}')\} = \overline{\mathscr{H}}_1(\underline{I}').$$

Let us expand χ in Fourier series, let the frequency be $\underline{\omega}_0 = \frac{\partial \mathscr{H}_0}{\partial \underline{I}'}$, then the generating function is given by

$$\chi(\underline{I}', \underline{\varphi}') = -i \sum_{\underline{k} \in \mathbb{Z}^n} \frac{b_{\underline{k}}(\underline{I}')}{\underline{k} \cdot \underline{\omega}_0} \ \exp(i \underline{k} \cdot \underline{\varphi}') \ ,$$

provided the following non-resonance condition is satisfied: $\underline{k} \cdot \underline{\omega}_0 \neq 0$. Normal forms of higher order are obtained by iterating the procedure described above.

4. Results

4.1. *Simulator of break-up events*

Using a simulator of break-up events developed within the ongoing collaboration in Apetrii, Celletti, Efthymiopoulos, Galeş, Vartolomei (2021), we produce synthetic data in order to show the effectiveness of the proper elements. This simulator reproduces the break-up model Evolve 4.0 provided by NASA (see Johnson (2001), Klinkrad (2006)) and allows us to determine the cross-sections, masses, and imparted velocities of the fragments after an explosion or a collision. Our procedure consists in the following steps:
(i) simulate a break-up event and obtain the Cartesian coordinates for all generated fragments;
(ii) compute the orbital elements for each fragment;
(iii) using the orbital elements after the break-up, we propagate each fragment for a given period of time, typically up to 150 years;
(iv) we use the position of the fragments after the propagation up to a given interval of time (e.g., 150 years) to compute the proper elements of each fragment;
(v) the distribution of the proper elements is then compared to that of the elements at the initial time after the break-up.

4.2. *Proper Elements*

We take an example concerning an explosion that generates 465 fragments. The event occurs at a medium altitude of the parent body with $a = 25200$ km and with a relatively

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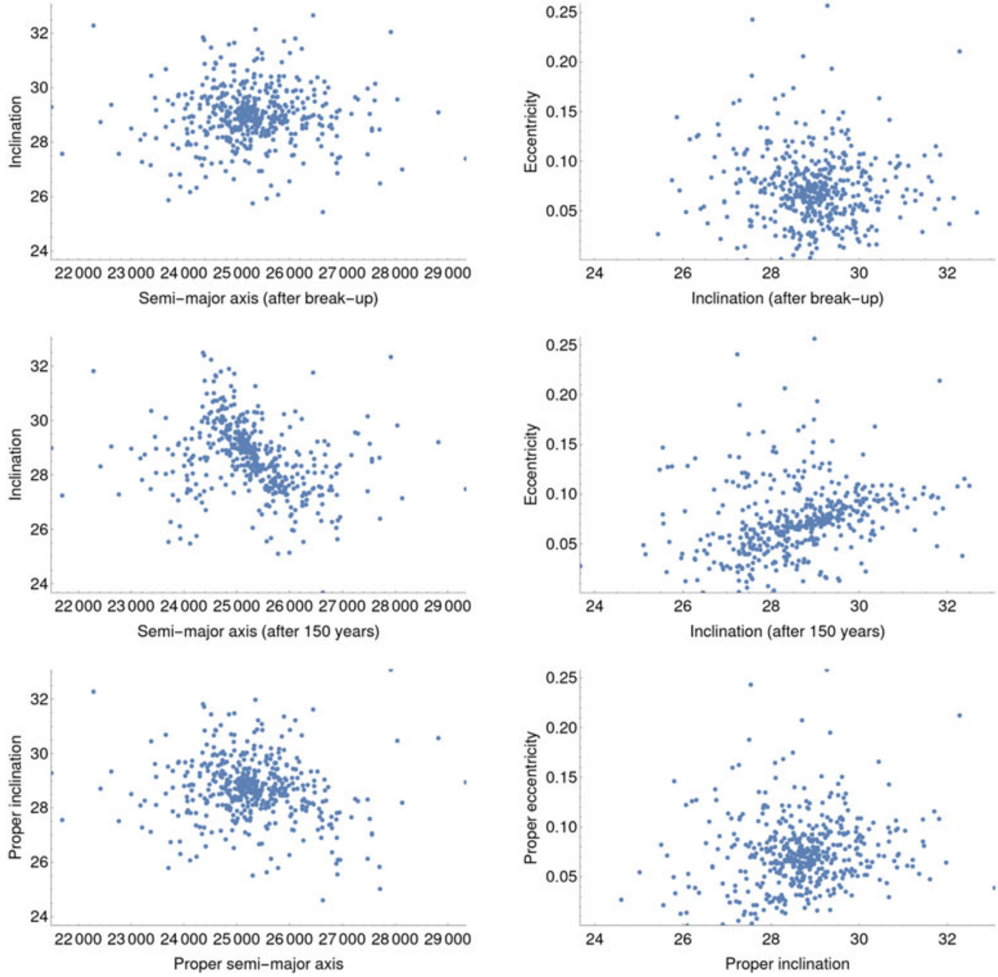

Figure 1. Distribution of initial osculating elements (first row), mean elements after 150 years (second row), and proper elements computed from mean elements after 150 years (third row) in the a-i plane (left), i-e plane (right). Initial osculating elements of the parent body: $a = 25200$ km, $e = 0.07$, $i = 29^o$, $\omega = 40^o$, $\Omega = 100^o$.

small inclination and eccentricity, $e = 0.07$, $i = 29^o$; the other elements are fixed as $\omega = 40^o$ and $\Omega = 100^o$.

Figure 1 shows the osculating elements after break-up (first row), mean elements after 150 years (second row), and proper elements computed after 150 years (third row) in the planes a-i and i-e. The scales have been fixed as the minimum and the maximum values of the evolution of the elements after 150 years.

While the distribution of the fragments in the mean elements is different from that in osculating elements, the first and third rows of Figure 1 show a connection between fragments at the break-up event and the distribution of the proper elements.

4.3. *Statistical analysis*

We compare the distributions of semimajor axis, eccentricity and inclination at break-up, after 150 years, and by computing the proper elements after 150 years, by implementing some statistical methods for data analysis Cowan (1997). Two of these

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methods are Kolmogorov-Smirnov (K-S) test that compares two distributions, and the Pearson correlation coefficient between the datasets.

As an example, we take the case of moderate orbits presented in Figure 1 and we implement the above methods to analyze the data. Since the semi-major axis is always constant, we are interested just in the analysis of eccentricity and inclination.

The K-S test for inclination gives a small probability, the so-called p-value, equal to 0.365534 when checking the similarity between the initial dataset and the mean elements after 150 years, while it gives a higher p-value equal to 0.968287 when looking at the initial data and the proper elements.

Pearson correlation coefficient provides evidence of the difference in inclination between the initial data and the data after 150 years, where its value turns out to be equal to 0.811243. Instead, a higher coefficient equal to 0.945672 is obtained when comparing the initial and proper elements.

5. Conclusions

In the present work we test the computation of the proper elements in the space debris problem. The Hamiltonian formulation of the model was used to describe the dynamics taking into account several perturbations: the potential of the Earth, the attraction of the Moon and the Sun, and the Solar radiation pressure. Using a break-up simulator, we analyzed the connection between proper elements and the initial osculating elements. In view of possible applications, we foresee several ways to improve the results, in primis the study of a more elaborated model including a larger number of spherical harmonics and a higher order expansion of the Hamiltonian.

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