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EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION TO A RAPIDLY GROWING PROBLEM VIA SUB-SUPERSOLUTION METHOD

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Abstract In this paper, we study the validity of the sub-supersolution method for the equation

$$\begin{cases} -\mathrm{div}(K(x)\nabla u) = K(x)|x|^{\alpha-2}f(x,u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $K(x) = exp(|x|^{\alpha}/4)$, $\alpha \geq 2$ and f is a continuous function, with hypotheses that will be given later. We apply the method to cases where f is singular, where f behaves like a logistic function, showing in both cases the existence and uniqueness of a positive solution.

Keywords: sub-supersolution method; singular elliptic problem; uniqueness

2010 Mathematics subject classification: Primary 35J15;
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1. Introduction

In this paper, we consider a general quasilinear elliptic problem given by

$$\begin{cases} -\Delta u - \dfrac{\alpha}{4}|x|^{\alpha-2}(x \cdot \nabla u) = |x|^{\alpha-2}f(x,u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $N \geq 3$, $\alpha \geq 2$ and f is a continuous function on $\mathbb{R}^N \times (0, \infty)$.

As shown by Haraux and Weissler in [14], the operator that appears in problem (1.1) with $\alpha = 2$ arises when we seek self-similar solutions, that is, solutions of the form $\omega(t, x) = t^{-1/(p-2)} u(t^{-1/2}x)$ for the parabolic problem

$$\omega_t - \Delta\omega = |\omega|^{p-2}\omega \text{ on } (0, \infty) \times \mathbb{R}^N. \tag{1.2}$$

In such a case, ω satisfies (1.2) whenever u satisfies

$$-\Delta u - (x \cdot \nabla u) = f(x, u) \text{ in } \mathbb{R}^N$$

with $f(x, u) = |u|^{p-2}u + \frac{1}{(p-2)}u$.

If we set $K(x) = exp(|x|^\alpha/4)$, a simple computation demonstrates that problem (1.1) is equivalent to

$$\begin{cases} -\text{div}(K(x)\nabla u) = K(x)|x|^{\alpha-2} f(x, u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Haraux and Weissler, in [14], examined problem (1.2) with the aim of establishing nonuniqueness results for the Cauchy problem associated with (1.2) in the case $N = 1$.

Escobedo and Kavian [4] investigated variational problems related to the existence of self-similar solutions of (1.1) with $\alpha = 2$. To achieve this, they proved compactness results for the embedding $H^1(K_\theta) \subset L^2(K_\theta)$, where $H^1(K_\theta)$ and $L^2(K_\theta)$ are the weighted Sobolev and Lebesgue spaces, respectively, with weight given by $K_\theta(x) = \exp(\theta(x))$, with $\theta \in C^2(\mathbb{R}^N, \mathbb{R}_+)$.

A more general formulation of this class of problems was studied by Catrina, Furtado and Montenegro in [3]. They established the existence of positive, rapidly decaying solutions for the equation

$$-\text{div}(K(x)\nabla u) = K(x)u^{2^*-1} + \lambda K(x)|x|^{\alpha-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$, $K(x) = \exp\left(\frac{1}{4}|x|^\alpha\right)$, $\alpha \geq 2$ and λ is a parameter.

Problems with nonlinearity of the concave-convex type for this class of problems were studied by [1, 8, 9, 16, 17].

Further works addressing the existence and multiplicity of positive or nodal solutions for this class of problems can be found in [5–7, 10–12, 13, 15] and the references therein.

In order to show the existence and uniqueness of a solution to problem (1.3), we define the space X as the completion of the space of smooth, compactly supported functions $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} K(x)|\nabla u|^2 \, dx.$$

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As quoted in [3], X is a Banach space and the weighted Lebesgue space

$$L^s_K(\mathbb{R}^N) = \left\{ u \text{ measurable in } \mathbb{R}^N : \|u\|_{s,K} = \left(\int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}|u|^s dx \right)^{1/s} < \infty \right\}$$

is such that the embedding $X \hookrightarrow L^s_K(\mathbb{R}^N)$ is continuous for $s \in [2, 2^*]$ and compact for $s \in [2, 2^*)$.

We say that $u \in X$ is a positive weak solution of problem (1.3) if $u > 0$ in $\mathbb{R}^N$ and satisfies

$$\int_{\mathbb{R}^N} K(x)\nabla u \cdot \nabla \phi \ dx - \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2} f(x, u)\phi dx = 0, \qquad \text{for all } \phi \in X.$$

Next, we define the concept of a sub-supersolution pair for (1.3), which we will later use to prove the existence of a weak solution of (1.3).

Definition 1.1. *We say that the pair* $(\underline{u}, \overline{u})$ *is a sub-supersolution for problem* (1.3)*, respectively, if* $\underline{u}, \overline{u} \in X$ *with*

(1) $0 < \underline{u} \leq \overline{u}$ *in* $\mathbb{R}^N$*,*
(2) For each $\phi \in X$ *with* $\phi \geq 0$*,*

$$\int_{\mathbb{R}^N} K(x)\nabla \underline{u} \cdot \nabla \phi \ dx \leq \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2} f(x, \underline{u})\phi \ dx$$

and

$$\int_{\mathbb{R}^N} K(x)\nabla \overline{u} \cdot \nabla \phi \ dx \geq \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2} f(x, \overline{u})\phi \ dx.$$

Now, we present our main results.

Theorem 1.1 *Assume that there exists a pair of sub-supersolution* $(\underline{u}, \overline{u})$ *of* (1.3) *and*

$$|f(x, u)| \leq G(x) \in L^2_K(\mathbb{R}^N) \quad \forall u \in [\underline{u}, \overline{u}]. \tag{1.4}$$

Then, (1.3) *has at least one positive weak solution* $u \in X \bigcap C^2(\mathbb{R}^N)$ *satisfying*

$$0 < \underline{u}(x) \leq u(x) \leq \overline{u}(x), \ a.e \ in \ \mathbb{R}^N.$$

As an application of the previous result, we present two interesting problems. First, we consider the following problem

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)|x|^{\alpha-2} h(x) u^{-\gamma} \text{in } \mathbb{R}^N, \tag{1.5}$$

where $\gamma > -1$ and h satisfies

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(h) There exist $0 < C_1 \le C_2$, $\delta_1 \ge \delta_2 > 0$ such that

$$C_1 e^{-\delta_1 |x|^\alpha} \le h(x) \le C_2 e^{-\delta_2 |x|^\alpha}.$$

Theorem 1.2 *Assume $\gamma > -1$, that h satisfies (h) and:*

(1) if $\gamma \ge 0$, suppose

$$\delta_1 \frac{\gamma}{1+\gamma} + \frac{1}{8} < \delta_2 \le \delta_1, \tag{1.6}$$

(2) if $-1 < \gamma < 0$, suppose

$$\frac{1+\gamma}{8} < \delta_2 \le \delta_1. \tag{1.7}$$

Then, there exists at least a positive solution of (1.5) in $X \cap C^2(\mathbb{R}^N)$ if and only if $\lambda > 0$. Moreover, there exist positive constants $C > 0$ and $\rho > \frac{1}{8}$ such that

$$u \le C e^{-\rho |x|^\alpha}. \tag{1.8}$$

Furthermore, there exists at most one positive solution of (1.5) for $\gamma \ge 0$ and $\rho > (\delta_2 - 1/8)/\gamma$ in (1.8) if $\gamma < 0$.

Second, we consider the logistic equation

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)|x|^{\alpha-2}(\lambda u - u^p) \text{in } \mathbb{R}^N, \tag{1.9}$$

where $p > 1$. For this equation, we need to analyse the eigenvalue problem

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)|x|^{\alpha-2} u \text{ in } \mathbb{R}^N. \tag{1.10}$$

We denote by λ_1 the principal eigenvalue of (1.10)

Theorem 1.3 *1.3 Assume $p > 1$. Then, there exists a positive solution of (1.9) in $X \bigcap C^2(\mathbb{R}^N)$ if, and only if, $\lambda > \lambda_1$. Moreover, there exists a unique positive solution of (1.9), and there exist positive constants $C > 0$ and $\rho > \frac{1}{8}$ such that*

$$u \le C e^{-\rho |x|^\alpha}. \tag{1.11}$$

In this article, we complement the existing literature in the sense that, to the best of our knowledge, this is the first paper to study this class of problems using subsolution and supersolution techniques. Moreover, in Theorem 1.2, when considering the case $\gamma \ge 0$, we address a strongly singular problem, which is a novelty in the study of this topic. Furthermore, when considering $-1 < \gamma < 0$ in the aforementioned theorem, we are dealing with the sublinear case, which had not been previously considered. On the other

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hand, equation (1.9) was analysed in [4] in the particular case $\alpha = 2$, using variational methods.

The paper is organised as follows. In Section 2, we apply the classical theory for compact and self-adjoint operators to study an eigenvalue problem more general than (1.10). Section 3 is devoted to the proof of an abstract result of the sub-supersolution type, which proves Theorem 1.1. The uniqueness of the positive solution of the problems considered in this paper is proved in Section 4. Finally, in Sections 5 and 6, we prove Theorems 1.2 and 1.3, respectively.

2. Eigenvalue problem

In this section, we study the following eigenvalue problem

$$-\mathrm{div}(K(x)\nabla u) + K(x)|x|^{\alpha-2}m(x)u = \lambda K(x)|x|^{\alpha-2}u \,\text{in}\, \mathbb{R}^N, \tag{2.1}$$

where $m \in L^\infty(\mathbb{R}^N)$.

We can apply the classical theory for compact and self-adjoint operators and deduce the following Theorem

Theorem 2.1 *There exists a sequence of eigenvalues of* (2.1)

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to +\infty \qquad \text{as } k \to +\infty.$$

Moreover,

$$\lambda_1(m) = \inf_{u\in X\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla u|^2 + \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}m(x)u^2}{\displaystyle\int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}u^2}. \tag{2.2}$$

Proof. Since $m \in L^\infty(\mathbb{R}^N)$ there exists a positive constant $C > 0$ such that $m(x)+C > 0$ for all $x \in \mathbb{R}^N$. Then, (2.1) is equivalent to

$$\mathcal{L}u := -\mathrm{div}(K(x)\nabla u) + K(x)|x|^{\alpha-2}(m(x) + C)u = (\lambda + C)K(x)|x|^{\alpha-2}u \,\text{in}\, \mathbb{R}^N.$$

Observe that $\mathcal{L}^{-1}$ is a self-adjoint, compact and positive operator, and then by the standard spectral theory for compact operators, there exists a sequence of eigenvalues of $\mathcal{L}$

$$0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_k \to \infty \qquad \text{as } k \to \infty.$$

Observe that $\lambda_i = \mu_i - C$ are eigenvalues of (2.1). This completes the proof. $\square$

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In general, the sign of $\lambda_1(m)$ depends on m. However, when $m \equiv 0$ we can see in ([3, Section 2]) that

$$\lambda_1 := \lambda_1(0) = \frac{1}{4}(\alpha - 2 + N) > 0,$$

and an eigenfunction associated with λ_1 is $\varphi_1(x) = exp(-|x|^\alpha/4)$.

In the following result, we show some properties of $\lambda_1(m)$.

Lemma 2.1.

(1) The map $m \mapsto \lambda_1(m)$ is increasing.

(2) Assume that φ is an eigenfunction associated with $\lambda_1(m)$. Then, φ does not change sign. In fact, we can take $\varphi(x) > 0$ for all $x \in \mathbb{R}^N$.

(3) Assume that $\lambda_1(m) > 0$ and that

$$-div(K(x)\nabla u) + K(x)|x|^{\alpha-2}m(x)u \geq 0 \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Then, $u \geq 0$ in $\mathbb{R}^N$.

(4) Assume that there exists $\varphi > 0$ in $\mathbb{R}^N$ such that

$$-div(K(x)\nabla\varphi) + K(x)|x|^{\alpha-2}m(x)\varphi = \lambda K(x)|x|^{\alpha-2}\varphi \quad \text{in } \mathbb{R}^N.$$

Then, $\lambda = \lambda_1(m)$.

Proof. 1. It follows from the variational characterisation of $\lambda_1(m)$, see (2.2).

2. Observe that if φ is an eigenfunction associated with $\lambda_1(m)$, then $w := |\varphi|$ is also an eigenfunction associated with $\lambda_1(m)$. Hence, $w \gneq 0$ in $\mathbb{R}^N$. Assume that there exists $x_0 \in \mathbb{R}^N$ such that $w(x_0) = 0$. Take $R > 0$ such that $w \gneq 0$ in $B(x_0, R)$. Then, we can apply the strong maximum principle in $B(x_0, R)$ to conclude that $w \equiv 0$ in $B(x_0, R)$, a contradiction. Then, $w(x) = |\varphi|(x) > 0$ for all $x \in \mathbb{R}^N$ and we can take $\varphi(x) > 0$ for all $x \in \mathbb{R}^N$.

3. By contradiction, assume that $u^- \not\equiv 0$. Taking u^- in (2.3) we get

$$\int_{\mathbb{R}^N} K(x)|\nabla u^-|^2 + \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}m(x)(u^-)^2 \leq 0.$$

On the other hand, by the variational characterisation of $\lambda_1(m)$, (2.2), we obtain

$$0 < \lambda_1(m)\int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}(u^-)^2 \leq \int_{\mathbb{R}^N} K(x)|\nabla u^-|^2 + \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}m(x)(u^-)^2 \leq 0,$$

a contradiction.

4. Taking as test function φ_1, a positive eigenfunction associated with $\lambda_1(m)$, we get

$$\lambda_1(m)\int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}\varphi\varphi_1 = \lambda\int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}\varphi\varphi_1.$$

Since $\varphi, \varphi_1 > 0$, we conclude that $\lambda = \lambda_1(m)$. □

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3. Sub-supersolution method - Proof of Theorem 1.1

Proof. Consider the function

$$g(x,t) := \begin{cases} f(x,\overline{u}(x)), & t > \overline{u}(x) \\ f(x,t), & \underline{u}(x) \le t \le \overline{u}(x) \\ f(x,\underline{u}(x)), & t < \underline{u}(x) \end{cases} \tag{3.1}$$

It is clear that g is a continuous function, and using (1.4) there exists a function H such that

$$|g(x,t)| \le H(x) \in L^2_K(\mathbb{R}^N), \quad \text{for all } t > 0. \tag{3.2}$$

Define the map $T : L^2_K(\mathbb{R}^N) \mapsto L^2_K(\mathbb{R}^N)$ such that $T(w) := u$, where u is the unique solution of

$$-\mathrm{div}(K(x)\nabla u) = K(x)|x|^{\alpha-2} g(x,w) \text{in } \mathbb{R}^N. \tag{3.3}$$

From the Lax–Milgran Theorem [2], we conclude that T is well defined. Now, we show T is compact and continuous. Indeed, let (w_n) be a bounded sequence in $L^2_K(\mathbb{R}^N)$ and set $u_n = T(w_n)$. Hence

$$\|u_n\|^2 = \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2} g(x,w_n)u_n dx \le \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2} H(x)u_n dx \le C\|H\|_{2,K}\|u_n\|_{2,K}.$$

Then, (u_n) is bounded in X and, in consequence, in $L^2_K(\mathbb{R}^N)$. Then, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in} \quad X$$

and

$$T(w_n) = u_n \to u \quad \text{in} \quad L^2_K(\mathbb{R}^N),$$

which implies that T is compact. Note that with the same previous arguments, we can prove that T is continuous.

Moreover, by (3.2), there exists $M > 0$ such that

$$\|T(w)\|_{2,K} \le M \quad \text{for all } w \in L^2_K(\mathbb{R}^N).$$

Then, if we define $\mathcal{B}(0,M) := \{u \in L^2_K(\mathbb{R}^N) : \|u\|_{2,K} \le M\}$, it is clear that

$$T(\mathcal{B}(0,M)) \subset \mathcal{B}(0,M).$$

The Schauder Fixed-Point Theorem shows that T has a fixed point u, and then u is solution of

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$$-\mathrm{div}(K(x)\nabla u) = K(x)|x|^{\alpha-2}g(x,u) \,\mathrm{in}\, \mathbb{R}^N. \tag{3.4}$$

We show that $\underline{u} \le u \le \overline{u}$ and then u is a solution of (1.3). Indeed, observe that

$$-\mathrm{div}(K(x)\nabla(\overline{u}-u)) \ge K(x)|x|^{\alpha-2}[f(x,\overline{u}) - g(x,u)].$$

Denote by $z := (\overline{u}-u)^-$, then

$$\int_{\mathbb{R}^N} K(x)|\nabla z|^2 \le \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}[f(x,\overline{u}) - g(x,u)](\overline{u}-u)^- \le 0.$$

The regularity of $u \in X$ follows by a direct adaptation of [4, Theorem 3.12]. This concludes the proof. □

4. Uniqueness of positive solution

In this section, we give two uniqueness results for positive solutions to (1.3) under the general condition

$$u > 0 \mapsto \frac{f(x,u)}{u} \quad \text{is decreasing for a. e. } x \in \mathbb{R}^N. \tag{4.1}$$

The first one assumes that the two solutions are ordered. Its proof follows the arguments of Proposition 3.13 of [4].

Proposition 4.1. *Assume that f satisfies (4.1) and that (1.3) possesses two positive solutions $u, v \in X$ such that $0 < u \le v$ in $\mathbb{R}^N$. Then, $u = v$.*

Proof. Assume that $0 < u \lneq v$. Taking u as a test function in the equation of v and vice versa, we obtain

$$\int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}\frac{f(x,u)}{u}uv = \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}\frac{f(x,v)}{v}uv$$

However, by (4.1), we have $f(x,u)/u \gneq f(x,v)/v$, which is a contradiction. □

In the second one, we do not need the solutions to be ordered, but we require some regularity on f and bounded solutions.

Proposition 4.2. *Assume that f is a continuous function, $f(x,0) = 0$, $f \in Lip_{loc}$ and satisfies (4.1). Let $\overline{u} \in L^\infty(\mathbb{R}^N)$ be a positive supersolution and $u \in L^\infty(\mathbb{R}^N)$ be a positive solution of (1.3). Then,*

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$$u \le \overline{u} \quad in\ \mathbb{R}^N.$$

Proof. First, observe that, since $f(x,0)=0$ and $f \in Lip_{loc}$, we get that $f(x,u)/u \in L^\infty(\mathbb{R}^N)$ and since u is a positive solution of (1.3), we have

$$-\mathrm{div}(K(x)\nabla u) + K(x)|x|^{\alpha-2}\left(-\frac{f(x,u)}{u}\right)u = 0,$$

and then by Lemma 2.1 4.,

$$\lambda_1\left(-\frac{f(x,u)}{u}\right) = 0. \tag{4.2}$$

Take now

$$z := \overline{u} - u.$$

Then,

$$-\mathrm{div}(K(x)\nabla z) + K(x)|x|^{\alpha-2}m(x)z \ge 0,$$

where

$$m(x) := -\frac{f(x,\overline{u}) - f(x,u)}{\overline{u} - u}.$$

Observe that $m \in L^\infty(\mathbb{R}^N)$ and by (4.1)

$$m(x) > -\frac{f(x,u)}{u}.$$

Using now Lemma 2.1 1 and (4.2), we obtain

$$\lambda_1(m) > \lambda_1\left(-\frac{f(x,u)}{u}\right) = 0.$$

Lemma 2.1 2 completes the proof. □

Corollary 4.1. *Assume that f is a continuous function, $f(x,0)=0$, $f \in Lip_{loc}$ and satisfies (4.1). Then, there exists at most one bounded positive solution of (1.3).*

Proof. Assume that u, v are two positive solutions of (1.3). Then, u is a supersolution and then $u \ge v$. Analogously, v is a supersolution and $v \ge u$. □

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5. Proof of Theorem 1.2

Before proving Theorem 1.2, we present a useful result. Observe that problem (1.5) is equivalent to

$$L_\alpha(u) = \lambda |x|^{\alpha-2} h(x) u^{-\gamma},$$

where

$$L_\alpha(u) := -\Delta u - \frac{\alpha}{4}|x|^{\alpha-2}(x \cdot \nabla u).$$

The next result follows from a straightforward computation.

Lemma 5.1. *Denote by*

$$\omega_\rho(x) := exp(-\rho |x|^\alpha),$$

where $\rho > 1/8$*. Then,* $\omega_\rho \in X$ *and*

$$L_\alpha(\omega_\rho) = \rho\alpha |x|^{\alpha-2}\omega_\rho \left[N + \alpha - 2 + |x|^\alpha \alpha \left(\frac{1}{4} - \rho\right)\right].$$

Proof of Theorem 1.1. We employ Theorem 1.1 with $f(x,u) = \lambda h(x) u^{-\gamma}$ and

$$\underline{u} = \varepsilon \omega_{\rho_1}, \quad \overline{u} := M\omega_{\rho_2}, \quad \text{with } \frac{1}{8} < \rho_2 \le \rho_1,$$

with ε and M positive constants to be chosen. Observe that $0 < \underline{u} \le \overline{u}$ for ε small or M large. On the other hand, taking $u \in [\underline{u}, \overline{u}]$, if $\gamma \ge 0$

$$|f(x,u)| = \lambda h(x) u^{-\gamma} \le \lambda h(x) \underline{u}^{-\gamma} \le C exp(-(\delta_2 - \gamma\rho_1)|x|^\alpha) := G_1(x),$$

while if $\gamma < 0$

$$|f(x,u)| = \lambda h(x) u^{-\gamma} \le \lambda h(x) \overline{u}^{-\gamma} \le C exp(-(\delta_2 - \gamma\rho_2)|x|^\alpha) := G_2(x).$$

Observe that $G_1 \in L^2_K(\mathbb{R}^N)$ provided that

$$\rho_1 \le \frac{\delta_2 - 1/8}{\gamma}. \tag{5.1}$$

On the other hand, $G_2 \in L^2_K(\mathbb{R}^N)$ if

$$\rho_2 > \frac{\frac{1}{8} - \delta_2}{-\gamma}. \tag{5.2}$$

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Moreover, $\underline{u}$ is a subsolution if

$$L_\alpha(\underline{u}) \leq \lambda|x|^{\alpha-2}h(x)\underline{u}^{-\alpha}.$$

Using Lemma 5.1, $\underline{u}$ is a subsolution provided that

$$\varepsilon^{1+\gamma}\rho_1\alpha exp((\delta_1-(1+\gamma)\rho_1)|x|^\alpha)\left[N+\alpha-2+|x|^\alpha\alpha\left(\frac{1}{4}-\rho_1\right)\right] \leq C_1\lambda$$

for which it is sufficient that

$$\frac{\delta_1}{1+\gamma} < \rho_1 \quad \text{and } \varepsilon > 0 \text{ small enough.} \tag{5.3}$$

On the other hand, $\overline{u}$ is a supersolution provided that

$$M^{1+\gamma}\rho_2\alpha exp((\delta_2-(1+\gamma)\rho_2)|x|^\alpha)\left[N+\alpha-2+|x|^\alpha\alpha\left(\frac{1}{4}-\rho_2\right)\right] \geq \lambda C_2.$$

for which it is sufficient that

$$\frac{\delta_2}{1+\gamma} > \rho_2, \quad \rho_2 \leq 1/4 \text{ and } M > 0 \text{ large.} \tag{5.4}$$

We separate the proof into two cases:

(1) Assume $\gamma \geq 0$ and (1.6). Then, we can take $\rho_2 \leq \rho_1$ satisfying (5.1), (5.3) and (5.4).
(2) Assume $-1 < \gamma < 0$ and (1.7). In this case, we can take $\rho_2 \leq \rho_1$ satisfying (5.2), (5.3) and (5.4).

Finally, we show the uniqueness results. We distinguish three cases:

(1) Assume that $\gamma = 0$. In this case, (1.5) is a linear equation, and then the uniqueness follows directly.
(2) Assume that $\gamma > 0$. Observe that $u^{-\gamma}$ is decreasing. Then, if there exist two positive solutions $u_1 \neq u_2$, then

$$\int_{\mathbb{R}^N} K(x)|\nabla(u_1-u_2)|^2 = \int_{\mathbb{R}^N} K(x)|x|^{\alpha-2}h(x)[u_1^{-\gamma}-u_2^{-\gamma}](u_1-u_2) \leq 0.$$

This implies that $u_1 \leq u_2$. Analogously, $u_2 \leq u_1$ and we conclude the uniqueness.
(3) Assume $-1 < \gamma < 0$. Due to the lack of regularity of the map $u^{-\gamma}$, we cannot apply Proposition 4.2. Hence, assume that there exist two positive solutions $u, v \in X$ such that

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$$u, v \leq Ke^{-\rho|x|^{\alpha}}, \qquad \rho > (\delta_2 - 1/8)/\gamma. \tag{5.5}$$

First, observe that

$$w := u + v \in X$$

is a supersolution of (1.5). Indeed, since $0 < -\gamma < 1$ we get that

$$(u+v)^{-\gamma} \leq u^{-\gamma} + v^{-\gamma}.$$

Now, we define the sequence w_n, with $w_0 := w$,

$$-\text{div}(K(x)\nabla w_n) = K(x)|x|^{\alpha-2} f(x, w_{n-1}) \text{ in } \mathbb{R}^N, \tag{5.6}$$

where

$$f(x, u) = \lambda h(x) u^{-\gamma}.$$

We show that

$$f(x, w_{n-1}) \in L^2_K(\mathbb{R}^N) \tag{5.7}$$

and

$$\{u, v\} \leq \cdots \leq w_{n+1} \leq w_n \leq \cdots \leq w_0 = w = u + v. \tag{5.8}$$

First, observe that $f(x, w_0) = f(x, w) = \lambda h(x)(u+v)^{-\gamma} \in L^2_K(\mathbb{R}^N)$ by (5.5), and then, w_1 is well-defined.

Observe that, using (5.6) and that w is supersolution of (1.5), we get

$$-\text{div}(K(x)\nabla(w_0 - w_1)) \geq 0,$$

and then $w_1 \leq w_0$ in $\mathbb{R}^N$.

Assume now that there exists w_{n-1} and $w_{n-1} \leq w_{n-2} \leq w_0 = w$. Then,

$$f(x, w_{n-1}) \in L^2_K(\mathbb{R}^N),$$

and so w_n is well defined. Moreover,

$$-\text{div}(K(x)\nabla(w_{n-1} - w_n)) = K(x)|x|^{\alpha-2}[f(x, w_{n-2}) - f(x, w_{n-1})] \geq 0,$$

and we conclude that $w_n \leq w_{n-1}$.

Analogously, we can show that $\{u, v\} \leq w_n$.

Hence, we can pass to the limit and obtain a solution W of (1.5 such that

$$\{u, v\} \leq W.$$

We can apply Proposition 4.1 and obtain that $u = W$, $v = W$, and we conclude the uniqueness of the positive solution.

□

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6. Proof of Theorem 1.3

Proof. Again we use Theorem 1.1 with $f(x,u) = \lambda u - u^p$ and

$$\underline{u} = \varepsilon\varphi_1, \quad \overline{u} := M\omega_\rho,$$

with ε and M positive constants to be chosen and $\rho > 1/8$. Observe that $0 < \underline{u} \le \overline{u}$ in $\mathbb{R}^N$.

On the other hand,

$$|f(x,u)| \le C(|\overline{u}| + |\overline{u}|^p) \le C(exp(-\rho|x|^\alpha) + exp(-\rho p|x|^\alpha)) \in L^2_K(\mathbb{R}^N).$$

Assume that $\lambda > \lambda_1$. It is clear that $\underline{u}$ is a subsolution for $\lambda > \lambda_1$. Indeed, this is equivalent to

$$\varepsilon^{p-1}\varphi_1^{p-1} \le \lambda - \lambda_1,$$

which is true for ε small.

On the other hand, $\overline{u}$ is supersolution of (1.9) provided that

$$\rho\alpha\left[N + \alpha - 2 + |x|^\alpha\alpha\left(\frac{1}{4} - \rho\right)\right] + M^{p-1}exp(\rho(1-p)|x|^\alpha) \ge \lambda \quad \text{in } \mathbb{R}^N. \tag{6.1}$$

Take $R > 0$. Then, in $B^c(0,R)$ we have that

$$\begin{aligned}&\rho\alpha\left[N + \alpha - 2 + |x|^\alpha\alpha\left(\frac{1}{4} - \rho\right)\right] + M^{p-1}exp(\rho(1-p)|x|^\alpha)\\ &\quad \ge \rho\alpha\left[N + \alpha - 2 + R^\alpha\alpha\left(\frac{1}{4} - \rho\right)\right] \ge \lambda,\end{aligned}$$

for R large. Now, in $B(0,R)$ we have

$$\begin{aligned}&\rho\alpha\left[N + \alpha - 2 + |x|^\alpha\alpha\left(\frac{1}{4} - \rho\right)\right] + M^{p-1}exp(\rho(1-p)|x|^\alpha)\\ &\quad \ge M^{p-1}exp(\rho(1-p)R^\alpha) \ge \lambda\end{aligned}$$

for M large. Hence, (6.1) is verified, and then $\overline{u}$ is supersolution of (1.9).

Now, assume that there exists a positive solution u of (1.9). Taking φ_1 as test function in (1.9) we get

$$(\lambda_1 - \lambda)\int_{\mathbb{R}^N} |x|^{\alpha-2}K(x)u\varphi_1 = -\int_{\mathbb{R}^N} |x|^{\alpha-2}K(x)u^p\varphi_1,$$

whence $\lambda > \lambda_1$.

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Now, we show that any solution positive solution u of (1.9) is $u \in L^\infty(\mathbb{R}^N)$. For that, we use a similar argument to Theorem 3.12 in [4], see also Lemma 2.2 in [12]. Define

$$w := exp(|x|^\alpha/8)u.$$

Then, w satisfies

$$-\Delta w + |x|^{\alpha-2}V(x)w = 0, \tag{6.2}$$

where

$$V(x) = \frac{\alpha^2}{64}|x|^\alpha - \lambda + \frac{\lambda_1}{2} + e^{-\frac{(p-1)}{8}|x|^\alpha}|w|^{p-1}.$$

Observe that $V > 0$ in $B^c(0,R)$ for some R large. Define

$$M := \sup_{x\in B(0,R)} w(x),$$

and define $\varphi := (w-M)^+$.

Then, taking φ as test function in (6.2), we get

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 + \int_{\mathbb{R}^N} |x|^{\alpha-2}V(x)w\varphi = 0.$$

It is direct to show that

$$\int_{\mathbb{R}^N} |x|^{\alpha-2}V(x)w\varphi \geq 0,$$

which implies that

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \leq 0$$

and then $\varphi \equiv 0$. This concludes that $w \in L^\infty(\mathbb{R}^N)$ and so $u \in L^\infty(\mathbb{R}^N)$.

Finally, the uniqueness and (1.11) follow by Corollary 4.1 and Proposition 4.2, respectively. □

Remark 6.1. In the particular case $\alpha = 2$, the uniqueness of a positive solution was proved in [4]. For that, the authors of [4] used Proposition 4.1 proving previously that a maximal solution exists.

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