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The Jordan Curve Theorem via Complex Analysis

Dedicated to the memory of Raghavan Narasimhan

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Abstract. The aim of this article is to give a proof of the Jordan Curve Theorem via complex analysis.

1 Introduction

1.1 Let K be a compact subset of $\mathbb{R}^2 = \mathbb{C}$. Fix a point b_k in each bounded connected component B_k of $\mathbb{C} \setminus K$ (the indexing set B for the k is at the most countable). Our focus will be on the following assertion:

Every continuous function $u: K \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ can be written in the form

$$u(z) = e(\alpha)(z) \prod_{k \in B} (z - b_k)^{m_k}, \tag{1.1}$$

where $\alpha: K \to \mathbb{C}$ is a continuous function, $e(\alpha) = e^{2\pi i \alpha}$ and the m_k are integers, non-zero for only finitely many k and uniquely determined by u.

We have introduced this assertion because it does not involve any new definitions, and a function theoretic proof suggests itself immediately. However, (1.1) can be formulated in several ways using standard definitions from Algebraic Topology. We do this in Section 1.5 and give the corollaries (which include the Jordan Curve Theorem). The function theoretic proof is given in Section 2. In Section 3, we make a digression to prove that the winding number of a Jordan curve around a point in its interior is ± 1; our proof uses the Jordan Curve Theorem. Finally, in Section 4, we give an elementary proof of (1.1).

Our exposition owes a great deal to Hurewicz and Wallman's beautiful book [2].

1.2 We make essential use of the following properties of the exponential covering.

Let $e: \mathbb{C} \to \mathbb{C}^*$ be the map sending z to $e^{2\pi i z}$. Let X be any topological space and let $f: X \times I \to \mathbb{C}^*$ be a continuous map; then given a lift $l(f_0): X \times \{0\} \to \mathbb{C}$ of $f_0 = f|_{X \times \{0\}}$ (*i.e.*, $e(l(f_0)) = f_0$), there exists a unique lift $l(f)$ of f extending f_0. This is called the *homotopic lifting property of the exponential covering.*

The special case X = singleton is the "unique path lifting property" of e (log can be analytically continued along any path in $\mathbb{C}^*$). If $\gamma: I \to \mathbb{C}^*$ is a closed path, then $l(\gamma(1)) - l(\gamma(0)) \in \mathbb{Z}$ is the winding number of γ around 0.

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1.3 A map $f: X \to \mathbb{C}^*$ has a lift $l(f): X \to \mathbb{C}$ if and only if it is homotopic to a constant. This follows from Section 1.2, since $\mathbb{C}$ is contractible.

1.4 For any topological space X, we denote by $\mathcal{C}(X)$ the set of all continuous functions from $X \to \mathbb{C}$, and by $\mathcal{C}^*(X)$ the subset of maps $X \to \mathbb{C}^*$.

1.5 Let $e: \mathcal{C}(X) \to \mathcal{C}^*(X)$ send f to $e^{2\pi i f}$. Then the image $e(\mathcal{C}(X))$ is a subgroup of the multiplicative group $\mathcal{C}^*(X)$. By Section 1.3, it consists precisely of the $f \in \mathcal{C}^*(X)$ that are homotopic to a constant.

Definition 1.1 $\pi^1(X) = \mathcal{C}^*(X)/e(\mathcal{C}(X))$.

Remark 1.2 $\pi^1(X)$ is a torsion free abelian group and $X \rightsquigarrow \pi^1(X)$ is a (contravariant) functor. It is probably the most easily defined algebraic-topological invariant of a topological space.

The assertion (1.1) is clearly equivalent to the following theorem.

Theorem 1.3 *If $K \subset \mathbb{C}$ is compact, then $\pi^1(K)$ is a free abelian group with the $(z - b_k)$, $k \in B$ as basis. In particular the cardinality of B,* i.e., *the "number" of connected components of $\mathbb{C} \setminus K$, is a topological invariant of K.*

In fact, $\pi^1(K)$ is an invariant of the homotopy type of K. This has the following corollaries.

Corollary 1.4 (Jordan Curve Theorem) $\pi^1(S^1) \cong \mathbb{Z}$. *If $\gamma \subset \mathbb{C}$ is a Jordan Curve (*i.e., *γ is homeomorphic to S^1), then $\mathbb{C} \setminus \gamma$ has precisely one bounded connected component.*

Corollary 1.5 *If $\gamma \subset \mathbb{C}$ in a Jordan arc (*i.e., *γ is homeomorphic to the unit interval in $\mathbb{R}$), then $\mathbb{C} \setminus \gamma$ is connected.*

Proof This is clear. Indeed $\mathbb{C} \setminus K$ is connected if K is contractible. In fact, $\mathbb{C} \setminus K$ is connected if and only if every map $K \to \mathbb{C}^*$ is homotopic to a constant. ■

For completeness, we state the following corollary.

Corollary 1.6 *If γ is a Jordan curve, then γ is the precise boundary of each of the two connected components of $\mathbb{C} \setminus \gamma$.*

Proof This is a standard and easy consequence of Corollaries 1.4 and 1.5. ■

Remark 1.7 For most spaces there is a natural isomorphism $\pi^1(X) \to H^1(X, \mathcal{Z})$, the later being the first cohomology group of the constant sheaf $\mathcal{Z}$ on X. Indeed, it follows from the exponential exact sequence of sheaves that $\pi^1(X)$ is always naturally isomorphic to the kernel of the natural map $H^1(X, \mathcal{Z}) \to H^1(X, \mathcal{C}_X)$, where $\mathcal{C}_X$ is the sheaf of germs of continuous complex-valued functions on X.

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1.6 If U is an open set in $\mathbb{C}$, then $H^1(U, \mathcal{O}_U) = 0$ (where $\mathcal{O}(U)$ is the sheaf of germs of holomorphic functions on U); hence, the holomorphic exponential exact sequence shows that

$$\pi^1(U) \approx H^1(U, \mathbb{Z}) \approx \mathcal{O}^*(U)/e(\mathcal{O}(U)).$$

It follows easily that every $u \in \mathcal{C}^*(K)$ is represented in $\pi^1(K)$ by a holomorphic function in a neighbourhood of K (see Section 2.2). This is "half" of the complex analytic proof of (1.1).

2 Complex Analytic Proof of (1.1)

2.1 Before beginning the proofs, we make a remark that will be used in the sequel without explicit mention.

Remark 2.1 Let $f \in \mathcal{C}^*(K)$, and let δ be the minimum of $|f|$ on K. Then, if $g \in \mathcal{C}(K)$ and $|f - g| < \delta$ on K, it follows that $g \in \mathcal{C}^*(K)$ and that g/f is an exponential; in particular, f and g are homotopic in $\mathcal{C}^*(K)$. This is because we have $|g/f - 1| < 1$, so that the logarithmic series can be used.

2.2 The right-hand side of (1.1) is reminiscent of Runge's Approximation Theorem, and (1.1) in fact follows from the easy part of Runge's theorem if u is the restriction to K of a holomorphic function v in a neighbourhood of K. Indeed, a rough version of the Cauchy's Integral Formula shows that, given $\epsilon > 0$, there exists a rational function

$$R(z) = \sum_{k=1}^{n} \frac{c_k}{z - a_k}, \quad a_k \notin K$$

such that $|R(z) - v(z)| < \epsilon$ on K (see Shastri [3, Lemma 8.6.1, p. 235], or Stein [5, Theorem 5.7, p. 61]). For variety, see Stein [5, Appendix B] for a proof of the Jordan Curve Theorem in the differentiable case.

If ϵ is small enough, the zeros of $R(z)$ will be outside K. To obtain (1.1) we now only have to observe that, for any $p \in \mathbb{C} \setminus K$, $(z - p)|_K$ is homotopic to $(z - b_k)|_K$ on K if $p \in B_k$, and $(z - p)|_K$is an exponential on K if p is in the unbounded component of $\mathbb{C} \setminus K$. It remains to show that every $u \in \mathcal{C}^*(K)$ is homotopic to (the restriction to K of) a holomorphic function v in a neighbourhood of K (see Section 1.6). By the Weierstrass approximation u can be approximated on K by $\mathcal{C}^\infty$ functions, hence we can assume up to homotopy that u is $\mathcal{C}^\infty$ in a neighbourhood of K, and we need to find a $\mathcal{C}^\infty$ λ in a neighbourhood of K such that $e^\lambda u$ is holomorphic, *i.e.*,

$$e^\lambda \frac{\partial \lambda}{\partial \bar{z}} u + e^\lambda \frac{\partial u}{\partial \bar{z}} = 0 \quad \textit{i.e.,} \quad \frac{\partial \lambda}{\partial \bar{z}} = -\frac{1}{u}\frac{\partial u}{\partial \bar{z}}.$$

This equation can be solved for λ in a neighbourhood of K by an integral formula (see Hörmander [1]), hence (1.1) is proved, except for uniqueness.

2.3 Uniqueness

For the uniqueness, we need to prove that if $u \equiv 1$ in (1.1), then $m_k = 0$ for all k. So suppose $m_1 \neq 0$. Then (1.1) can be written as

$$(z - b_1)^{m_1} = e(-\alpha) \prod_{k \neq 1} (z - b_k)^{-m_k}$$

on K, hence on $\partial B_1 \subset K$. But the right-hand side clearly extends from ∂B_1 into B_1 without zeros, and it is easy to see (in many ways) that this is impossible.

3 Winding Numbers

In this section we prove that the winding number $w_\gamma(0)$ of a Jordan curve in $\mathbb{C}^*$ around 0 is ± 1 if 0 is in the bounded component of $\mathbb{C} \setminus \gamma$.

Lemma 3.1 *Suppose $w_\gamma(0) = 0$. Then 0 is in the unbounded component $\mathbb{C} \setminus \gamma$.*

Proof We think of γ as a closed curve. Let $e\colon \mathbb{C} \to \mathbb{C}^*$ be the exponential covering. $\gamma\colon I \to \mathbb{C}^*$ with $\gamma(t_1) = \gamma(t_2)$ if and only if $|t_1 - t_2| = 1$. If $w_\gamma(0) = 0$, any lift $l(\gamma)$ of γ for e is also a closed curve $l(\gamma)(1) = l(\gamma)(0)$, and of course it is a Jordan curve. Let $\widetilde{B}$ be the bounded component of $\mathbb{C} \setminus l(\gamma)$. Then $B = e(\widetilde{B})$ is a bounded connected open set in $\mathbb{C}^*$ (it is contained in the compact set $e(B \cup l(\gamma))$ with boundary $\subset e(l(\gamma)) = \gamma$, hence a component of $\mathbb{C} \setminus \gamma$. Hence B is the bounded connected component of $\mathbb{C} \setminus \gamma$. But $0 \notin B$, so the lemma is proved. ■

Theorem 3.2 *Let γ be a Jordan curve in $\mathbb{C}^*$ and suppose 0 belongs to the bounded component of $\mathbb{C} \setminus \gamma$. Then $|w_\gamma(0)| = 1$.*

Proof We need only prove $|w_\gamma(0)| < 2$. If not, suppose $|w_\gamma(0)| \geq 2$, say. We can assume $|\gamma(t)| \leq 1$ for all $t \in I$ and $\gamma(0) = \gamma(1) = 1$. Then the lift $l(\gamma)$ of γ for e starting at 0 is a Jordan arc in the upper-half plane from 0 to $w_\gamma(0) \geq 2$. Hence, its translate by 1, $l(\gamma)+1$, is a Jordan arc in the upper half plane, from 1 to $w_\gamma(0)+1$, and disjoint from $l(\gamma)$. But this is impossible. For example, if we complete $l(\gamma)$ to a Jordan curve C by adding the dotted line segments in the lower half plane as in Figure 1, then $l(\gamma) + 1$ and C are disjoint, but 1 and $w_\gamma(0) + 1$ are in different components of $\mathbb{C} \setminus C$. ■

4 Elementary Proof of (1.1)

Lemma 4.1 (1.1) *is true for $K = S^1$.*

Proof For $K = S^1$, the Weierstrass (Trigonometric) Polynomial Approximation Theorem shows immediately that any $f \in \mathcal{C}(S^1)$ is approximable on S^1 by rational functions of the form $P(z)/z^m$, $P(z) \in \mathbb{C}[z]$. The rest of the proof is as in Subsection 2.2. ■

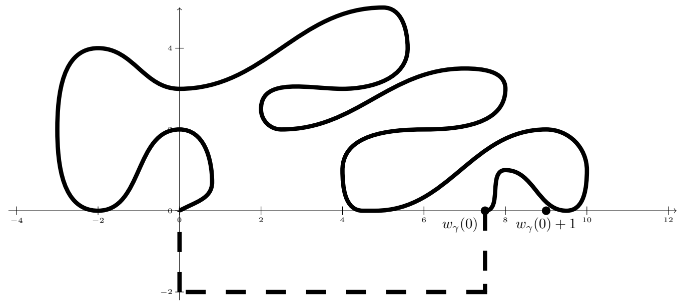

Figure 1

Lemma 4.2 *Let $K \subset \mathbb{C}$ be compact and let D be a closed disc containing K in its interior. Let $u \in \mathcal{C}^*(K)$. Then for any $\epsilon > 0$ there exists $f \in \mathcal{C}(\overline{D})$ such that*

(i) *$|f - u| < \epsilon$ on K and*

(ii) *f has only finitely many zeros in $\overline{D}$, all contained in $D \smallsetminus K$.*

Proof Let Δ be a triangle such that $\overline{D} \subset$ interior of Δ. Let $G \in \mathcal{C}(\Delta)$, with $G|_K = u$ and $G|_{\partial D} \equiv 1$. Then Δ can be subdivided into triangles T_i such that the oscillation of G on each T_i is less than η say, $\eta > 0$, to be chosen. Let (e_j) be an enumeration of the vertices of all the T_i and let $f_j \in \mathbb{C}$ be such that $|G(e_j) - f_j| < \eta$ for each j, and no three f_j are collinear. Let $F\colon \Delta \to \mathbb{C}$ be the function with $F(e_j) = f_j$ for all j and F affine linear on each T_i. Then for η sufficiently small, $F|_{\overline{D}}$ clearly has the desired properties. ■

4.1 End of Elementary Proof (1.1)

Given $u \in \mathcal{C}^*(K)$, find $f \in \mathcal{C}(\overline{D})$ as in Lemma 4.2. Let $(D_k) \subset D$ be discs with the zeros c_k of f as centres such that the $\overline{D}_k$ are disjoint and disjoint from K. If ϵ in Lemma 4.2 is small, $f|_K$ is homotopic to u on K, hence it is enough to prove (1.1) for f on $\overline{D} \smallsetminus \cup D_k$.

By Lemma 4.1, $f|_{\partial D_1} = e^{\lambda_1}(z - c_1)^{m_1}, \lambda_1 \in \mathcal{C}(\partial D_1)$. Thus, $f(z - c_1)^{-m_1} = f_1$ can be extended into $\overline{D}_1$ without zeros (extended λ_1). Repeating the process with f_1 on ∂D_2 and so on, we see that $f \prod_i (z - c_i)^{-m_i}$ extends to $\overline{D}$ without zeros, hence is an exponential. Finally, if c_i is in the unbounded component of $\mathbb{C} \smallsetminus K$, then $(z - c_i)$ is an exponential on K and, if c_i is contained in some bounded component of $\mathbb{C} \smallsetminus K$, then D_i is contained in B_k, and we can move the c_i up to homotopy on K to the b_k and (1.1) is proved.

Remark 4.3 The fact that the number of bounded connected components of $X = R^2 \smallsetminus K$ is the rank of the group $H^1(X, \mathbb{Z})$, and is hence in particular a topological invariant of K, is a very special case of the classical Alexander Duality Theorem (see E. H. Spanier [4, Chapt. 6, §2, Thrm. 16], the special case $n = 2$ and $q = 0$).

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