Bull. Aust. Math. Soc. **87** (2013), 400–405
doi:10.1017/S0004972712000639

ON THE POSITIVITY OF RIEMANN–STIELTJES INTEGRALS

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(Received 13 April 2012; accepted 22 June 2012; first published online 2 August 2012)

Abstract

We study the question whether a Riemann–Stieltjes integral of a positive continuous function with respect to a nonnegative function of bounded variation is positive.

2010 *Mathematics subject classification*: primary 26A42; secondary 26A45.

Keywords and phrases: Riemann–Stieltjes integral, positivity, function of bounded variation, Grönwall's inequality.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $g : [a, b] \to \mathbb{R}$ a function of bounded variation. It is a classical result that for such f and g, and for any $y \in (a, b]$, the Riemann–Stieltjes integral

$$\int_a^y f(x)\, dg(x) \tag{1.1}$$

exists (see, for example, [7, pp. 316–317]). While the basic properties of Riemann–Stieltjes integrals (and related Lebesgue–Stieltjes integrals) are covered in classical textbooks on real analysis and integration [4, 5, 7, 8]—Protter and Morrey [7] offering a particularly comprehensive account—the following simple question is not addressed in them.

Question 1.1. If f is positive and g is nonnegative, nonvanishing and satisfies $g(a) = 0$, can we then select the upper limit of integration y so that the integral (1.1) is positive?

The answer to the question is obviously yes for Riemann integrals ($g(x) = x - a$), which could help to explain why it has hitherto been overlooked. In fact, having conducted an extensive search of the literature, we believe that Question 1.1 has not been answered before in full generality. (We shall comment later, in Remark 3.3,

J. Lukkarinen was supported by the Academy of Finland. M. S. Pakkanen acknowledges support from the Finnish Cultural Foundation, from CREATES, funded by the Danish National Research Foundation, and from the Aarhus University Research Foundation regarding the project 'Stochastic and Econometric Analysis of Commodity Markets'.

on why the present generality may be relevant in applications.) The only reference known to us is a note by Satyanarayana [9] where an affirmative answer is proved under a slightly different set of assumptions—most importantly, assuming that g is *nondecreasing*.

Let us stress that *positive* will here always mean *strictly positive*. In particular, since f is continuous, under our assumptions $0 < \min f \le \max f < \infty$. Recall that, since g is of bounded variation, there exist nondecreasing functions g^+ and g^- such that $g = g^+ - g^-$. Therefore, finite $\lim_{x\to y^+} g(x)$ and $\lim_{x\to y^-} g(x)$ exist for any $y \in [a, b]$ (apart from $\lim_{x\to a^-} g(x)$ and $\lim_{x\to b^+} g(x)$, obviously). Moreover, the integral (1.1) is equal to

$$\int_a^y f(x)\, dg^+(x) - \int_a^y f(x)\, dg^-(x).$$

Since $g(a) = 0$, we may assume that g^+ and g^- are nonnegative and satisfy $g^\pm(a) = 0$.

We may now distinguish two special cases, where it is evident that the answer to Question 1.1 is yes.

(1) If $\lim_{x\to x_L^+} g(x) > 0$, where $x_L := \inf\{x : g(x) > 0\}$, then there exists $\varepsilon > 0$ such that (1.1) is positive for $y = x_L + \varepsilon$.

(2) If $g^+(y) > 0$ and $g^-(y) = 0$, then (1.1) is at least $g^+(y) \min_{x\in[a,y]} f(x) > 0$.

Item (1) follows from the elementary lower bound, valid for $0 < \varepsilon \le b - x_L$,

$$\int_a^{x_L+\varepsilon} f(x)\, dg(x) \ge g^+(x_L+\varepsilon) \min_{a,x_L-\varepsilon\le x\le x_L+\varepsilon} f(x) - g^-(x_L+\varepsilon) \max_{a,x_L-\varepsilon\le x\le x_L+\varepsilon} f(x),$$

whereas item (2) is a straightforward consequence of $g^\pm(a) = 0$.

In general, the integral (1.1) is positive if and only if

$$\int_a^y f(x)\, dg^+(x) > \int_a^y f(x)\, dg^-(x). \tag{1.2}$$

Obviously, the condition $g \ge 0$ is equivalent to $g^+ \ge g^-$, and we have $g(x) > 0$ if and only if $g^+(x) > g^-(x)$. It would be tempting to conjecture that, as a continuous function, f is 'nearly constant' in some neighborhood of x_L and, hence, that (1.2) ought to hold for $y = x_L + \varepsilon$ with some 'small' $\varepsilon > 0$, suggesting an affirmative answer to Question 1.1 in general.

REMARK 1.2. The proviso $g(a) = 0$ may seem superfluous as the value of the integral (1.1) does not depend on $g(a)$. However, together with the condition $g \ge 0$ it constrains the behaviour of g near x_L, which is a key part of the formulation of Question 1.1. It should also be stressed that the continuity of f is equally important—aside from the possible nonexistence of the integral, there is no reason to expect the answer to Question 1.1 to be yes when f fails to be continuous.

2. Negative answer to Question 1.1

Unfortunately, the heuristic above is too simplistic since mere continuity does not restrict the *fine properties* of the integrand f and leaves it with enough room to vary 'too much' for our purposes. Indeed, the general answer to Question 1.1 is no. We show that for any f that exhibits 'enough' variation, there exists a suitable g such that the integral (1.1) is less than zero for all $y \in (a, b]$.

THEOREM 2.1. *Let $f : [a, b] \to (0, \infty)$ be a continuous function. Suppose that there exist two sequences $(\underline{x}_n)$ and $(\overline{x}_n)$ with*

$$a < \cdots < \underline{x}_n < \overline{x}_n < \cdots < \underline{x}_2 < \overline{x}_2 < \underline{x}_1 < \overline{x}_1 \leq b$$

such that for some $\alpha > 0$ and $\gamma \in (0, 1)$,

$$f(\overline{x}_n) - f(\underline{x}_n) \geq \alpha n^{-\gamma} \quad \text{for all } n \in \mathbb{N}.$$

Then, there exists a nonvanishing function $g : [a, b] \to [0, \infty)$ of bounded variation such that $g(a) = 0$ and

$$\int_a^y f(x)\, dg(x) < 0 \quad \text{for all } y \in (a, b].$$

PROOF. Let $\beta > 1$ and consider $h : [a, b] \to [0, \infty)$ defined by

$$h(x) := \sum_{n \in \mathbb{N}} n^{-\beta} \chi_{[\underline{x}_n, \overline{x}_n)}(x),$$

where χ_E denotes the characteristic function of a set E. (Figure 1 illustrates the definition for Example 2.2 below.) This is, by construction, a function of bounded variation such that $h(a) = 0$. We have for any $n \in \mathbb{N}$,

$$\int_a^{\underline{x}_n} f(x)\, dh(x) = n^{-\beta} f(\underline{x}_n) - \sum_{k=n+1}^{\infty} k^{-\beta}(f(\overline{x}_k) - f(\underline{x}_k)),$$

where

$$\begin{aligned}\sum_{k=n+1}^{\infty} k^{-\beta}(f(\overline{x}_k) - f(\underline{x}_k)) &\geq \alpha \sum_{k=n+1}^{\infty} k^{-(\beta+\gamma)} \\ &\geq \alpha \int_{n+2}^{\infty} x^{-(\beta+\gamma)}\, dx \\ &= \frac{\alpha}{\beta+\gamma-1}(n+2)^{-(\beta+\gamma-1)}.\end{aligned}$$

Since $\gamma < 1$ and $\sup_{n \in \mathbb{N}} f(\underline{x}_n) < \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_a^{\underline{x}_n} f(x)\, dh(x) < 0 \quad \text{for all } n \geq n_0.$$

We may also note that for all $n \geq n_0$, with $n \geq 2$, and $y \in (\underline{x}_n, \underline{x}_{n-1})$,

$$\int_a^y f(x)\, dh(x) \leq \int_a^{\underline{x}_n} f(x)\, dh(x) < 0.$$

Thus, defining $g := h\chi_{[0, \overline{x}_{n_0})}$ yields a function with all the properties stated in the theorem. □

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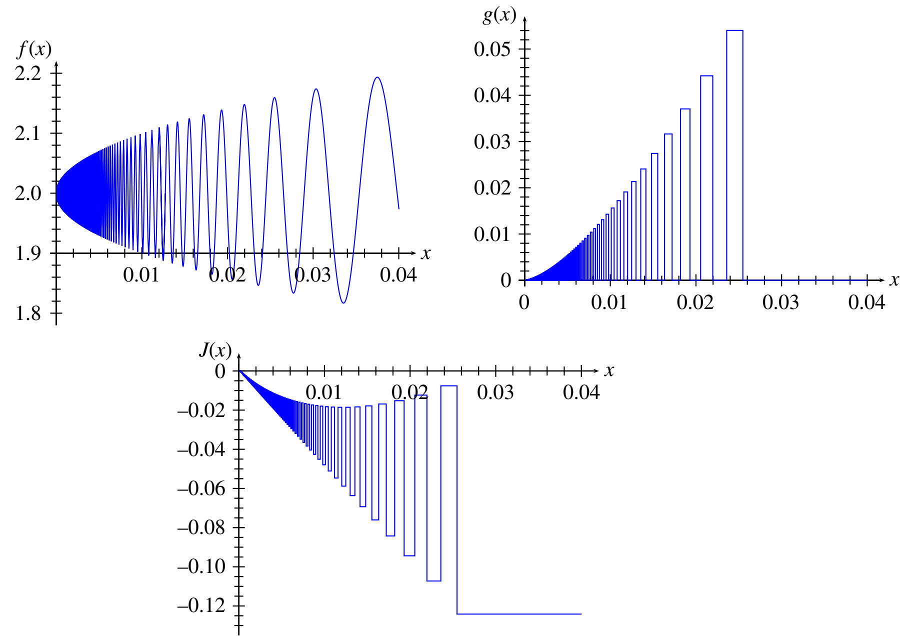

FIGURE 1. Plot of f, as defined in Example 2.2 for $\gamma = \frac{1}{2}$, and the corresponding functions g and J, $J(x) := \int_0^x f(x')\, dg(x')$, as defined in the proof of Theorem 2.1 using $\beta = \frac{3}{2}$ and $n_0 = 7$. The values have been computed numerically by considering only contributions with $n \leq 1000$. For clarity, only the region with $x \in [0, 0.04]$ is shown.

EXAMPLE 2.2. The function $f : [0, 1] \to (0, \infty)$ given by

$$f(x) := \begin{cases} x^\gamma \sin(1/x) + 2 & \text{if } x > 0, \\ 2 & \text{if } x = 0, \end{cases}$$

where $\gamma \in (0, 1)$, satisfies the condition of Theorem 2.1 with

$$\overline{x}_n = \frac{1}{4n-3}\frac{2}{\pi}, \quad \underline{x}_n = \frac{1}{4n-1}\frac{2}{\pi}, \quad n \in \mathbb{N},$$

using $\alpha = 2(2\pi)^{-\gamma}$. Figure 1 illustrates the behaviour of f and of the corresponding g and the Riemann–Stieltjes integral, as defined in the proof of Theorem 2.1.

3. Integrands of bounded variation

Any integrand f that satisfies the condition of Theorem 2.1 is clearly of *unbounded variation*. This prompts us to ask whether we could actually obtain an affirmative answer to Question 1.1 if f varied ‘less’. In fact, we are able to show that, if f is of *bounded variation*, the answer to Question 1.1 is yes. Bounds for Riemann–Stieltjes integrals under these assumptions have been derived by Beesack [1], Ganelius [3], and

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Knowles [6], but instead of building our argument on them, we give a direct proof which relies on some elementary measure theory and the measure-theoretic version of *Grönwall's inequality*.

THEOREM 3.1. *Let $f : [a, b] \to (0, \infty)$ be a continuous function and $g : [a, b] \to [0, \infty)$ a nonvanishing function of bounded variation such that $g(a) = 0$. If f is of bounded variation, then*

$$\int_a^y f(x)\, dg(x) > 0 \quad \textit{for some } y \in (a, b].$$

PROOF. Let us denote by $V_c^d f$ the total variation of f on $[c, d] \subset [a, b]$. Recall that total variation is additive in the sense that $V_a^d f = V_a^c f + V_c^d f$ for $a \le c \le d$ [7, Theorem 12.1]. Moreover, since f is continuous, the mapping $x \mapsto V_a^x f$ from $[a, b]$ to $[0, \infty)$ is continuous [7, Theorem 12.2]. Thus, there exists a finite, positive Borel measure ν on $[a, b]$ such that $\nu([c, d)) = V_c^d f$ for any c and d such that $a \le c < d \le b$. Let us define another finite, positive Borel measure by $\mu(dx) := f(x)^{-1}\nu(dx)$. By construction, we then have

$$|f(d) - f(c)| \le \int_{[c,d)} f(x)\mu(dx).$$

By an approximation with suitable Riemann–Stieltjes sums, where the values of g are chosen to be sufficiently close to their respective infima on all subintervals of the partitions, we can prove that

$$\left|\int_a^y g(x)\, df(x)\right| \le \int_{[a,y)} f(x)g(x)\mu(dx). \tag{3.1}$$

The Riemann–Stieltjes integral on the left-hand side is well defined by the integration by parts formula [7, Theorem 12.14]

$$\int_a^y g(x)\, df(x) = f(y)g(y) - f(a)g(a) - \int_a^y f(x)\, dg(x), \tag{3.2}$$

whenever $\int_a^y f(x)\, dg(x)$ exists and this is always true under the present assumptions.

Now suppose that, contrary to our assertion,

$$\int_a^y f(x)\, dg(x) \le 0 \quad \text{for all } y \in (a, b]. \tag{3.3}$$

Rearranging the integration by parts formula (3.2) and using the assumption $g(a) = 0$ and inequalities (3.1) and (3.3), we obtain, for any $y \in (a, b]$,

$$f(y)g(y) \le \int_a^y g(x)\, df(x) \le \int_{[a,y)} f(x)g(x)\mu(dx).$$

But the measure-theoretic version of Grönwall's inequality [2, Theorem A.5.1] implies that then $f(x)g(x) \le 0$ for all $x \in [a, b]$, whence $g = 0$, a contradiction. □

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REMARK 3.2. There are two straightforward refinements to Theorem 3.1. Firstly, it clearly suffices that f is of bounded variation on $[x_L, x_L + \varepsilon]$ for some $\varepsilon > 0$. Secondly, if g is right-continuous, then the mapping

$$y \mapsto \int_a^y f(x)\,dg(x)$$

is also right-continuous and, under the assumptions of Theorem 3.1, there exists an interval $[c, d] \subset [a, b]$ such that

$$\int_a^y f(x)\,dg(x) > 0 \quad \text{for all } y \in [c, d].$$

REMARK 3.3. Theorem 3.1 explains why the answer to Question 1.1 is perhaps elusive. Experimentation with nicely behaving integrands will not suffice, since a 'pathological' f is required in order to discover the general answer. However, such integrands need not be mere curiosities. In fact, our study of Question 1.1 was originally motivated by an application in financial mathematics involving as the integrand a *path* of a continuous-time *stochastic process*, which is typically of unbounded variation.

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