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Low-Pass Filters and Scaling Functions for Multivariable Wavelets

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Abstract. We show that a characterization of scaling functions for multiresolution analyses given by Hernández and Weiss and that a characterization of low-pass filters given by Gundy both hold for multivariable multiresolution analyses.

1 Introduction

In this paper we investigate low-pass filters and scaling functions associated with multivariable multiresolution analyses. In the multivariable setting, instead of the standard dilation by 2 we use a dilation matrix.

Definition 1.1 A *dilation matrix* is an $n \times n$ matrix A with integer entries, all of whose eigenvalues λ satisfy $|\lambda| > 1$.

Note that $q := |\det A|$ is an integer with $q \geq 2$. A dilation matrix A gives a mapping of the lattice $\mathbb{Z}^n$ into itself with nontrivial cokernel. The definition of a dilation matrix does not ensure that all singular values of A are strictly greater than 1, so we may not have $\|Ax\|_{\ell_2} > \|x\|_{\ell_2}$ for all $x \in \mathbb{Z}^n$. However there exists an integer $M \geq 1$ such that $\|A^j x\|_{\ell_2} > \|x\|_{\ell_2}$ for all $j \geq M$ (see [1]). It can also be shown that $\mathbb{Z}^n / A(\mathbb{Z}^n)$ has q cosets [13].

Definition 1.2 Let A be an $n \times n$ dilation matrix. A *digit set* for A is a set containing exactly one representative of each coset of $\mathbb{Z}^n / A(\mathbb{Z}^n)$.

In n dimensions, a multiresolution analysis is defined as follows.

Definition 1.3 Let A be an $n \times n$ dilation matrix. A *multiresolution analysis* (MRA) associated with A is a nested sequence of subspaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

of $L^2(\mathbb{R}^n)$ satisfying the following:

(i) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;
(ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
(iii) $f(x) \in V_0$ if and only if $f(x-k) \in V_0$ for all $k \in \mathbb{Z}^n$;

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(iv) $f(x) \in V_j$ if and only if $f(Ax) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(v) there exists a function $\phi \in V_0$, called a *scaling function*, such that the set

$$\{\phi_{0,k}(x) := \phi(x-k) : k \in \mathbb{Z}^n\}$$

is a complete orthonormal basis for V_0.

This generalization of multiresolution analyses beyond the case of dilation by 2 was originally introduced by Gröchenig and Madych [5], who showed that Haar-like scaling functions for multivariable MRAs (that is, scaling functions that can be written as the characteristic function of a set, $\phi = \chi_Q$) are associated with lattice self-affine tilings of $\mathbb{R}^n$ by sets of the form

$$T = \Big\{ \sum_{j=1}^{\infty} A^{-j} d_j : d_j \in D \Big\}$$

where D is a digit set for A. The existence of such tilings has been studied by Lagarias and Wang [9–12]), He and Lau [7], Belock and Dobric [3], and the author [1, 2]. This paper investigates general multivariable MRAs. In Section 2 we generalize a characterization theorem for scaling functions given by Hernández and Weiss [8] to the multivariable setting.

As in the case of dilation by 2, a scaling function for a multivariable MRA satisfies a scaling equation

$$\phi(A^{-1}x) = \sum_{k \in \mathbb{Z}^n} h_k \phi(x-k)$$

for some coefficients h_k. Taking the Fourier transform of both sides, there exists a periodic function $m(\xi) \in L^2(\mathbb{T}^n)$ such that $\hat{\phi}(\xi) = m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi)$.

Definition 1.4 A periodic function $m(\xi) \in L^2(\mathbb{T}^n)$ such that

$$\hat{\phi}(\xi) = m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi)$$

is called a *low-pass filter* for the scaling function ϕ and associated multiresolution analysis.

In Section 3 we prove a characterization of low-pass filters, formulated by Gundy [6] in the case of dilation by 2, in the multivariable setting.

2 Scaling Functions

Hernández and Weiss give a characterization of scaling functions for multiresolution analyses associated with dilation by 2 [8, Theorem 5.2, Ch. 7]. Their characterization can be extended to the multivariable setting with few changes.

Theorem 2.1 *Let A be an $n \times n$ dilation matrix. A function $\phi \in L^2(\mathbb{R}^n)$ is a scaling function for a multiresolution analysis under dilation by A if and only if*

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(i) $\sum_{k\in\mathbb{Z}^n}|\hat{\phi}(\xi+k)|^2 = 1$ *almost everywhere,*
(ii) $\lim_{j\to\infty}|\hat{\phi}((A^*)^{-j}\xi)| = 1$ *almost everywhere,*
(iii) *there exists a periodic function* $m(\xi) \in L^2(\mathbb{T}^n)$ *such that*

$$\hat{\phi}(\xi) = m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi)$$

almost everywhere (note that $m(\xi)$ *is then a low-pass filter for* ϕ*).*

Before proving this theorem, we need to know that the result of [8, Theorem 1.6, Ch. 2] still holds. This lemma shows that condition (ii) of the definition of an MRA is redundant.

Lemma 2.2 *Let* $\{V_j : j \in \mathbb{Z}\}$ *be a sequence of subspaces of* $L^2(\mathbb{R}^n)$ *satisfying*
(i) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$ *(the subspaces are nested),*
(ii) $f(x) \in V_j$ *if and only if* $f(Ax) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$ *(condition* (iv) *of the definition of an MRA),*
(iii) *there exists a function* $\phi \in V_0$*, called a* scaling function*, such that the set*

$$\{\phi_{0,k}(x) := \phi(x-k) : k \in \mathbb{Z}^n\}$$

is a complete orthonormal basis for V_0 *(condition* (v) *of the definition of an MRA).*
Then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ *(condition* (ii) *of the definition of an MRA holds).*

Proof Suppose that there exists a non-zero function f in $\bigcap_{j\in\mathbb{Z}} V_j$. We may assume $\|f\|_2 = 1$. Also, since $f \in V_j$ for each $j \in \mathbb{Z}$, if we let $f_j(x) = q^{j/2} f(A^j x)$ (where $q = |\det A|$), then $f_j \in V_0$, by condition (ii) of the statement of the lemma. A change of variables shows that $\|f_j\|_2 = \|f\|_2 = 1$. Since $\{\phi(\cdot - k) : k \in \mathbb{Z}^n\}$ is a basis for V_0, we may write

$$f_j(x) = \sum_{k\in\mathbb{Z}^n} \alpha_k^j \phi(x-k)$$

for some constant coefficients α_k^j, with convergence in $L^2(\mathbb{R}^n)$ such that

$$\sum_{k\in\mathbb{Z}^n} |\alpha_k^j|^2 = \|f_j\|_2^2 = 1.$$

Taking Fourier transforms, we get

$$q^{-j/2}\hat{f}((A^*)^{-j}\xi) = \hat{f}_j(\xi) = \sum_{k\in\mathbb{Z}^n} \alpha_k^j e^{-2\pi i k\cdot\xi}\hat{\phi}(\xi) = m_j(\xi)\hat{\phi}(\xi),$$

where $m_j(\xi) = \sum_{k\in\mathbb{Z}^n} \alpha_k^j e^{-2\pi i k\cdot\xi}$. Note that $m_j(\xi)$ is a $\mathbb{Z}^n$-periodic function, belonging to $L^2(\mathbb{T}^n)$, with L^2-norm 1. Thus

$$\hat{f}(\xi) = q^{j/2} m_j((A^*)^j\xi)\hat{\phi}((A^*)^j\xi),$$

and, for $j \geq 1$,

$$\int_{[1,2)^n} |\hat{f}(\xi)|\, d\xi \leq q^{j/2} \Big(\int_{[1,2)^n} |m_j((A^*)^j \xi)|^2\, d\xi \Big)^{1/2} \Big(\int_{[1,2)^n} |\hat{\phi}((A^*)^j \xi)|^2\, d\xi \Big)^{1/2}.$$

Let D be a digit set for A as in Definition 1.2, and let

$$T = \{ \sum_{j=1}^{\infty} A^{-j} d_j : \ d_j \in D \}.$$

Either the set T is congruent modulo $\mathbb{Z}^n$ to the unit cube $[1,2)^n$ (up to a set of measure zero), or a subset of T is congruent to $[1,2)^n$ [13]. Thus we can use the periodicity of $m_j(\xi)$ to rewrite the first integral above, to see that

$$\int_{[1,2)^n} |\hat{f}(\xi)|\, d\xi \leq q^{j/2} \Big(\int_{T} |m_j((A^*)^j \xi)|^2\, d\xi \Big)^{1/2} \Big(\int_{[1,2)^n} |\hat{\phi}((A^*)^j \xi)|^2\, d\xi \Big)^{1/2}.$$

Using a change of variables, $\mu = (A^*)^j \xi$, we then have

$$\begin{aligned} \int_{[1,2)^n} |\hat{f}(\xi)|\, d\xi &\leq q^{-j/2} \Big(\int_{(A^*)^j T} |m_j(\mu)|^2\, d\mu \Big)^{1/2} \Big(\int_{(A^*)^j [1,2)^n} |\hat{\phi}(\mu)|^2\, d\mu \Big)^{1/2} \\ &= \Big(q^{-j} \sum_{k \in D_{A,j}} \int_{T+k} |m_j(\mu)|^2 d\mu \Big)^{1/2} \Big(\int_{(A^*)^j [1,2)^n} |\hat{\phi}(\mu)|^2\, d\mu \Big)^{1/2} \\ &\leq (1)^{1/2} \Big(\int_{(A^*)^j [1,2)^n} |\hat{\phi}(\mu)|^2\, d\mu \Big)^{1/2}, \end{aligned}$$

where $D_{A,l} = \{k = \sum_{i=0}^{l-1} A^i d_i\}$ with the d_i in a digit set D for A. Note that there are q^j distinct elements in $D_{A,j-1}$. The last line in the above calculation follows from the $\mathbb{Z}^n$-periodicity of $m_j(\xi)$ and $(\int_F |m_j(\xi)|^2\, d\xi)^{1/2} \leq 1$.

We recall that $\lim_{j\to\infty} \min \{\sigma : \sigma \text{ a singular value of } A^j\} = \infty$ and thus

$$\lim_{j\to\infty} \min \{ \|\mu\|_{\ell_2} : \mu \in (A^*)^j [1,2)^n \} = \infty.$$

In particular, we can take j sufficiently large so that the set $(A^*)^j [1,2)^n$ contains an arbitrarily small amount of the mass of the function $|\hat{\phi}(\mu)|^2$. Thus the integral in the last line of the above calculation tends to 0 as $j \to \infty$. A similar calculation shows that $\int_{(A^*)^l [1,2)^n} |\hat{f}(\xi)|\, d\xi = 0$ for any fixed $l \in \mathbb{Z}$. Thus we obtain that $\hat{f}(\xi) = 0$ almost everywhere on $\bigcup_{l\in\mathbb{Z}} (A^*)^l [1,2)^n$. We may apply this argument to any other set congruent to F and such that $|(A^*)^j \mu| \to \infty$ for every μ in the set. For example, for each $\xi \in \mathbb{R}^n$, $\xi \neq 0$, we may take the unit cube translated so that its closest vertex to the origin is at ξ. Then $\hat{f}(\xi) = 0$ for almost every $\xi \in \mathbb{R}^n$. This completes the proof of the lemma. ■

Proof of Theorem 2.1 First, suppose that ϕ is a scaling function for an MRA $\{V_j : j \in \mathbb{Z}\}$. Then $\{\phi(\cdot - k) : k \in \mathbb{Z}^n\}$ is an orthonormal system in $L^2(\mathbb{R}^n)$, implying (i) (*e.g.*, by [13, Proposition 5.7(ii)], which states that $\{f(\cdot - k) : k \in \mathbb{Z}^n\}$ is an orthonormal system if and only if $\sum_{k\in\mathbb{Z}^n} |\hat{f}(\xi + k)|^2 = 1$ almost everywhere).

Let $F = [-\frac{1}{2}, \frac{1}{2})^n$ and $q = |\det A|$. We claim that

$$\lim_{j\to\infty} \int_F |\hat{\phi}((A^*)^{-j}\xi)|^2 \, d\xi = 1. \tag{1}$$

To see this, let f be the function such that $\hat{f} = \chi_F$, and let P_j be the projection onto V_j. Write $\phi_{j,k}(x) = q^{j/2}\phi(A^j x - k)$ for $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. Then

$$\begin{aligned}
\|P_j f\|_2^2 &= \Big\| \sum_{k\in\mathbb{Z}^n} \langle f, \phi_{j,k}\rangle \phi_{j,k} \Big\|_2^2 \\
&= \sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{\phi_{j,-k}}(\xi)} \, d\xi \Big|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} \hat{f}(\xi) q^{-j/2} e^{-2\pi i k\cdot\xi} \overline{\hat{\phi}((A^*)^{-j}\xi)} \, d\xi \Big|^2 \\
&= q^j \sum_{k\in\mathbb{Z}^n} \Big| \int_{(A^*)^{-j}F} \overline{\hat{\phi}(\mu)} e^{-2\pi i k\cdot\mu} \, d\mu \Big|^2.
\end{aligned}$$

The last expression is q^j times the sum of the squares of the absolute values of the Fourier coefficients of the function $\chi_{(A^*)^{-j}F}\overline{\hat{\phi}}$. Therefore, by the Plancherel theorem, it is equal to $q^j \int_{(A^*)^{-j}F} |\hat{\phi}(\mu)|^2 d\mu$. However, since $\{V_j : j \in \mathbb{Z}\}$ is dense in $L^2(\mathbb{R}^n)$, $\lim_{j\to\infty} \|P_j f\|_2^2 = \|f\|_2^2$. Thus this last expression tends to $\|\chi_F\|_2^2 = 1$ as $j \to \infty$.

A change of variables $\xi = (A^*)^{-j}\mu$ then gives us (1).

Since $|m(\xi)| \le 1$ for almost every ξ and $\hat{\phi}(\xi) = m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi)$, we must have $|\hat{\phi}((A^*)^{-j}\xi)|$ nondecreasing for almost every $\xi \in \mathbb{R}^n$ as $j \to \infty$. Let

$$g(\xi) = \lim_{j\to\infty} |\hat{\phi}((A^*)^{-j}\xi)|.$$

By condition (i) of the statement of the theorem, $|\hat{\phi}(\xi)| \le 1$ almost everywhere. Together with (1) and the Lebesgue dominated convergence theorem, this gives us $\int_F g(\xi) d\xi = 1$. We now have condition (ii) of the statement of the theorem, since $0 \le g(\xi) \le 1$ almost everywhere.

Lastly, we have condition (iii) of the statement of the theorem by [13, Lemma 5.8], which states that a function f belongs to V_1 if and only if $\hat{f}(A^*\xi) = m_f(\xi)\hat{\phi}(\xi)$ for some $\mathbb{Z}^n$-periodic function $m_f(\xi)$.

Conversely, suppose that ϕ satisfies conditions (i), (ii), and (iii). We want to show that the definition of a multiresolution analysis is satisfied. Proposition 5.7(ii) of [13] together with (i) imply that $\{\phi(\cdot - k) : k \in \mathbb{Z}^n\}$ is an orthonormal system. We define V_0 as the closure of the span of this system. Thus conditions (iii) and (v) of the definition of an MRA are satisfied. We define each V_j for $j \in \mathbb{Z}$ by

$$V_0 = \overline{\operatorname{span}\{\hat{\phi}(\cdot - k) : k \in \mathbb{Z}^n\}}; \qquad V_j = \{f : f(A^{-j}\cdot) \in V_0\} \text{ for } j \neq 0.$$

Then condition (iv) of the definition of an MRA is satisfied.

We claim furthermore that

$$V_j = \{f : \hat{f}((A^*)^j \xi) = \mu_j(\xi)\hat{\phi}(\xi) \text{ for some } \mathbb{Z}^n\text{-periodic } \mu_j \in L^2(\mathbb{T}^n)\},$$

since we may write $f(A^{-j}\cdot) \in V_0$ as a linear combination of $\phi(\cdot - k)$, $k \in \mathbb{Z}^n$, and then take Fourier transforms. By the periodicity of m and by (i), we have

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}^n} |\hat{\phi}(\xi + k)|^2 \\ &= \sum_{k\in\mathbb{Z}^n} |m((A^*)^{-1}\xi + (A^*)^{-1}k)|^2 |\hat{\phi}((A^*)^{-1}\xi + (A^*)^{-1}k)|^2 \\ &= \sum_{d\in D} |m((A^*)^{-1}(\xi + d))|^2 \sum_{\gamma\in\mathbb{Z}^n} |\hat{\phi}((A^*)^{-1}(\xi + d) + l)|^2 \\ &= \sum_{d\in D} |m((A^*)^{-1}(\xi + d))|^2 \end{aligned}$$

for almost every $\xi \in \mathbb{T}^n$ (where D is a digit set for A). In particular, $|m(\xi)| \leq 1$ for almost every $\xi \in \mathbb{T}^n$. Now in order to show that the subspaces $\{V_j\}$ are nested, we only need to show that $V_0 \subset V_1$. Given $f \in V_0$, we may write $\hat{f}(\xi) = \mu_0(\xi)\hat{\phi}(\xi)$ for some $\mathbb{Z}^n$-periodic function μ_0. Then

$$\hat{f}(A^*\xi) = \mu_0(A^*\xi)\hat{\phi}(A^*\xi) = \mu_0(A^*\xi)m(\xi)\hat{\phi}(\xi).$$

Note that $\mu_0(A^*\xi)m(\xi)$ is $\mathbb{Z}^n$-periodic and is in $L^2(\mathbb{T}^n)$, since $|m(\xi)| \leq 1$ for almost every $\xi \in \mathbb{T}^n$. Thus $f \in V_1$.

Next we would like to show that $L^2(\mathbb{R}^n) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$. Let P_j be the projection onto V_j. It suffices to show that

$$\|P_j f - f\|_2^2 = \|f\|_2^2 - \|P_j f\|_2^2 \to 0 \quad \text{as } j \to \infty.$$

We may also assume that $f \in L^2(\mathbb{R}^n)$ is such that $\hat{f}$ has compact support, so that for sufficiently large $j \in \mathbb{Z}$, $\hat{f}((A^*)^j \cdot)$ has support in F (see [1, Lemma 5]). Then

$$\begin{aligned}\|P_j f\|_2^2 &= q^{-j} \sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} f(t)\overline{\phi(A^j t - k)} q^j \, dt \Big|^2 \\ &= q^{-j} \sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} f(A^{-j} t)\overline{\phi(t-k)} \, dt \Big|^2 \\ &= q^{j} \sum_{k\in\mathbb{Z}^n} \Big| \int_{F} \hat{f}((A^*)^j \xi)\overline{\hat{\phi}(\xi)} e^{2\pi i k\cdot\xi} \, d\xi \Big|^2 \\ &= q^{j} \int_{F} \Big| \hat{f}((A^*)^j \xi)\hat{\phi}(\xi) \Big|^2 \, d\xi \\ &= \int_{A^j F} \Big| \hat{f}(\eta)\hat{\phi}((A^*)^{-j}\eta) \Big|^2 \, d\eta.\end{aligned}$$

By the dominated convergence theorem, since $|\hat{\phi}(\xi)| \leq 1$, and by condition (ii),

$$\int_{A^j F} |\hat{f}(\eta)|^2 |\hat{\phi}((A^*)^{-j}\eta)|^2 d\eta \to \int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta = \|f\|_2^2$$

as $j \to \infty$. Thus we have condition (i) of the definition of an MRA.

Lastly, condition (ii) of the definition of an MRA is a consequence of Lemma 2.2. ■

3 Low-Pass Filters

Let ϕ be a scaling function for a multiresolution analysis associated with an $n \times n$ dilation matrix A, and let $m(\xi)$ be a low-pass filter for ϕ. By Theorem 2.1(iii),

$$\hat{\phi}(\xi) = m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi) \text{ a.e. } \xi. \tag{2}$$

Also, Wojtasczyk [13] has shown that if D is a digit set for A, then

$$\sum_{d\in D} |m((A^*)^{-1}(\xi + d))|^2 = 1 \text{ a.e. } \xi.$$

These two equations lead us to consider the operators

$$\mathcal{P}\colon\ f(\xi) \to \sum_{d\in D} |m((A^*)^{-1}(\xi+d))|^2 f((A^*)^{-1}(\xi+d)),$$

$$\mathbf{p}\colon\ f(\xi) \to |m((A^*)^{-1}\xi)|^2 f((A^*)^{-1}\xi),$$

defined on $L^1 \cap L^2(\mathbb{R}^n)$ and $L^\infty(\mathbb{T}^n)$, respectively. We see that when $m(\xi) \in L^2(\mathbb{T}^n)$ is a low-pass filter associated with a scaling function $\phi(x)$, $|\hat{\phi}(\xi)|^2$ is a fixed point of the operator $\mathcal{P}$. Additionally, from Theorem 2.1(i), the function

$$e(\xi) := \sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\xi + k)|^2$$

is 1 almost everywhere, and thus is a fixed point of the operator **p**.

Condition (ii) of Theorem 2.1 states that a necessary condition for $\phi(x)$ to be a scaling function is $\lim_{j\to\infty}|\hat{\phi}((A^*)^{-j}\xi)| = 1$ almost everywhere. If $m(\xi)$ were continuous and sufficiently regular, for example if $m(\xi)$ were a trigonometric polynomial, then $\hat{\phi}(\xi)$ would be continuous at the origin, and we would need

$$1 = |\hat{\phi}(0)|^2 = \prod_{j=1}^{\infty}|m((A^*)^{-j}0)|^2 = \prod_{j=1}^{\infty}|m(0)|^2,$$

and thus $|m(0)|^2 = 1$ (see [8,13]). In general, however, we are considering $m(\xi)$ to be an equivalence class of functions in $L^2(\mathbb{T}^n)$, and so cannot specify $m(\xi)$ or $|m(\xi)|^2$ for given ξ. As we show below, a low-pass filter must satisfy a weak form of continuity at the origin, however.

Definition 3.1 Let $g(\xi) \in L^1 \cap L^\infty(\mathbb{R}^n)$. A function $f(\xi)$ is *almost everywhere A-adically g-continuous at the origin* if

$$\lim_{j\to\infty} \frac{f((A^*)^{-j}\xi)}{|g((A^*)^{-j}\xi)|^2}$$

exists and is constant almost everywhere. We denote the value of the limit by $\frac{f(0)}{|g(0)|^2}$.

We take $g = \hat{\phi}$ below.

Note that $e(\xi)$ is almost everywhere A-adically $\hat{\phi}$-continuous at the origin when $\phi(x)$ is a scaling function, as well. This leads us to consider the following space of functions.

Definition 3.2 $D_\infty(\hat{\phi})$ is the space of functions $h(\xi)$ satisfying the following:

(i) both $h(\xi)$ and its reciprocal $h^{-1}(\xi)$ are in $L^\infty(\mathbb{T}^n)$;

(ii) $h(\xi)$ is almost everywhere A-adically $\hat{\phi}$-continuous at the origin with $\frac{h(0)}{|\hat{\phi}(0)|^2} = 1$.

Gundy [6] gave a characterization of low-pass filters in the case of dilation by 2, using the ideas presented above. The same characterization holds in the multivariable setting, and in fact characterizes low-pass filters associated with *pre-scaling functions*.

Definition 3.3 A *pre-scaling function* associated with a multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$ is a function $\phi(x)$ such that the set of translates $\{\phi(x-k) : k \in \mathbb{Z}^n\}$ forms a Riesz basis for the space V_0.

As shown in Wojtasczyk [13], a pre-scaling function can be normalized by $e(\xi) = \sum_{k\in\mathbb{Z}^n}|\hat{\phi}(\xi+k)|^2$ so that the function $\gamma(x)$ defined by

$$\hat{\gamma}(\xi) = \frac{\hat{\phi}(\xi)}{(e(\xi))^{1/2}}$$

is a scaling function for the multiresolution analysis $\{V_j\}$.

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Theorem 3.4 *If $m(\xi)$ is a low-pass filter associated with a pre-scaling function $\phi(x)$, then we have the following statements.*

(i) *The function $m(\xi)$ is $\mathbb{Z}^n$-periodic and in $L^2(\mathbb{T}^n)$, and $|m(\xi)|^2$ is a.e. A-adically $\hat{\phi}$-continuous at the origin with $\frac{|m(0)|^2}{|\hat{\phi}(0)|^2} = 1$, that is,*

$$\lim_{j\to\infty} |m((A^*)^{-j}\xi)|^2 = 1 \quad a.e.\ \xi.$$

(ii) *The operators $\mathbf{p}$ and $\mathcal{P}$ have nontrivial fixed points, $|\hat{\phi}(\xi)|^2 \in L^1 \cap L^2(\mathbb{R}^n)$ and $e(\xi) \in L^\infty(\mathbb{T}^n)$, respectively.*

(iii) *The function $e(\xi)$ is the unique fixed point of the operator $\mathcal{P}$ in the class $D_\infty(\hat{\phi})$.*

Conversely, if a function $m(\xi)$ satisfies these three conditions, then there is a $\mathbb{Z}^n$-periodic, $L^2(\mathbb{T}^n)$ function $m_0(\xi)$, with $|m(\xi)| = |m_0(\xi)|$ almost everywhere, such that $m_0(\xi)$ is the low-pass filter associated with a pre-scaling function.

We prove the converse direction first in Section 4. We then complete the proof in Section 5.

4 Finding Square Roots

Proof of Theorem 3.4(i) The operators $\mathcal{P}$ and $\mathbf{p}$ depend only on $M(\xi) := |m(\xi)|^2$. Our problem is to find a suitable square root $m_0(\xi)$ for $M(\xi)$ and a square root $\hat{\phi}(\xi)$ for the fixed point $|\hat{\phi}(\xi)|^2$ of $\mathbf{p}$ such that $\hat{\phi}(\xi)$ is a pre-scaling function with low-pass filter $m_0(\xi)$. Since $M(\xi)$, $|\hat{\phi}(\xi)|^2$, and $e(\xi)$ are all real-valued and strictly positive, we can take a real valued square root of each function, $M^{1/2}(\xi)$, $|\hat{\phi}(\xi)|$, and $e^{1/2}(\xi)$, respectively.

Define

$$m_0(\xi) := m(\xi)(e^{1/2}(\xi)/e^{1/2}(A^*\xi)) = \operatorname{sgn} m(\xi) M^{1/2}(\xi)(e^{1/2}(\xi)/e^{1/2}(A^*\xi)).$$

We want to use $\mu(\xi) = \operatorname{sgn} m(\xi)$ to find a function for $\operatorname{sgn}\hat{\phi}(\xi)$. However $m(\xi)$ and $\operatorname{sgn} m(\xi)$ are defined on $\mathbb{T}^n$, whereas we need $\hat{\phi}(\xi)$ and $\operatorname{sgn}\hat{\phi}(\xi)$ to be defined on $\mathbb{R}^n$. To extend $\operatorname{sgn} m(\xi)$, we observe [6] that any unimodular, $\mathbb{Z}^n$-periodic function $\mu(\xi)$ may be written in terms of a non-periodic (not necessarily unique) unimodular function $t(\xi)$ as

$$\mu(\xi) = t(A^*\xi)t^{-1}(\xi).$$

To show this, first partition $\mathbb{R}^n\backslash\{0\}$ as follows. Let Q be the region between the sphere $C = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ and the set

$$(A^*)^{-1}C = \{y = (A^*)^{-1}x \in \mathbb{R}^n : \|x\|_2 = 1\},$$

including C but excluding $(A^*)^{-1}C$. For each $j \in \mathbb{Z}$, let $(A^*)^jQ$ denote the region between $(A^*)^jC$ (inclusive) and $(A^*)^{j-1}C$ (exclusive), where these sets are defined similarly to $(A^*)^{-1}C$. Thus the sets $(A^*)^jQ$ for $j \in \mathbb{Z}$ are mutually disjoint, and their union is $\mathbb{R}^n\backslash\{0\}$.

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Define $t(\xi) = 1$ for $\xi \in Q$. Consider $\mu(\xi)$ as a periodic function on $\mathbb{R}^n$, and define $t(\xi)$ for ξ in successive sets $(A^*)^j S$, $j \neq 0$, by

$$t(\xi) = \begin{cases} t((A^*)^{-1}\xi)\mu((A^*)^{-1}\xi) & \text{for } \xi \in (A^*)^j Q \text{ with } j \geq 1, \\ t(A^*\xi)\mu^{-1}(\xi) & \text{for } \xi \in (A^*)^j Q \text{ with } j \leq -1. \end{cases}$$

Also set $t(0) = 1$. Then $\mu(\xi) = t(A^*\xi)t^{-1}(\xi)$ for all $\xi \in \mathbb{R}^n$.

Now define $\hat{\phi}(\xi)$ by $\hat{\phi}(\xi) = t(\xi)|\hat{\phi}(\xi)|$. To show that $\hat{\phi}(\xi)$ so defined is a pre-scaling function, we refer to Theorem 2.1. Observe that

$$\sum_{k\in\mathbb{Z}^n} |\hat{\gamma}(\xi + k)|^2 = \sum_{k\in\mathbb{Z}^n} \frac{|\hat{\phi}(\xi + k)|^2}{e(\xi + k)} = \frac{\sum_{k\in\mathbb{Z}^n} |\hat{\phi}(\xi + k)|^2}{e(\xi)} = \frac{e(\xi)}{e(\xi)} = 1 \quad \text{a.e.},$$

where the second equality follows from the periodicity of $e(\xi)$. Thus condition (i) is satisfied.

For condition (ii), note that

$$\lim_{j\to\infty} |\hat{\gamma}((A^*)^{-j}\xi)|^2 = \lim_{j\to\infty} \frac{|\hat{\phi}((A^*)^{-j}\xi)|^2}{e((A^*)^{-j}\xi)} = 1 \quad \text{a.e.},$$

since $e(\xi) \in D_\infty(\hat{\phi})$.

To show condition (iii), first note that

$$\begin{aligned} \hat{\phi}(\xi) &= t(\xi)|\hat{\phi}(\xi)| = t(\xi)|m((A^*)^{-1}\xi)||\hat{\phi}((A^*)^{-1}\xi)| \\ &= t(\xi)t^{-1}((A^*)^{-1}\xi)|m((A^*)^{-1}\xi)|t((A^*)^{-1}\xi)|\hat{\phi}((A^*)^{-1}\xi)| \\ &= \operatorname{sgn} m((A^*)^{-1}\xi)|m((A^*)^{-1}\xi)|\hat{\phi}((A^*)^{-1}\xi) \\ &= m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi) \end{aligned}$$

almost everywhere. Now

$$\begin{aligned} \hat{\gamma}(\xi) &= \frac{\hat{\phi}(\xi)}{e^{1/2}(\xi)} = \frac{m((A^*)^{-1}\xi)\hat{\phi}((A^*)^{-1}\xi)}{e^{1/2}(\xi)} \\ &= \frac{m((A^*)^{-1}\xi)e^{1/2}((A^*)^{-1}\xi)}{e^{1/2}(\xi)} \frac{\hat{\phi}((A^*)^{-1}\xi)}{e^{1/2}((A^*)^{-1}\xi)} \\ &= m_0((A^*)^{-1}\xi)\hat{\gamma}((A^*)^{-1}\xi) \end{aligned}$$

almost everywhere. ∎

5 Proving Uniqueness

Proof of Theorem 3.4(ii) From the scaling equation (2) and condition (ii) of Theorem 2.1,

$$\lim_{j\to\infty} \frac{|m((A^*)^{-j}\xi)|^2}{|\hat{\phi}((A^*)^{-j}\xi)|^2} = \lim_{j\to\infty} \frac{|\hat{\phi}((A^*)^{-(j-1)}\xi)|^2/|\hat{\phi}((A^*)^{-j}\xi)|^2}{|\hat{\phi}((A^*)^{-j}\xi)|^2} = 1 \quad \text{a.e.}.$$

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Thus a low-pass filter $m(\xi)$ for a scaling or pre-scaling function $\phi(x)$ satisfies condition (i). By definition, $|\hat{\phi}(\xi)|$ is a fixed point of the operator $\mathbf{p}$. A standard calculation shows that $e(\xi)$ is a fixed point for the operator $\mathcal{P}$. Thus condition (ii) is also satisfied.

To show that condition (iii) is satisfied, we again refer to Theorem 2.1. If $\hat{\phi}(\xi)$ is a pre-scaling function, then

$$\hat{\gamma}(\xi) := \frac{\hat{\phi}(\xi)}{e^{1/2}(\xi)}$$

is a scaling function. Then

$$\lim_{j\to\infty} \frac{e((A^*)^{-j}\xi)}{|\hat{\phi}((A^*)^{-j}\xi)|^2} = \lim_{j\to\infty} |\hat{\gamma}^{-1}((A^*)^{-j}\xi)| = 1 \quad \text{a.e.},$$

so $e(\xi)$ is almost everywhere A-adically $\hat{\phi}$-continuous at the origin, with $e(0)/|\hat{\phi}(0)|^2 = 1$. Also, since $\hat{\phi}(\xi)$ is a pre-scaling function, there exist constants $c, C > 0$ such that $c < e(\xi) < C$ almost everywhere [13], so that both $e(\xi)$ and $e^{-1}(\xi)$ are in $L^\infty(\mathbb{T}^n)$. Thus $e(\xi)$ is in the class $D_\infty(\hat{\phi})$. It remains to show that $e(\xi)$ is the unique function in $D_\infty(\hat{\phi})$. That is, if $h(\xi) \in D_\infty(\hat{\phi})$, then $h(\xi) = e(\xi)$ almost everywhere.

Since $\sum_{k\in\mathbb{Z}^n} |\hat{\gamma}(\xi + k)|^2 = 1$ almost everywhere, we may interpret $|\hat{\gamma}(\xi + k)|^2$ as a probability distribution on $\mathbb{Z}^n$ for almost every ξ. The low-pass filter associated with $\hat{\gamma}(\xi)$ is

$$m_0(\xi) := m(\xi)\Big(\frac{e^{1/2}(\xi)}{e^{1/2}(A^*\xi)}\Big).$$

Set $M(\xi) := |m_0(\xi)|^2$ (note that $M(\xi) > 0$ almost everywhere). We can then write $|\hat{\gamma}(\xi + k)|^2$ as a limit of partial products

$$|\hat{\gamma}(\xi + k)|^2 = \lim_{N\to\infty} |\hat{\gamma}_N(\xi + k)|^2 \quad \text{a.e.}$$

where

$$|\hat{\gamma}_N(\xi + k)|^2 := \prod_{j=1}^{N} M((A^*)^{-j}(\xi + k)).$$

By [1, Corollary 7], there exists an integer $\beta \geq 1$ such that for $B := A^\beta$ we may represent each $k \in \mathbb{Z}^n$ by a radix representation. That is, there exists an integer $n = n(k) \geq 1$ such that $k = \sum_{j=0}^{n(k)} B^j d_j$ where each $\omega_j(k)$ is in the digit set $D_B = B([-\frac{1}{2}, \frac{1}{2})^n) \cap \mathbb{Z}^n$ for B. We then identify $k \in \mathbb{Z}^n$ with the sequence $(\omega_0(k), \omega_1(k), \omega_2(k), \dots)$ where $\omega_j(k) = d_j$ for $0 \leq j \leq n(k)$ and $\omega_j(k) = 0$ for $j > n(k)$.

Let Ω be the set of all such sequences (for arbitrary n), $\Omega = D_B \times D_B \times \cdots$. We have identified $\mathbb{Z}^n$ with the subset of Ω consisting of all finite sequences. Given $k \in \mathbb{Z}^n$, let $\mathbf{k}_N$ denote the cylinder set in Ω composed of sequences beginning with $(\omega_0(k), \dots, \omega_{N-1}(k))$. Define a measure P_ξ^N on cylinder sets in Ω by

$$P_\xi^N(\mathbf{k}_N) := \prod_{j=0}^{N-1} Q_{\xi,j}(\mathbf{k}_N),$$

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where

$$Q_{\xi,j}(\mathbf{k}_N) := \prod_{i=0}^{\beta-1} M(((A^*)^{-1})^{-\beta j-i-1}(\xi + k)).$$

We claim the following.

Lemma 5.1

$$\sum_{\substack{k\in\mathbb{Z}^n \\ \omega_j(k)=0 \text{ for } j\geq N}} P_\xi^N(\mathbf{k}_N) = 1 \quad \text{a.e. } \xi.$$

We delay the proof of Lemma 5.1 to the end of this present proof.

By a theorem of Kolmogorov, the family P_ξ^N extends to a probability P_ξ on Borel sets of Ω. Since $\hat{\gamma}(\xi)$ is a scaling function,

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}^n} |\hat{\gamma}(\xi + k)|^2 = \sum_{k\in\mathbb{Z}^n} \lim_{N\to\infty} \prod_{j=1}^{N} M((B^*)^{-j}(\xi + k)) \\ &= \sum_{k\in\mathbb{Z}^n} \lim_{N\to\infty} P_\xi^N(k) \end{aligned}$$

for almost every ξ. Since $\mathbb{Z}^n$ corresponds to the set of finite sequences in Ω, the family P_ξ^N is tight on $\mathbb{Z}^n$. That is, for every $\epsilon > 0$, there exists an $r(\epsilon, \xi)$ such that $\sum P_\xi^N(\mathbf{k}_N) \leq \epsilon$ for all $N \geq 1$, where the sum is taken over N-dimensional cylinder sets $\mathbf{k}_N$ such that the largest index $\tilde{j}$ with $\omega_{\tilde{j}}(k) \neq 0$ satisfies $\tilde{j} \geq r(\epsilon, \xi)$. This implies that P_ξ is concentrated on finite sequences. We say that $P_\xi(\mathbb{Z}^n) = 1$ for almost every ξ.

Consider $X_j(k) = \omega_j(k)$ as a sequence of random variables taking values in the digit set D_B, with the probability that $X_j = d$ given $X_0, \ldots, X_{j-1}$ being

$$M((B^*)^{-1}(\xi_j + d))$$

for each $j \geq 0$ and $d \in D_B$, with $\xi_0(k) := \xi$ and $\xi_{j+1} := (B^*)^{-1}(\xi_j + X_j)$ for $j \geq 0$. That P_ξ is concentrated on finite sequences for almost every ξ means that the sequence $\{X_j\}_{j\geq 0}$ converges to 0 relative to P_ξ for every $k \in \mathbb{Z}^n$ and almost every ξ. Now

$$\begin{aligned} P_\xi(\xi_{j+1} \parallel \xi_j, \ldots, \xi_0) &= P_\xi((B^*)^{-1}(\xi_j + X_j) \parallel \xi_j, \ldots, \xi_0) \\ &= M\big((B^*)^{-1}(\xi_j + X_j)\big). \end{aligned}$$

By construction, $P_\xi(\xi_{j+1} \parallel \xi_j, \ldots, \xi_0) = P_\xi(\xi_{j+1} \parallel \xi_j)$, thus $\{\xi_j\}_{j\geq 0}$ is a Markov process. Furthermore, since P_ξ is concentrated on finite sequences (for almost every ξ), $\lim_{j\to\infty} \xi_j = 0$ almost surely, for almost every ξ.

Now consider $r(\xi) := \frac{h(\xi)}{e(\xi)}$. We wish to show that $r(\xi) = 1$ for almost every ξ, so that $h(\xi) = e(\xi)$ almost everywhere. Since $e(\xi)$ and $h(\xi)$ are fixed points of $\mathcal{P}$, $r(\xi)$ satisfies

$$r(\xi) = \sum_{d\in D} M\big((A^*)^{-1}(\xi + d)\big)\, r\big((A^*)^{-1}(\xi + d)\big) \quad \text{a.e.}$$

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Using this, and that the sequence $\{\xi_j\}$ is a Markov process with transition probabilities $P_\xi(\xi_{j+1} = d \parallel \xi_j) = M((B^*)^{-1}(\xi_j + d))$, we find that for almost every ξ,

$$\begin{aligned}\mathbb{E}[r(\xi_{j+1}) \parallel r(\xi_j), \dots, r(\xi_0)] &= \mathbb{E}[r((B^*)^{-1}(\xi_j + X_j)) \parallel r(\xi_j), \dots, r(\xi_0)] \\ &= \mathbb{E}[r((B^*)^{-1}(\xi_j + X_j)) \parallel r(\xi_j)] \\ &= \sum_{d \in D_B} M\big((B^*)^{-1}(\xi_j + d)\big)\, r\big((B^*)^{-1}(\xi_j + d)\big) \\ &= r(\xi_j).\end{aligned}$$

Thus $r(\xi_j)$ is a martingale. Note that $r(\xi_j)$ is strictly positive and bounded, and converges P_ξ-almost surely to $r(0) = 1$ for almost every ξ, since $\xi_j \to 0$. Using the Lebesgue dominated convergence theorem,

$$r(0) = \mathbb{E}[r(0) \parallel r(\xi_j)] = \mathbb{E}[\lim_{l\to\infty} r(\xi_l) \parallel r(\xi_j)] = \lim_{l\to\infty} \mathbb{E}[r(\xi_l) \parallel r(\xi_j)] = r(\xi_j)$$

for every $j \geq 0$. Thus

$$1 = r(0) = r(\xi) = \frac{h(\xi)}{e(\xi)}$$

for almost every ξ, and $e(\xi)$ is the unique fixed point of the operator $\mathcal{P}$ in the class $D_\infty(\hat{\phi})$. ■

The proof of Lemma 5.1 relies on the following lemma.

Lemma 5.2 *Let A by a dilation matrix and let $B = A^\beta$ for some integer $\beta \geq 1$. If $m(\xi)$ is a low-pass filter under dilation by A, then $m_B(\xi) := \prod_{i=0}^{\beta-1} m((A^*)^i\xi)$ is a low-pass filter under dilation by B.*

Proof Let $\hat{\phi}(\xi)$ be the Fourier transform of the scaling function associated with $m(\xi)$. Then

$$\prod_{j=1}^{\infty} m_B((B^*)^{-j}\xi) = \prod_{j=1}^{\infty}\prod_{i=0}^{\beta-1} m((A^*)^{-j\beta+i}\xi) = \prod_{j'=1}^{\infty} m((A^*)^{-j'}\xi) = \hat{\phi}(\xi) \quad \text{a.e.},$$

$$\hat{\phi}(\xi) = m_B((B^*)^{-1}\xi)\hat{\phi}((B^*)^{-1}\xi) \quad \text{a.e.}$$

Since $\phi(x)$ is a scaling function under dilation by A, we know that

$$\sum_{k\in\mathbb{Z}^n} |\hat{\phi}(\xi + k)|^2 = 1 \quad \text{a.e.}$$

Also, $\lim_{j\to\infty} |\hat{\phi}((A^*)^{-j}\xi)| = 1$ a.e., and thus the limit along a subsequence is the same: $\lim_{j\to\infty} |\hat{\phi}((B^*)^{-j}\xi)| = 1$ a.e. Since all three conditions of Theorem 2.1 are satisfied, $\phi(x)$ is a scaling function under dilation by $B = A^\beta$, with low-pass filter $m_B(\xi)$ by construction. ■

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Proof of Lemma 5.1 Since $m_B(\xi)$ is a low-pass filter under dilation by B, it satisfies the relation

$$\sum_{d \in D_B} |m_B((B^*)^{-1}(\xi + d))|^2 = 1 \quad \text{a.e.},$$

from which the desired result follows:

$$\sum_{\substack{k \in \mathbb{Z}^n \\ \omega_j(k)=0 \ \text{for}\, j \geq N}} P_\xi^N(\mathbf{k}_N) = \sum_{d \in D_{B,N}} |m_B((B^*)^{-N}(\xi + d))|^2 = 1 \quad \text{a.e. } \xi.$$

■

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