Basic notions

In this chapter we shall deal with semigroups \( \{ V_t, t \in \mathbb{R}^+ = [0, +\infty) \} \) of continuous operators \( V_t : X \to X \) acting on a complete metric space \( X \). We shall denote them \( \{ V_t, t \in \mathbb{R}^+, X \} \) or simply \( \{ V_t \} \).

In what follows, the term *semigroup* refers to any family of continuous operators \( V_t : X \to X \) depending on a parameter \( t \in \mathbb{R}^+ \) and enjoying the semigroup property: \( V_{t_1}(V_{t_2}(x)) = V_{t_1+t_2}(x) \) for all \( t_1, t_2 \in \mathbb{R}^+ \) and \( x \in X \).

A semigroup \( \{ V_t \} \) is called *pointwise continuous* if the mapping \( t \to V_t(x) \) from \( \mathbb{R}^+ \) to \( X \) is continuous for each \( x \in X \). A semigroup is called *continuous* if the mapping \( (t, x) \to V_t(x) \) from \( \mathbb{R}^+ \times X \) to \( X \) is continuous.

Given a semigroup \( \{ V_t \} \) the following notation will be frequently used:

\[
\gamma^+(x) := \{ y \in X \mid y = V_t(x), t \in \mathbb{R}^+ \} \equiv \{ V_t(x), t \in \mathbb{R}^+ \};
\]

\[
\gamma^+_{[t_1, t_2]}(x) := \{ V_t(x), t \in [t_1, t_2] \};
\]

\[
\gamma^+_t(x) := \gamma^+_{[t, \infty)}(x) \equiv \{ V_\tau(x), \tau \in [t, \infty) \};
\]

\[
\gamma^+(A) := \bigcup_{x \in A} \gamma^+(x);
\]

\[
\gamma^+_{[t_1, t_2]}(A) := \bigcup_{x \in A} \gamma^+_{[t_1, t_2]}(x);
\]

\[
\gamma^+_t(A) := \bigcup_{x \in A} \gamma^+_t(x).
\]

It is easy to verify that \( V_t(\gamma^+(A)) = \gamma^+_t(A) \).

The curve \( \gamma^+(x) \) is called the positive semi-trajectory of \( x \).

The collection of all bounded subsets of \( X \) is denoted by \( \mathcal{B} \).

We use the letter \( B \) (with or without indices) to denote the elements of \( \mathcal{B} \), i.e. the bounded subsets of \( X \).
A semigroup \( \{V_t\} \) is called *locally bounded* if \( \gamma_{[0,t]}(B) \in \mathcal{B} \) for all \( B \in \mathcal{B} \) and all \( t \in \mathbb{R}^+ \). \( \{V_t\} \) is a *bounded semigroup* if \( \gamma^+(B) \in \mathcal{B} \) for each \( B \in \mathcal{B} \).

Let \( A \) and \( M \) be subsets of \( X \). We say that \( A \) attracts \( M \) or \( M \) is attracted to \( A \) by semigroup \( \{V_t\} \) if for every \( \epsilon > 0 \) there exists a \( t_1(\epsilon, M) \in \mathbb{R}_+^+ \) such that \( V_t(M) \subset O_\epsilon(A) \) for all \( t \geq t_1(\epsilon, M) \). Here \( O_\epsilon(A) \) is the \( \epsilon \)-neighbourhood of \( A \) (i.e. the union of all open balls of radii \( \epsilon \) centered at the points of \( A \)). We say that the set \( A \subset X \) attracts the point \( x \in X \) if \( A \) attracts the one-point set \( \{x\} \).

If \( A \) attracts each point \( x \) of \( X \) then \( A \) is called a *global attractor* (for the semigroup). \( A \) is called a *global \( B \)-attractor* if \( A \) attracts each bounded set \( B \in \mathcal{B} \).

A semigroup is called *pointwise dissipative* (respectively, *\( B \)-dissipative*) if it has a bounded global attractor (respectively a bounded global \( B \)-attractor).

Our main purpose here is to find those semigroups for which there is a minimal closed global \( B \)-attractor and investigate properties of such attractors. These attractors will be denoted by \( \mathcal{M} \). We shall examine also the existence of a minimal closed global attractor \( \hat{\mathcal{M}} \). It is clear that \( \hat{\mathcal{M}} \subset \mathcal{M} \) and later on we will also verify that \( \hat{\mathcal{M}} \) might be just a small part of \( \mathcal{M} \).

The concept of invariant sets is closely related to these subjects. We call a set \( A \subset X \) invariant (relative to semigroup \( \{V_t\} \)) if \( V_t(A) = A \) for all \( t \in \mathbb{R}_+^+ \).

A set \( A \subset X \) is called absorbing if for every \( x \in X \) there exists a \( t_1(x) \in \mathbb{R}_+^+ \) such that \( V_t(x) \in A \) for all \( t \geq t_1(x) \). A set \( A \) is called \( B \)-absorbing if for every \( B \in \mathcal{B} \) there exists a \( t_1(B) \in \mathbb{R}_+^+ \) such that \( V_t(B) \subset A \) for all \( t \geq t_1(B) \).

In our investigation of the problems concerning the attractors \( \mathcal{M} \) and \( \hat{\mathcal{M}} \) the concept of \( \omega \)-limit sets will play a fundamental role. For \( x \in X \) the \( \omega \)-limit set \( \omega(x) \) is, by definition, the set of all \( y \in X \) such that \( y = \lim_{k \to \infty} V_{t_k}(x) \) for a sequence \( t_k \nearrow +\infty \).

The \( \omega \)-limit set \( \omega(A) \) for a set \( A \subset X \) is the set of the limits of all converging sequences of the form \( V_{t_k}(x_k) \), where \( x_K \in A \) and \( t_k \nearrow +\infty \).

An equivalent description of the \( \omega \)-limit sets is given by

**Lemma 1.1**

\[
\omega(x) = \bigcap_{t \geq 0} [\gamma_t^+(x)]_X; \quad \omega(A) = \bigcap_{t \geq 0} [\gamma_t^+(A)]_X. \tag{1.1}
\]

Here the symbol \([ \ ]_X\) means the closure in the topology of the metric space \( X \).

The proof of Lemma 1.1 is traditional and so is omitted. Since, \( \gamma_{t_2}^+(A) \subset \gamma_{t_1}^+(A) \) whenever \( t_2 > t_1 \), the intersection over all \( t \in \mathbb{R}_+^+ \) in (1.1) may be replaced by \( \bigcap_{t \geq T} \) with any \( T \in \mathbb{R}_+^+ \).
It is necessary to have in mind that for locally non-compact spaces $X$ the use of the concept of limit sets requires some caution since the intersection $A_0 = \bigcap_{k=1}^{\infty} A_k$ of $A_k = [A_k)_X \supset A_{k+1} = [A_{k+1})_X$ in them may be empty (and therefore unhelpful).