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A NOTE ON (m, n)-JORDAN DERIVATIONS OF RINGS AND BANACH ALGEBRAS

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Abstract

In this paper we prove the following result: let $m, n \geq 1$ be distinct integers, let R be an $mn(m+n)|m-n|$-torsion free semiprime ring and let $D : R \to R$ be an (m, n)-Jordan derivation, that is an additive mapping satisfying the relation $(m+n)D(x^2) = 2mD(x)x + 2nxD(x)$ for $x \in R$. Then D is a derivation which maps R into its centre.

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1. Introduction

Throughout, R will represent an associative ring with centre $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if, for $x \in R$, $nx = 0$ implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the commutator identity $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. A ring R is prime if, for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and it is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by char(R) the characteristic of a prime ring R.

An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [14] asserts that any Jordan derivation on a prime ring with char$(R) \neq 2$ is a derivation. A brief proof of Herstein's theorem can be found in [6]. Cusack [9] generalised Herstein's theorem to 2-torsion free semiprime rings (see [2] for an alternative proof). For generalisations of Herstein's theorem, we refer to [1, 8, 11]. An additive mapping $D : R \to R$ is called a left derivation if $D(xy) = yD(x) + xD(y)$ holds for all pairs $x, y \in R$, and a left Jordan derivation (or Jordan left derivation) if $D(x^2) = 2xD(x)$ for all $x \in R$. The concepts of left derivation and left Jordan derivation were introduced by Brešar and Vukman [7]. One can easily prove that any left derivation on a noncommutative prime ring is zero. Moreover, we have the following result.

THEOREM 1.1. *Let R be a noncommutative 2-torsion free prime ring. Then the only left Jordan derivation $D : R \to R$ is $D = 0$.*

This result was first proved by Brešar and Vukman [7] under the additional assumption that R is 3-torsion free. Deng [10] removed the assumption that R is 3-torsion free. Theorem 1.1 is related to the theory of commuting and centralising mappings. A mapping F which maps a ring R into itself is called centralising on R if $[F(x), x] \in Z(R)$ for all $x \in R$. In the special case when $[F(x), x] = 0$ for all $x \in R$, F is called commuting on R. A classical result of Posner [16] (Posner's second theorem) states that the existence of a nonzero centralising derivation $D : R \to R$, where R is a prime ring, forces the ring to be commutative. Posner's second theorem cannot be generalised to semiprime rings, as shown by the following example. Take R_1 to be a noncommutative prime ring and let R_2 be a commutative prime ring that admits a nonzero derivation $d : R_2 \to R_2$. Then $R = R_1 \oplus R_2$ is a noncommutative semiprime ring, and the mapping $D(r_1, r_2) = (0, d(r_2))$ is a nonzero derivation which maps R into $Z(R)$. This example also shows that Theorem 1.1 cannot be proved for general semiprime rings. However, Vukman [21] proved the following result.

THEOREM 1.2 [21, Theorem 2]. *Let R be a 2-torsion free semiprime ring and let D be a left Jordan derivation on R. Then D is a derivation which maps R into $Z(R)$.*

Let $m, n \geq 0$ be fixed integers with $m + n \neq 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called an (m, n)-Jordan derivation if

$$(m + n)D(x^2) = 2mD(x)x + 2nxD(x), \quad x \in R. \tag{1.1}$$

The concept of (m, n)-Jordan derivation was introduced by Vukman [22]. It embraces both Jordan derivations and left Jordan derivations, because a $(1, 1)$-Jordan derivation on a 2-torsion free ring is a Jordan derivation and a $(0, 1)$-Jordan derivation is a left Jordan derivation. Vukman made the following conjecture.

CONJECTURE 1.3 [22, Conjecture 1]. *Let $m, n \geq 0$ be distinct integers with $m + n \neq 0$ and $D : R \to R$ be an (m, n)-Jordan derivation, where R is a semiprime ring with suitable torsion restrictions. Then D is a derivation which maps R into $Z(R)$.*

Fošner and Vukman [13] recently proved the following result.

THEOREM 1.4 [13, Theorem 2]. *Let $m, n \geq 1$ be distinct integers and let R be a prime ring with $\mathrm{char}(R) > (m + n)^2$. Suppose that $D : R \to R$ is a nonzero additive mapping satisfying the relation*

$$(m + n)^2 D(x^3) = m(3m + n)D(x)x^2 + 4mnxD(x)x + n(3n + m)x^2 D(x) \tag{1.2}$$

for all $x \in R$. Then D is a derivation and R is commutative.

One can easily prove that any (m, n)-Jordan derivation on an arbitrary ring satisfies (1.2), which means that Theorem 1.4 proves Conjecture 1.3 for the case of a prime ring.

The aim of this paper to prove the following result.

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Theorem 1.5. *Let $m, n \geq 1$ be distinct integers, R an $mn(m+n)|m-n|$-torsion free semiprime ring and $D : R \to R$ an (m, n)-Jordan derivation. Then D is a derivation which maps R into $Z(R)$.*

The methods used in the proof of Theorem 1.4 differ from those used in Theorem 1.5. The main tool in the proof of Theorem 1.4 is the theory of generalised functional identities (Brešar–Beidar–Chebotar theory). See [4] for an introductory account of functional identities and their applications and [5] for a full treatment of this theory. The proof of Theorem 1.5 is, as we shall see, elementary in the sense that one needs no specific knowledge concerning semiprime rings.

2. Proof of the main theorem

In the proof of Theorem 1.5, we shall use the following three results.

Theorem 2.1 [22, Proposition 1]. *Let $m, n \geq 0$ be integers with $m + n \neq 0$, R a 2-torsion free ring and $D : R \to R$ an (m, n)-Jordan derivation. Then the relation*

$$\begin{aligned}(m+n)^2 D(xyx) = m(n-m)D(x)xy + m(m-n)D(y)x^2 + n(n-m)x^2 D(y) \\ + n(m-n)yxD(x) + m(3m+n)D(x)yx \\ + 4mnxD(y)x + n(3n+m)xyD(x)\end{aligned}$$

holds for all pairs $x, y \in R$.

Theorem 2.2 [23, Theorem 4]. *Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $F : R \to R$ satisfies the relation*

$$[[F(x), x], x] = 0$$

for all $x \in R$. Then F is commuting on R.

Theorem 2.2 generalises a result of Brešar [3]. The proof of Theorem 2.2 is also due to Brešar. Theorem 2.2 has recently been generalised by Fošner *et al.* [12].

Lemma 2.3 [20, Lemma 1]. *Let R be a semiprime ring. Suppose that the relation*

$$axb + bxc = 0$$

holds for all $x \in R$ and some $a, b, c \in R$. Then

$$(a + c)xb = 0$$

is satisfied for all $x \in R$.

Proof of Theorem 1.5. From the assumption that R is $mn(m+n)\,|m-n|$-torsion free, it follows that R is 2-torsion free. The linearisation of the relation (1.1) gives

$$(m+n)D(xy+yx) = 2mD(x)y + 2mD(y)x + 2nxD(y) + 2nyD(x), \quad x, y \in R. \quad (2.1)$$

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From Theorem 2.1,

$$(m+n)^2D(xyx) = m(n-m)D(x)xy + m(m-n)D(y)x^2 + n(n-m)x^2D(y) \\ + n(m-n)yxD(x) + m(3m+n)D(x)yx + 4mnxD(y)x + n(3n+m)xyD(x) \tag{2.2}$$

for $x, y \in R$. Putting $(m+n)xyx$ for y in (2.1) and using (2.2),

$$(m+n)^3D(x^2yx + xyx^2) = 2m(3mn + n^2)D(x)xyx + 2m^2(m-n)D(y)x^3 \\ + 2mn(5n-m)x^2D(y)x + 2mn(m-n)yxD(x)x + 2m^2(3m+n)D(x)yx^2 \\ + 2mn(5m-n)xD(y)x^2 + 2mn(3n+m)xyD(x)x + 2mn(n-m)xD(x)xy \\ + 2n^2(n-m)x^3D(y) + 2mn(m+3n)xyxD(x) + 2mn(3m+n)xD(x)yx \\ + 2n^2(3n+m)x^2yD(x), \quad x, y \in R. \tag{2.3}$$

On the other hand, by substituting $(m+n)(xy+yx)$ for y in (2.2) and applying (2.1),

$$(m+n)^3D(x^2yx + xyx^2) = m(m+n)(n-m)D(x)x^2y + 2m(m+n)^2D(x)xyx \\ + m(5m^2 + 2mn + n^2)D(x)yx^2 + 2m^2(m-n)D(y)x^3 + 2mn(5m-n)xD(y)x^2 \\ + 2mn(m-n)yD(x)x^2 + 2mn(n-m)x^2D(x)y + 2mn(5n-m)x^2D(y)x \\ + 2n^2(n-m)x^3D(y) + n(5n^2 + 2mn + m^2)x^2yD(x) + 2n(m+n)^2xyxD(x) \\ + n(m+n)(m-n)yx^2D(x) + 8m^2nxD(x)yx + 8mn^2xyD(x)x \tag{2.4}$$

for $x, y \in R$. Since R is $|m-n|$-torsion free, comparing (2.3) and (2.4) yields

$$2m^2D(x)xyx - m(m+n)D(x)x^2y - m(m+n)D(x)yx^2 + 2mnyD(x)x^2 \\ - 2mnx^2D(x)y + n(m+n)x^2yD(x) - 2n^2xyxD(x) + n(m+n)yx^2D(x) \\ - 2mnxyD(x)x + 2mnxD(x)yx - 2mnyxD(x)x + 2mnxD(x)xy = 0 \tag{2.5}$$

for $x, y \in R$. Put yx for y in the relation (2.5), multiply the relation (2.5) on the right-hand side by x and subtract the relations so obtained one from another. This yields

$$2mny[D(x), x]x^2 + n(m+n)x^2y[D(x), x] - 2n^2xyx[D(x), x] + n(m+n)yx^2[D(x), x] \\ - 2mnxy[D(x), x]x - 2mnyx[D(x), x]x = 0 \tag{2.6}$$

for $x, y \in R$. From the above relation,

$$n(m+n)[D(x), x^2]y[D(x), x] - 2n^2[D(x), x]yx[D(x), x] - 2mn[D(x), x]y[D(x)x]x = 0,$$

for $x, y \in R$, which can be written in the form

$$n(m+n)[D(x), x^2]y[D(x), x] + [D(x), x]y(-2n^2x[D(x), x] - 2mn[D(x), x]x) = 0$$

for $x, y \in R$. From the last relation and Lemma 2.3,

$$(n(m+n)[D(x), x]x + n(m+n)x[D(x), x] - 2n^2x[D(x), x] \\ - 2mn[D(x), x]x)y[D(x), x] = 0$$

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for $x, y \in R$, which reduces to

$$n(m-n)[[D(x), x], x]y[D(x), x] = 0, \quad x, y \in R,$$

and finally to

$$[[D(x), x], x]y[D(x), x] = 0, \quad x, y \in R.$$

By following the approach used to derive (2.6) from (2.5), the last relation yields

$$[[D(x), x], x]y[[D(x), x], x] = 0, \quad x, y \in R,$$

whence it follows that

$$[[D(x), x], x] = 0, \quad x \in R,$$

by using the semiprimeness of R. According to Theorem 2.2, it follows that

$$[D(x), x] = 0, \quad x \in R. \tag{2.7}$$

The relation (2.7) makes it possible to replace $D(x)x$ with $xD(x)$ in (1.1). Therefore, $(m+n)D(x^2) = 2(m+n)xD(x)$ for $x \in R$, which reduces to $D(x^2) = 2xD(x)$ for $x \in R$. Applying the relation (2.7) again, we arrive at $D(x^2) = D(x)x + xD(x)$ for $x \in R$. In other words, D is a Jordan derivation, whence it follows that D is a derivation by Cusack's generalisation of Herstein's theorem.

It is well known that any commuting derivation on a semiprime ring maps the ring into its centre, but we shall proceed with the proof for the sake of completeness. The linearisation of the relation (2.7) gives

$$[D(x), y] + [D(y), x] = 0, \quad x, y \in R. \tag{2.8}$$

The substitution of xy for y in (2.8) gives

$$0 = [D(x), xy] + [D(x)y + xD(y), x] = x[D(x), y] + D(x)[y, x] + x[D(y), x] = D(x)[y, x]$$

for $x, y \in R$. Therefore,

$$D(x)[y, x] = 0, \quad x, y \in R.$$

The linearisation of the above relation gives

$$D(x)[y, z] + D(z)[y, x] = 0, \quad x, y, z \in R,$$

and, in particular, for $y = D(x)$,

$$D(x)[D(x), z] = 0, \quad x, z \in R.$$

Substituting yz for z in the above relation gives

$$D(x)y[D(x), z] = 0, \quad x, y, z \in R. \tag{2.9}$$

By substituting zy for y in (2.9), multiplying (2.9) on the left-hand side by z and subtracting the two relations so obtained one from the other, we arrive at

$$[D(x), z]y[D(x), z] = 0, \quad x, y, z \in R,$$

whence $[D(x), z] = 0$ for $x, z \in R$ by the semiprimeness of R. The proof of the theorem is complete. □

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Johnson and Sinclair [15] proved that any linear derivation on a semisimple Banach algebra is continuous. By combining this result with the Singer–Wermer theorem [18], which states that a continuous linear derivation on a commutative Banach algebra maps the algebra into its radical, one sees that there are no nonzero continuous linear derivations on commutative semisimple Banach algebras. Thomas [19] proved the Singer–Wermer theorem without the continuity assumption, whence it follows immediately that there are no nonzero linear derivations on commutative semisimple Banach algebras. Vukman [21] has proved the following result, which can be considered an extension of the result we have just mentioned.

Theorem 2.4 [21]. *Let A be a complex semisimple Banach algebra and $D : A \to A$ a linear left Jordan derivation. Then $D = 0$.*

Theorem 1.5 will be used in the proof of our next result, which is in the spirit of Theorem 2.4.

Theorem 2.5. *Let A be a complex semisimple Banach algebra, let $m, n \geq 1$ be distinct integers and let $D : A \to A$ be a linear (m, n)-Jordan derivation. Then $D = 0$.*

Proof. Semisimple Banach algebras are semiprime, which means that all the assumptions of Theorem 1.5 are fulfilled. Therefore, we have a linear derivation on A, such that $[D(x), y] = 0$ for all pairs $x, y \in A$. Johnson and Sinclair [15] proved that any linear derivation on a semisimple complex Banach algebra is continuous. Sinclair [17] proved that any continuous linear derivation on a complex Banach algebra leaves the primitive ideals invariant. Therefore, for any primitive ideal $P \subset A$, one can introduce a derivation $D_P : A/P \to A/P$, where A/P is the factor algebra, by $D_P(\hat{x}) = D(x)$, $\hat{x} = x + P$. It is well known that a complex commutative primitive Banach algebra is isomorphic to the complex field. Since A/P is primitive, in case A/P is commutative, we have $D_P = 0$. It remains to prove that $D_P = 0$ also in case A/P is noncommutative. From $[D(x), y] = 0$ for $x, y \in A$, it follows that $[D_P(\hat{x}), \hat{y}] = 0$ for all pairs $\hat{x}, \hat{y} \in A/P$. Since A/P is prime, it follows from Posner's second theorem that $D_P = 0$. Hence, $D_P = 0$ in any case. In other words, $D(x)$ is in the intersection of all primitive ideals of A for any $x \in A$. Since the intersection of all primitive ideals is the radical, and since A is semisimple, the proof of the theorem is complete. □

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