THE MULTIPLIER ALGEBRA AND BSE PROPERTY OF THE DIRECT SUM OF BANACH ALGEBRAS

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Abstract

The notion of BSE algebras was introduced and first studied by Takahasi and Hatori and later studied by Kaniuth and Ülger. This notion depends strongly on the multiplier algebra $M(A)$ of a commutative Banach algebra $A$. In this paper we first present a characterisation of the multiplier algebra of the direct sum of two commutative semisimple Banach algebras. Then as an application we show that $A \oplus B$ is a BSE algebra if and only if $A$ and $B$ are BSE. We also prove that if the algebra $A \times_\theta B$ with $\theta$-Lau product is a BSE algebra and $B$ is unital then $B$ is a BSE algebra. We present some examples which show that the BSE property of $A \times_\theta B$ does not imply the BSE property of $A$, even in the case where $B$ is unital.

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1. Introduction

Let $\mathcal{A}$ be a commutative Banach algebra. Throughout this paper $\Delta(\mathcal{A})$ denotes the set of all nonzero multiplicative linear functionals on $\mathcal{A}$. Then $\Delta(\mathcal{A})$ is a topological space with the Gelfand topology, called the Gelfand spectrum of $\mathcal{A}$.

A bounded continuous function $\sigma$ on $\Delta(\mathcal{A})$ is called a BSE function if there exists a constant $C > 0$ such that for every finite number of $\phi_1, \ldots, \phi_n$ in $\Delta(\mathcal{A})$ and the same number of complex numbers $c_1, \ldots, c_n$, the inequality

$$
\left| \sum_{j=1}^{n} c_j \sigma(\phi_j) \right| \leq C \left\| \sum_{j=1}^{n} c_j \phi_j \right\|_{\mathcal{A}}
$$

holds. The BSE norm of $\sigma$, $\| \sigma \|_{BSE}$, is defined to be the infimum of all such $C$. The set of all BSE functions is denoted by $C_{BSE}(\Delta(\mathcal{A}))$. Takahasi and Hatori [14] showed that under the norm $\| \cdot \|_{BSE}$, $C_{BSE}(\Delta(\mathcal{A}))$ is a commutative semisimple Banach algebra.

A bounded linear operator on $\mathcal{A}$ is called a multiplier if it satisfies $xT(y) = T(xy)$ for all $x, y \in \mathcal{A}$. The set $M(\mathcal{A})$ of all multipliers of $\mathcal{A}$ is a closed unital commutative subalgebra of the operator algebra $B(\mathcal{A})$, called the multiplier algebra of $\mathcal{A}$.
For each \( T \in M(\mathcal{A}) \) there exists a unique continuous function \( \hat{T} \) on \( \Delta(\mathcal{A}) \) such that \( \hat{T}(a)(\varphi) = T(\varphi)a(\varphi) \) for all \( a \in \mathcal{A} \) and \( \varphi \in \Delta(\mathcal{A}) \). See [9] for a proof. Write
\[
\hat{M}(\mathcal{A}) = \{ \hat{T} : T \in M(\mathcal{A}) \}.
\]
We say that a commutative Banach algebra \( \mathcal{A} \) without order is a \emph{BSE algebra} (or it is said to have the \emph{BSE property}) if \( \mathcal{A} \) satisfies the condition
\[
C_{BSE}(\Delta(\mathcal{A})) = \hat{M}(\mathcal{A}).
\]

**Remark 1.1.** Let \( \mathcal{A} \) be a semisimple Banach algebra and \( \Phi : \Delta(\mathcal{A}) \to \mathbb{C} \) be a continuous function such that \( \Phi \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}} \). We call \( \Phi \) a multiplier of \( \mathcal{A} \). This is another definition of a multiplier of a Banach algebra. In the presence of semisimplicity this definition is equivalent to the above definition, by considering \( \Phi = \hat{T} \); see [9] for more details. Define
\[
\mathcal{M}(\mathcal{A}) = \{ \Phi : \Delta(\mathcal{A}) \to \mathbb{C} : \Phi \text{ is continuous and } \Phi \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}} \}.
\]
When \( \mathcal{A} \) is a semisimple Banach algebra, \( \mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \).

The abbreviation BSE stands for Bochner–Schoenberg–Eberlein and refers to the famous theorem, proved by Bochner and Schoenberg [2, 13] for the additive group of real numbers and by Eberlein [3] for general locally compact abelian groups \( G \), saying that, in the above terminology, the group algebra \( L^1(G) \) is a BSE algebra. (See [11] for a proof.)

The notions of BSE algebra and BSE functions were introduced and studied by Takahasi and Hatori [14, 15] and later by Kaniuth and Ülger [8].

A bounded net \( (e_a)_a \) in \( \mathcal{A} \) is called a bounded approximate identity for \( \mathcal{A} \) if it satisfies \( ||e_a a - a|| \to 0 \) for all \( a \in \mathcal{A} \). A bounded net \( (e_a)_a \) in \( \mathcal{A} \) is called a \( \Delta \)-weak bounded approximate identity for \( \mathcal{A} \) if it satisfies \( \varphi(e_a) \to 1 \) (equivalently, \( \varphi(e_a a) \to \varphi(a) \) for every \( a \in \mathcal{A} \)) for all \( \varphi \in \Delta(\mathcal{A}) \). Such approximate identities were studied in [6], where the first example was given of a semisimple commutative Banach algebra which has a \( \Delta \)-weak approximate identity but does not possess a bounded approximate identity. As is shown in [14, Corollary 5], \( \mathcal{A} \) has a \( \Delta \)-weak bounded approximate identity if and only if \( \mathcal{M}(\mathcal{A}) \subseteq C_{BSE}(\Delta(\mathcal{A})) \).

In this paper we first present a characterisation of the multiplier algebra of the direct sum of two semisimple Banach algebras. Then as an application we show that for two semisimple Banach algebras \( \mathcal{A} \) and \( \mathcal{B} \), \( \mathcal{A} \oplus \mathcal{B} \) is BSE if and only if \( \mathcal{A} \) and \( \mathcal{B} \) are.

We also prove that if \( \mathcal{A} \) and \( \mathcal{B} \) are Banach algebras such that \( \mathcal{B} \) is unital and \( \mathcal{A} \times_\theta \mathcal{B} \) is BSE, then \( \mathcal{B} \) is a BSE algebra, and we present some examples showing that the BSE property of \( \mathcal{A} \times_\theta \mathcal{B} \) does not imply that of \( \mathcal{A} \).

**2. Direct sum of Banach algebras**

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two commutative Banach algebras. The direct sum algebra \( \mathcal{A} \oplus \mathcal{B} \) of \( \mathcal{A} \) and \( \mathcal{B} \) is defined as the Cartesian product \( \mathcal{A} \times \mathcal{B} \) with the algebra multiplication
\[
(a, a') \cdot (b, b') = (aa', bb') \quad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}),
\]
and with norm
\[||(a, b)|| = ||a|| + ||b|| \quad (a \in \mathcal{A}, b \in \mathcal{B}).\]

In this section we will give a characterisation of the multiplier algebra of the direct sum algebra \(\mathcal{A} \oplus \mathcal{B}\), \(\mathcal{M}(\mathcal{A} \oplus \mathcal{B})\), and then prove that for two semisimple Banach algebras \(\mathcal{A}\) and \(\mathcal{B}\), \(\mathcal{A} \oplus \mathcal{B}\) is a BSE algebra if and only if \(\mathcal{A}\) and \(\mathcal{B}\) are BSE algebras. First we need to prove the following lemma.

**Lemma 2.1.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be two Banach algebras and
\[E = \{(\varphi, 0) : \varphi \in \Delta(\mathcal{A})\}, \quad F = \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}.
\]
Then \(\Delta(\mathcal{A} \oplus \mathcal{B}) = E \cup F\).

**Proof.** It is obvious that \(E \cup F \subseteq \Delta(\mathcal{A} \oplus \mathcal{B})\). For the reverse inclusion suppose that \((\varphi, \psi) \in \Delta(\mathcal{A} \oplus \mathcal{B}) \subseteq \mathcal{A}' \oplus \mathcal{B}'\). Then, for every \(a_1, a_2 \in \mathcal{A}\) and \(b_1, b_2 \in \mathcal{B}\),
\[(\varphi, \psi)(a_1a_2, b_1b_2) = (\varphi, \psi)(a_1, b_1) \cdot (\varphi, \psi)(a_2, b_2).\]
This means that, for all \(a_1, a_2 \in \mathcal{A}\) and \(b_1, b_2 \in \mathcal{B}\),
\[\varphi(a_1a_2) + \psi(b_1b_2) = (\varphi(a_1) + \psi(b_1)) \cdot (\varphi(a_2) + \psi(b_2))\]
\[= \varphi(a_1)\varphi(a_2) + \varphi(a_1)\psi(a_2) + \psi(b_1)\varphi(a_2) + \psi(b_1)\psi(b_2).\] (I)

If we take \(b_1 = b_2 = 0\), it follows that \(\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)\) for all \(a_1, a_2 \in \mathcal{A}\). And similarly, if we take \(a_1 = a_2 = 0\) it follows that \(\psi(b_1b_2) = \psi(b_1)\psi(b_2)\). Then \(\varphi \in \Delta(\mathcal{A}) \cup \{0\}\) and \(\psi \in \Delta(\mathcal{B}) \cup \{0\}\). Now if \(\varphi = 0\), then \((\varphi, \psi) = (0, \psi) \in F\). If \(\varphi \neq 0\), then (I) implies that
\[\varphi(a_1)\psi(a_2) + \psi(b_1)\varphi(a_2) = 0,
\]
for all \(a_1, a_2 \in \mathcal{A}\) and \(b_1, b_2 \in \mathcal{B}\). If we set \(a_1 = 0\) and \(a_2\) such that \(\varphi(a_2) \neq 0\), it follows that \(\psi(b_1) = 0\) for all \(b_1 \in \mathcal{B}\) and then \(\psi = 0\). This means that \((\varphi, \psi) = (\varphi, 0) \in E\). So \(\Delta(\mathcal{A} \oplus \mathcal{B}) \subseteq E \cup F\).

**Remark 2.2.** Since \(E \cup F \subset (\mathcal{A} \oplus \mathcal{B})^* = \mathcal{A}' \oplus \mathcal{B}'\), its topology is the one induced from \(\mathcal{A}' \oplus \mathcal{B}'\) and is precisely the Gelfand topology of \(\Delta(\mathcal{A} \oplus \mathcal{B})\).

Note that Lemma 2.1 implies that \(\mathcal{A} \oplus \mathcal{B}\) is semisimple if and only if both \(\mathcal{A}\) and \(\mathcal{B}\) are semisimple.

**Theorem 2.3.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be semisimple Banach algebras. Then
\[\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) = \{(\Phi, \Psi) : \Phi \in \mathcal{M}(\mathcal{A}), \Psi \in \mathcal{M}(\mathcal{B})\},\]
where \((\Phi, \Psi)(\varphi, 0) = \Phi(\varphi)\) and \((\Phi, \Psi)(0, \psi) = \Phi(\psi)\) for all \((\varphi, 0) \in E\) and \((0, \psi) \in F\).
PROOF. Let $\Phi \in \mathcal{M}(\mathcal{A})$ and $\Psi \in \mathcal{M}(\mathcal{B})$. Since $\Phi \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$ and $\Psi \widehat{\mathcal{B}} \subseteq \widehat{\mathcal{B}}$, then for all $(\varphi, 0) \in E$ and $(a, b) \in \mathcal{A} \oplus \mathcal{B}$, there are elements $a' \in \mathcal{A}$ and $b' \in \mathcal{B}$ such that

$$(\Phi, \Psi) \cdot (a, b))(\varphi, 0) = (\Phi, \Psi)\varphi = (a, b),$$

for all $\varphi \in \Delta(\mathcal{A})$, and

$$(\Phi, \Psi) \cdot (a, b))(0, \psi) = (\Phi, \Psi)(0, \psi) = (a, b)(0, \psi) = \Psi a'(\varphi, 0) = \varphi,$$

for all $\psi \in \Delta(\mathcal{B})$. Then $((\Phi, \Psi) \cdot (a, b))(\varphi, 0) = (a', b')(\varphi, 0)$ and $((\Phi, \Psi) \cdot (a, b))(0, \psi) = (a', b')(0, \psi)$. This implies that

$$(\Phi, \Psi) \cdot \widehat{\mathcal{A}} \oplus \widehat{\mathcal{B}} \subseteq \widehat{\mathcal{A}} \oplus \widehat{\mathcal{B}}$$

and $(\Phi, \Psi) \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$.

Now let $F \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$. Define $\Phi(\varphi) = F(\varphi, 0)$ and $\Psi(\psi) = F(0, \psi)$, for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. So $F = (\Phi, \Psi)$. It is enough to show that $\Phi \in \mathcal{M}(\mathcal{A})$ and $\Psi \in \mathcal{M}(\mathcal{B})$. For all $a \in \mathcal{A}$, there exists $(a', b') \in \mathcal{A} \oplus \mathcal{B}$ such that

$$\Phi a(\varphi, 0) = \Phi(\varphi) a(0, 0) = (a', b')(\varphi, 0) = \varphi.$$

Then $\Phi \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$ and $\Phi \in \mathcal{M}(\mathcal{A})$. Similarly, $\Psi \in \mathcal{M}(\mathcal{A})$.

\[\Box\]

**Theorem 2.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be two semisimple Banach algebras. Then $\mathcal{A} \oplus \mathcal{B}$ is BSE if and only if $\mathcal{A}$ and $\mathcal{B}$ are BSE.

**Proof.** First suppose that $\mathcal{A}$ and $\mathcal{B}$ are BSE. Then by [14, Corollary 5] $\mathcal{A}$ and $\mathcal{B}$ have $\Delta$-weak bounded approximate identities. Let $\{e_a\}_a$ and $\{f_b\}_b$ be $\Delta$-weak bounded approximate identities of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\{(e_a, f_b)\}_{(a,b)}$ is a $\Delta$-weak bounded approximate identity for $\mathcal{A} \oplus \mathcal{B}$. Indeed, for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$,

$$\lim_{(a, b)} (e_a, f_b) = \lim_{a} \varphi(e_a) = 1$$

and

$$\lim_{(a, b)} (0, \psi) = \lim_{b} \psi(f_b) = 1,$$

so that, for all $\Phi \in E \cup F = \Delta(\mathcal{A} \oplus \mathcal{B})$,

$$\lim_{(a, b)} F(e_a, f_b) = 1,$$

and $\{(e_a, f_b)\}_{(a,b)}$ is a $\Delta$-weak approximate identity for $\mathcal{A} \oplus \mathcal{B}$. Then by [14, Corollary 5]

$$\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) \subseteq C_{\text{BSE}}(\Delta(\mathcal{A} \oplus \mathcal{B})).$$

For the reverse conclusion, let $\sigma \in C_{\text{BSE}}(\Delta(\mathcal{A} \oplus \mathcal{B}))$. Then, by [14, Theorem 4(ii)], $\sigma \in C_b(\Delta(\mathcal{A} \oplus \mathcal{B})) \cap (\mathcal{A} \oplus \mathcal{B})^{**} = C_b(\Delta(\mathcal{A} \oplus \mathcal{B})) \cap (\mathcal{A}^{**} \oplus \mathcal{B}^{**})^{**}(\Delta(\mathcal{A} \oplus \mathcal{B}))$. Then
there are \( \sigma_1 \in \mathcal{A}^{**} \) and \( \sigma_2 \in \mathcal{B}^{**} \) such that \( \sigma_1|_{\Delta(\mathcal{A})} \in C_b(\Delta(\mathcal{A})) \cap \mathcal{A}^{**}|_{\Delta(\mathcal{A})} \), \( \sigma_2|_{\Delta(\mathcal{B})} \in C_b(\Delta(\mathcal{B})) \cap \mathcal{B}^{**}|_{\Delta(\mathcal{B})} \) and \( \sigma = (\sigma_1, \sigma_2)|_{\Delta(\mathcal{A} \oplus \mathcal{B})} \). On the other hand, since \( \sigma \in C_{BS E}(\Delta(\mathcal{A} \oplus \mathcal{B})) \), there exists \( \beta > 0 \) such that, for every finite number of \( c_1, \ldots, c_n \in \mathbb{C} \) and \( (\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n) \in \Delta(\mathcal{A} \oplus \mathcal{B}) \),

\[
\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i, \psi_i) \right| \leq \beta \left| \sum_{i=1}^{n} c_i (\varphi_i, \psi_i) \right|_{\mathcal{A} \oplus \mathcal{B}^*}.
\]

In particular, for every \( (\varphi_1, 0), \ldots, (\varphi_n, 0) \in E \) and \( c_1, \ldots, c_n \in \mathbb{C} \),

\[
\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i, 0) \right| = \left| \sum_{i=1}^{n} c_i \sigma_1(\varphi_i) \right| \\
\leq \beta \left| \sum_{i=1}^{n} c_i (\varphi_i, 0) \right|_{\mathcal{A} \oplus \mathcal{B}^*} \\
= \beta \sup\left\{ \left| \sum_{i=1}^{n} c_i (\varphi_i, 0)(a, b) \right| : ||a|| + ||b|| \leq 1 \right\} \\
\leq \beta \sup\left\{ \left| \sum_{i=1}^{n} c_i \varphi_i(a) \right| : ||a|| \leq 1 \right\} \\
= \beta \left| \sum_{i=1}^{n} c_i \varphi_i \right|_{\mathcal{A}}.
\]

This means that \( \sigma_1 \in C_{BS E}(\Delta(\mathcal{A})) \). Now since \( \mathcal{A} \) is a semisimple BSE algebra, \( \sigma_1 \in M(\mathcal{A}) \). In a similar way (by considering \( (0, \psi_1), \ldots, (0, \psi_n) \)) we conclude that \( \sigma_2 \in M(\mathcal{B}) \). So \( \sigma = (\sigma_1, \sigma_2) \in M(\mathcal{A} \oplus \mathcal{B}) \). Then \( C_{BS E}(\Delta(\mathcal{A} \oplus \mathcal{B})) \subseteq M(\mathcal{A} \oplus \mathcal{B}) \) and \( \mathcal{A} \oplus \mathcal{B} \) is a BSE algebra.

Now suppose that \( \mathcal{A} \oplus \mathcal{B} \) is BSE and let \( \{(e_\alpha, f_\alpha)\}_\alpha \) be a \( \Delta \)-weak bounded approximate identity for \( \mathcal{A} \oplus \mathcal{B} \). Then, for all \( \varphi \in \Delta(\mathcal{A}) \),

\[
\lim_{\alpha} \varphi(e_\alpha) = \lim_{\alpha} \varphi(0)(e_\alpha, f_\alpha) = 1.
\]

So \( \{e_\alpha\}_\alpha \) is a \( \Delta \)-weak bounded approximate identity for \( \mathcal{A} \), and similarly \( \{f_\alpha\}_\alpha \) is a \( \Delta \)-weak bounded approximate identity for \( \mathcal{B} \), and [14, Corollary 5] implies that \( M(\mathcal{A}) \subseteq C_{BS E}(\Delta(\mathcal{A})) \) and \( M(\mathcal{B}) \subseteq C_{BS E}(\Delta(\mathcal{B})) \).

Now let \( \sigma_1 \in C_{BS E}(\Delta(\mathcal{A})) \) and \( \sigma_2 \in C_{BS E}(\Delta(\mathcal{B})) \). Then by [14, Theorem 4(i)] there are nets \( \{x_\lambda\}_\lambda \in \mathcal{A} \) and \( \{y_\mu\}_\mu \in \mathcal{B} \) such that \( \lim_{\lambda} \widehat{x}_\lambda(\varphi) = \sigma_1(\varphi) \) and \( \lim_{\mu} \widehat{y}_\mu(\psi) = \sigma_2(\psi) \), for all \( \varphi \in \Delta(\mathcal{A}) \) and \( \psi \in \Delta(\mathcal{B}) \). If we consider the net \( \{(x_\lambda, y_\mu)\}_{(\lambda,\mu)} \subset \mathcal{A} \oplus \mathcal{B} \), then

\[
\lim_{(\lambda,\mu)} (\widehat{x}_\lambda, \widehat{y}_\mu)(\varphi, 0) = \lim_{(\lambda,\mu)} (\varphi, 0)(x_\lambda, y_\mu) = \lim_{(\lambda,\mu)} \varphi(x_\lambda) + 0 \\
= \lim_{(\lambda,\mu)} \widehat{x}_\lambda(\varphi) + 0 = \sigma_1(\varphi) + 0 \\
= (\sigma_1,! \sigma_2)(\varphi, 0),
\]
for all $\varphi \in \Delta(\mathcal{A})$. And similarly, for all $\psi \in \Delta(\mathcal{B})$,

$$
\lim_{(\lambda,\mu)} \widehat{(x_\lambda, y_\mu)}(0, \psi) = (\sigma_1, \sigma_2)(0, \psi).
$$

This means that if we let $\sigma = (\sigma_1, \sigma_2)$ then

$$
\lim_{(\lambda,\mu)} \widehat{(x_\lambda, y_\mu)}(\Phi) = \sigma(\Phi),
$$

for all $\Phi \in E \cup F = \Delta(\mathcal{A} \oplus \mathcal{B})$. Then $\sigma = (\sigma_1, \sigma_2) \in C_{BS}(\Delta(\mathcal{A} \oplus \mathcal{B}))$. Now since $\mathcal{A} \oplus \mathcal{B}$ is a BSE algebra, $\sigma = (\sigma_1, \sigma_2) \in M(\mathcal{A} \oplus \mathcal{B})$ and, by Theorem 2.3, $\sigma_1 \in M(\mathcal{A})$ and $\sigma_2 \in M(\mathcal{B})$. So $\mathcal{A}$ and $\mathcal{B}$ are BSE algebras.

Let $G$ be a locally compact abelian group and $M(G)$ the Banach algebra of bounded regular measures on $G$. The set of continuous measures in $M(G)$ is denoted by $M_c(G)$. This is a closed ideal in $M(G)$, and

$$
M(G) = M_d(G) \oplus M_c(G) = L^1(G) \oplus M_c(G).
$$

When $G$ is not discrete, $M_c(G) \neq \{0\}$. It is shown in [14] that $M(G)$ is a BSE algebra if and only if $G$ is discrete. So we have the following result.

**Corollary 2.5.** For a nondiscrete locally compact abelian group $G$, $M_c(G)$ is not a BSE algebra.

**Corollary 2.6.** Let $G$ be a locally compact abelian group and $\mu \in M(G)$ be a measure which factors as a product of an invertible measure and an idempotent measure. Then $\mu * L^1(G)$ is a BSE algebra.

**Proof.** Define $T_\mu(f) = \mu * f$. Then $T_\mu$ is a multiplier of the Banach algebra $L^1(G)$. It is shown in [16] that $T(L^1(G)) = \mu * L^1(G)$ is closed in $L^1(G)$. By [4, Theorem 3.4], since $L^1(G)$ is a commutative semisimple amenable Banach algebra, it factors as follows:

$$
L^1(G) = T_\mu(L^1(G)) \oplus \text{Ker}(T_\mu) = (\mu * L^1(G)) \oplus \text{Ker}(T_\mu).
$$

Since $L^1(G)$ is a BSE algebra, by Theorem 2.4, $\mu * L^1(G)$ is BSE as well.

3. **$\theta$-Lau product of Banach algebras**

The products $\mathcal{A} \times_\theta \mathcal{B}$ of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ were first introduced and studied by Lau [10]. The Banach algebra $\mathcal{B}$ inherits some properties of $\mathcal{A} \times_\theta \mathcal{B}$. For instance, from [7, Proposition 2.1], for $n \in \mathbb{N}$, $n$-ideal amenability of $\mathcal{A} \times_\theta \mathcal{B}$ implies that of $\mathcal{B}$. In this section we prove that if the $\theta$-Lau product of two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \times_\theta \mathcal{B}$, with $\mathcal{B}$ unital, is BSE, then $\mathcal{B}$ is a BSE algebra. Before that we need to present some definitions and preliminaries. More results on this product can be found in [12].
**Definition 3.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebras for which $\Delta(\mathcal{B}) \neq \emptyset$. Let $\theta \in \Delta(\mathcal{B})$. The $\theta$-Lau product $\mathcal{A} \times_{\theta} \mathcal{B}$ is defined as the Cartesian product $\mathcal{A} \times \mathcal{B}$ with the algebra multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + \theta(b_1)a_2 + \theta(b_2)a_1, b_1b_2)$$

and norm $\|(a, b)\| = \|a\| + \|b\|$.

Obviously if $\mathcal{B}$ is a unital Banach algebra, then $\mathcal{A} \times_{\theta} \mathcal{B}$ is a unital Banach algebra for any Banach algebra $\mathcal{A}$.

**Remark 3.2.** The space $\mathcal{A} \times_{\theta} \mathcal{B}$ is a Banach algebra. If we allow $\theta = 0$, we obtain the usual direct sum of Banach algebras. If $\mathcal{B} = \mathbb{C}$ and $\theta : \mathbb{C} \to \mathbb{C}$ is the identity map, then $\mathcal{A} \times_{\theta} \mathbb{C}$ coincides with the unitisation $\mathcal{A}_e$ of $\mathcal{A}$.

The dual of the space $\mathcal{A} \times_{\theta} \mathcal{B}$ can be identified with $\mathcal{A}^* \oplus \mathcal{B}^*$ in the natural way: $(\varphi, \psi)(a, b) = \varphi(a) + \psi(b)$. The dual norm on $\mathcal{A}^* \oplus \mathcal{B}^*$ is the maximum norm $\|\varphi, \psi\| = \max\{\|\varphi\|, \|\psi\|\}$. On $\mathcal{A}^* \oplus \mathcal{B}^*$, the weak* topology coincides with the product of the weak* topologies of $\mathcal{A}^*$ and $\mathcal{B}^*$. The following theorem, which is proved in [12], identifies the Gelfand spectrum $\Delta(\mathcal{A} \times_{\theta} \mathcal{B})$ of $\mathcal{A} \times_{\theta} \mathcal{B}$.

**Theorem 3.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with $\Delta(\mathcal{B}) \neq \emptyset$. Let $\theta \in \Delta(\mathcal{B})$ and

$$E = \{(\varphi, \theta) : \varphi \in \Delta(\mathcal{A})\}, \quad F = \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}.$$

Set $E = \emptyset$ if $\Delta(A) = \emptyset$. Then $\Delta(A \times_{\theta} B) = E \cup F$.

Note that the topology on $E \cup F = \Delta(\mathcal{A} \times_{\theta} \mathcal{B})$ is the weak* topology induced from $\Delta(\mathcal{A} \times_{\theta} \mathcal{B})^* = \mathcal{A}^* \oplus \mathcal{B}^*$ and it is precisely the Gelfand topology on $\Delta(\mathcal{A} \times_{\theta} \mathcal{B})$.

**Theorem 3.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebras. Suppose that $\mathcal{A} \times_{\theta} \mathcal{B}$ is a BSE algebra. Then $\mathcal{B}$ is a BSE algebra.

**Proof.** Let $\sigma \in C_{\text{BSE}}(\Delta(\mathcal{B}))$. Then by [14, Theorem 4(i)], there exists a bounded net $(y_\lambda)_\lambda \subset \mathcal{B}$ such that $\lim_\lambda \widehat{y}_\lambda(\psi) = \sigma(\psi)$, for all $\psi \in \Delta(\mathcal{B})$. If we consider the bounded net $\{(0, y_\lambda)_\lambda \subset \mathcal{A} \times_{\theta} \mathcal{B}$,

$$\lim_\lambda \widehat{(0, y_\lambda)}(0, \psi) = \lim_\lambda \widehat{(0, y_\lambda)}(0, y_\lambda) = \lim_\lambda \widehat{y}_\lambda(\psi) = \sigma(\psi) = (0, \sigma)(0, \psi),$$

for all $(0, \psi) \in E$. Also, for all $(\varphi, \theta) \in F$,

$$\lim_\lambda \widehat{(0, y_\lambda)}(\varphi, \theta) = \lim_\lambda \widehat{(0, y_\lambda)}(\varphi, \theta)(0, y_\lambda) = \lim_\lambda \widehat{y}_\lambda(\theta) = \sigma(\theta) = (0, \sigma)(\varphi, \theta).$$

Consequently, for all $\Phi \in E \cup F = \Delta(\mathcal{A} \times_{\theta} \mathcal{B})$,

$$\lim_\lambda \widehat{(0, y_\lambda)}(\Phi) = (0, \sigma)(\Phi).$$
Then \((0, \sigma) \in C_{BSE}(\Delta(\mathcal{A} \times_\theta \mathcal{B}))\) and since \(\mathcal{A} \times_\theta \mathcal{B}\) is BSE, \((0, \sigma) \in (\mathcal{A} \times_\theta \mathcal{B})\). So there exists \((a, b) \in \mathcal{A} \times_\theta \mathcal{B}\) such that \((a, b) = (0, \sigma)\). Then, for all \(\psi \in \Delta(\mathcal{B})\),

\[
\sigma(\psi) = (0, \sigma)(0, \psi) = (a, b)(0, \psi) = (0, \psi)(a, b) = \tilde{b}(\psi).
\]

This means that \(\sigma \in \hat{\mathcal{B}}\) and since \(\mathcal{B}\) is unital, \(\mathcal{B}\) is a BSE algebra. \(\square\)

The following examples show that if \(\mathcal{A} \times_\theta \mathcal{B}\), for \(\mathcal{B}\) unital, is BSE we cannot conclude that \(\mathcal{A}\) is BSE in general. Before that we need to present a result proved by Kaniuth and Ülger, [8, Theorem 4.8].

**Theorem 3.5.** Let \(\mathcal{A}\) be a nonunital commutative Banach algebra. Then the unitisation \(\mathcal{A}_e\) of \(\mathcal{A}\) is a BSE algebra if and only if

\[
C_{BSE}(\Delta(\mathcal{A})) \cap C_0(\Delta(\mathcal{A})) = \hat{\mathcal{A}}.
\]

**Example 3.6.**

1. Let \(G\) be a second countable noncompact locally compact group whose regular representation is not completely reducible and \(A(\mathcal{G})\) be the Fourier algebra of \(G\). Then \(A(\mathcal{G}) \neq B(\mathcal{G}) \cap C_0(\mathcal{G})\) (see [1, 5, 8]). Thus if in addition \(G\) is amenable, then \(A(\mathcal{G})\) is a BSE algebra [8], but \(A(\mathcal{G})_e = A(\mathcal{G}) \times_\theta \mathbb{C}\), such that \(\theta : \mathbb{C} \to \mathbb{C}\) is the identity map, is not BSE, by Theorem 3.5.

2. Let \(l^1(\mathbb{N})\) be the semigroup algebra of the additive semigroup of natural numbers. Then \(l^1(\mathbb{N})\) is not a BSE algebra [14]. However, the semigroup algebra \(l^1(\mathbb{N} \cup \{0\}) = l^1(\mathbb{N})_e = l^1(\mathbb{N}) \times_\theta \mathbb{C}\), such that \(\theta : \mathbb{C} \to \mathbb{C}\) is the identity map, is a BSE algebra by [15, Theorem 6].

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**References**


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