ON THE SEMIGROUP OF HADAMARD DIFFERENTIABLE MAPPINGS

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The main purpose of this paper is to prove that every automorphism of the semigroup of all Hadamard-differentiable mappings of a separable real Banach space into itself is inner. This generalizes the result of [7] which is a generalization of a result proved by Magill, Jr. [5].

We start with a brief account on the Hadamard differentiation.

1. The Hadamard differentiation

The following method of defining derivatives has been given by Averbukh and Smolyanov [1, 2], where it was also proved that the Hadamard differentiability defined below is equivalent to the quasi-differentiability defined by Dieudonné [3, p. 151].

Let E be a real Banach space, and let $\boldsymbol{M}$ be a class of some subsets of E. We denote by $\mathscr{L} = \mathscr{L}(E)$ the Banach algebra of all continuous linear mappings of E into itself with the usual algebraic structure and the upper bound norm. Then, a mapping $f: E \to E$ is said to be *$\boldsymbol{M}$-differentiable at* $a \in E$ if there exists $u \in \mathscr{L}$ such that, for any $M \in \boldsymbol{M}$,

$$\sup_{x \in M} \| \varepsilon^{-1} r(f, a, \varepsilon x) \| \to 0 \quad \text{if} \quad \varepsilon \to 0,$$

where

$$r(f, a, x) = f(a + x) - f(a) - u(x).$$

We denote by $\mathscr{D}_{\boldsymbol{M}}$ the set of all $f: E \to E$ which are $\boldsymbol{M}$-differentiable at every point of E.

(1) If $\boldsymbol{M}$ is the class of all single point sets, then $f \in \mathscr{D}_{\boldsymbol{M}}$ is said to be *Gâteaux-differentiable*. In this case, we denote $\mathscr{D}_{\boldsymbol{M}}$ by $\mathscr{D}_G$.

(2) If $\boldsymbol{M}$ is the class of all compact subsets, then $f \in \mathscr{D}_{\boldsymbol{M}}$ is said to be *Hadamard-differentiable*. In this case, we denote $\mathscr{D}_{\boldsymbol{M}}$ by $\mathscr{D}_H$.

(3) If $\boldsymbol{M}$ is the class of all bounded subsets, then $f \in \mathscr{D}_{\boldsymbol{M}}$ is said to be *Fréchet-differentiable*. In this case, we denote $\mathscr{D}_{\boldsymbol{M}}$ by $\mathscr{D}_F$.

In each of these cases, the continuous linear mapping u is determined uniquely and is denoted by $f'(a)$. It is called the *Gâteaux-*, the *Hadmarad-* or the *Fréchet-derivative of f at a* respectively.

Obviously,

$$\mathscr{L} \subset \mathscr{D}_F \subset \mathscr{D}_H \subset \mathscr{D}_G,$$

and the inclusions are generally strict. If E is finite-dimensional, we have $\mathscr{D}_F = \mathscr{D}_G$, and if E is one dimensional we have $\mathscr{D}_F = \mathscr{D}_H = \mathscr{D}_G$.

The following theorem will be used in the following section. We shall denote by $\mathscr{K}$ the set of all completely continuous (i.e,. continuous and compact) mapping of E into itself. We also denote by $\mathscr{K}_1$ the set of all $f: E \to E$ such that

$$f(x) = \mu(\langle x, \bar{a}\rangle) \quad \text{for every} \quad x \in E,$$

where μ is any differentiable E-valued function of a real variable, $\bar{a} \in \bar{E}$ (the conjugate space of E and $\langle x, \bar{a}\rangle$ is the value of $\bar{a}$ at x. Obviously, $\mathscr{K}_1 \subset \mathscr{K}$.

In the sequel, the composition of two mappings $f, g: E \to E$ is denoted by fg that is,

$$(fg)(x) = f(g(x)) \quad \text{for every} \quad x \in E.$$

THEOREM 1. 1) *If $f \in \mathscr{D}_H$, then $fk \in \mathscr{D}_F$ for any $k \in \mathscr{D}_F \cap \mathscr{K}$ and*

$$(*) \qquad (fk)'(a) = f'(k(a))k'(a) \quad \textit{for any} \quad a \in E.$$

2) *If $f \in \mathscr{D}_G$, and if for every $k \in \mathscr{D}_F \cap \mathscr{K}_1$ it is true that $fk \in \mathscr{D}_G$ and (*) is satisfied, then $f \in \mathscr{D}_H$.*

PROOF. 1) For $f \in \mathscr{D}_H$ and $k \in \mathscr{D}_F \cap \mathscr{K}$,

$$\begin{aligned} &fk(a+x) - fk(a) - f'(k(a))k'(a)(x) \\ &\quad = f'(k(a))[k(a+x) - k(a)] + r(f, k(a), k(a+x) - k(a)) - \\ &\qquad\qquad - f'(k(a))k'(a)(x) \\ &\quad = f'(k(a))r(k, a, x) + r(f, k(a), k(a+x) - k(a)). \end{aligned}$$

Then, for a bounded set B, since $k \in \mathscr{D}_F$,

$$\begin{aligned} &\sup_{x \in B} \| \varepsilon^{-1} f'(k(a)) r(k, a, \varepsilon x) \| \\ &\qquad \leqq \|f'(k(a))\| \sup_{x \in B} \| \varepsilon^{-1} r(k, a, \varepsilon x)\| \to 0 \quad \text{if } \varepsilon \to 0, \end{aligned}$$

and

$$\begin{aligned} &\sup_{x \in B} \| \varepsilon^{-1} r(f, k(a), k(a+\varepsilon x) - k(a)) \| \\ &\quad = \sup_{x \in B} \| \varepsilon^{-1} r(f, k(a), \varepsilon[\varepsilon^{-1}(k(a+\varepsilon x) - k(a))]) \| \to 0 \quad \text{if } \varepsilon \to 0, \end{aligned}$$

because, since $k \in \mathscr{K}$, for any $\varepsilon_n \to 0$, the set

$$\{\varepsilon_n^{-1}(k(a+\varepsilon_n x)-k(a)) \mid x \in B, \quad n=1,2,\cdots\}$$

is contained in a compact set. In fact, since

$$\varepsilon_n^{-1}(k(a+\varepsilon_n x)-k(a)) = k'(a)(x)+\varepsilon_n^{-1}r(k,a,\varepsilon_n x),$$

the fact that $k'(a) \in \mathscr{K}$ ([6, p. 27]) implies that $\{k'(a)(x) \mid x \in B\}$ is contained in a compact set and the fact that $k \in \mathscr{D}_F$ implies that the second term converges to 0 as $n \to \infty$. Therefore,

$$fk \in \mathscr{D}_F \quad \text{and} \quad (fk)'(a) = f'(k(a))k'(a).$$

2) Let us assume that $f \notin \mathscr{D}_H$. Then, there exist $\varepsilon_n \downarrow 0$, $a \in E$ and $x_n \to x_0$ such that

$$\varepsilon_n^{-1}r(f,a,\varepsilon_n x_n) \not\to 0 \quad \text{as} \quad n \to \infty.$$

Now, the method used in [2, p. 92] supplies a differentiable E-valued function $\mu(\xi)$ of a real variable such that

$$\mu(0) = a, \qquad \mu(\varepsilon_n) = a+\varepsilon_n x_n \quad \text{and} \quad \mu'(0) = x_0.$$

Then, consider the mapping $k \in \mathscr{K}_1$ defined by

$$k(x) = \mu(\langle x, \bar{a}\rangle),$$

where $\bar{a} \in \bar{E}$ and $\langle a, \bar{a}\rangle = 1$. By the assumption,

$$fk \in \mathscr{D}_G \quad \text{and} \quad (fk)'(0) = f'(k(0))k'(0).$$

On the other hand,

$$\begin{aligned} k'(0)(a) &= \lim_{\varepsilon\to 0} \varepsilon^{-1}[k(\varepsilon a)-k(0)] \\ &= \lim_{\varepsilon\to 0} \varepsilon^{-1}[\mu(\varepsilon)-\mu(0)] = \mu'(0) = x_0, \end{aligned}$$

and

$$\begin{aligned} &\varepsilon_n^{-1}r(f,a,\varepsilon_n x_n) \\ &\quad = \varepsilon_n^{-1}[f(a+\varepsilon_n x_n)-f(a)-f'(a)(\varepsilon_n x_n)] \\ &\quad = \varepsilon_n^{-1}[fk(\varepsilon_n a)-fk(0)-(fk)'(0)(\varepsilon_n a)]+(fk)'(0)(a)-f'(a)(x_n) \\ &\quad = \varepsilon_n^{-1}r(fk,0,\varepsilon_n a)+f'(a)(x_0-x_n) \to 0 \quad \text{if} \quad n\to\infty, \end{aligned}$$

which is a contradiction.

2. $\mathscr{D}_H$ as a semigroup

It is well-known that if $f, g \in \mathscr{D}_F$ then $fg \in \mathscr{D}_F$. In other words, $\mathscr{D}_F$ is a semigroup with respect to the composition. For any semigroup $\mathscr{D}$, a one-to-one mapping ϕ of $\mathscr{D}$ onto itself is called an *automorphism* if

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$$\phi(fg) = \phi(f)\phi(g) \quad \text{for} \quad f, g \in \mathscr{D}.$$

If there exists $h \in \mathscr{D}$ such that it has the two-sided inverse h^{-1} in $\mathscr{D}$ and

$$\phi(f) = hfh^{-1} \quad \text{for every} \quad f \in \mathscr{D}$$

then the automorphism is said to be *inner*.

Eidelheit [4] has proved that every continuous automorphism of the semigroup $\mathscr{L}$ is inner.

On the other hand, Magill, Jr. [5] has proved that, if E is one-dimensional, every automorphism of the semigroup $\mathscr{D}_F$ ($= \mathscr{D}_H = \mathscr{D}_G$) is inner.

These two results take us naturally to the question whether every automorphism of the semigroup $\mathscr{D}_F$ on a general Banach space is inner.

Eidelhiet's result suggests that we may need some continuity assumptions. In fact, in [9], we have shown that, in the semigroup of all boundely and continuously differentiable mappings, where the topology is defined by

$$\|f\|_n = \sup_{\|x\| \leq n} \{\|f(x)\| + \|f'(x)\|\}, \quad n = 1, 2, \cdots,$$

an automorphism is inner if and only if it is continuous.

On the other hand, in [8] we have given a necessary and sufficient condition for an automorphism ϕ of $\mathscr{D}_F$ to be inner. The method used there has been refined in [7], where we have generalized the Magill's result mentioned above to arbitrary finite-dimensional Banach spaces.

Now, we turn to the set $\mathscr{D}_H$. As Averbukh and Smolyanov [1, 2] have pointed out, the Hadamard differentiation is, in a sense, the weakest differentiation which has the composition property: if $f, g \in \mathscr{D}_H$, then $fg \in \mathscr{D}_H$ and

$$(fg)'(a) = f'(g(a))g'(a) \quad \text{for every} \quad a \in E.$$

Moreover, if E is finite-dimensional, then $\mathscr{D}_F = \mathscr{D}_H$. Therefore, the following result is a generalization of that of [7]:

THEOREM 2. *Let E be separable. Then, every automorphism of the semigroup $\mathscr{D}_H$ is inner.*

PROOF. Let ϕ be an automorphism. Exactly the same argument as in [7], if $\mathscr{D}_F$ there is replaced by $\mathscr{D}_H$, gives the following facts:

(1) there exists a unique one-to-one mapping h of E onto E such that

$$\phi(f) = hfh^{-1} \quad \text{for every} \quad f \in \mathscr{D}_H.$$

(2) $h \in \mathscr{D}_G$ and $h^{-1} \in \mathscr{D}_G$;

(3) $(a \otimes \bar{a})h \in \mathscr{D}_H$ for any $a \in E$ and $\bar{a} \in \bar{E}$, where $a \otimes \bar{a}$ is an element of $\mathscr{L}$ defined by

$$(a \otimes \bar{a})(x) = \langle x, \bar{a} \rangle a \quad \text{for every} \quad x \in E;$$

and

(4) $((a \otimes \bar{a})h)'(x)(y) = \langle h'(x)(y), \bar{a}\rangle a$

We shall prove that $h \in \mathscr{D}_H$. Since we may start with ϕ^{-1} instead of ϕ, we use the fact that any result containing h remains true if we replace h by h^{-1}.

Now, by Theorem 1, we have only to prove that

$$hk_1 \in \mathscr{D}_G \quad \text{for any } k_1 \in \mathscr{K}_1 \cap \mathscr{D}_F$$

and

$$(hk_1)'(x) = h'(k_1(x))k_1'(x).$$

Let us take an arbitrary $k_1 \in \mathscr{K}_1$:

$$k_1(x) = \mu(\langle x, \bar{a}\rangle),$$

and let $a \in E$ be such that $\langle a, \bar{a}\rangle = 1$. Then, we have $k_1 = k_1(a \otimes \bar{a})$. Since Since $a \otimes \bar{a} \in \mathscr{L} \subset \mathscr{D}_H$, there exists $k \in \mathscr{D}_H$ such that

$$\phi(k) = a \otimes \bar{a}.$$

Since

$$k(x) = h^{-1}(\langle h(x), \bar{a}\rangle a),$$

where $\langle h(x), \bar{a}\rangle$ is continuous by [8, p. 506] and $h^{-1}(\xi a)$ is continuous with respect to ξ by (2) above, we see that $k \in \mathscr{K}$. Therefore, from (3) it follows that

$$(a \otimes \bar{a})hk \in \mathscr{D}_H.$$

Since

$$(a \otimes \bar{a})hk(x) = \langle hk(x), \bar{a}\rangle a,$$

the mapping $\langle hk(x), \bar{a}\rangle$ of E into the set of real numbers is Hadamard-differentiable. Therefore, the mapping $\mu(\langle hk(x), \bar{a}\rangle)$ of E into E is Hadamard-differentiable and obviously,

$$\mu(\langle hk(x), \bar{a}\rangle) = k_1hk(x) \quad \text{for every} \quad x \in E.$$

In other words,

$$k_1hk \in \mathscr{D}_H.$$

Therefore,

$$\phi(k_1hk) \in \mathscr{D}_H$$

and

$$\phi(k_1hk) = hk_1hkh^{-1} = hk_1\phi(k) = hk_1(a \otimes \bar{a}) = hk_1,$$

from which it follows that

$$hk_1 \in \mathscr{D}_H.$$

Thus, it only remains to prove the equality (*) of Theorem 1. First, since $a \otimes \bar{a} \in \mathscr{D}_H$ and $hk_1 \in \mathscr{D}_H$, we have

$$(a \otimes \bar{a})hk_1 \in \mathscr{D}_H \quad \text{and} \quad ((a \otimes \bar{a})hk_1)'(x)(y) = \langle (hk_1)'(x)(y), \bar{a} \rangle a.$$

Also, by applying Theorem 1, 1) to $(a \otimes \bar{a})h$ and k_1 we have

$$((a \otimes \bar{a})hk_1)'(x)(y) = ((a \otimes \bar{a})h)'(k_1(x))k_1'(x)(y)$$

and by (3) and (4) the right hand side here is $\langle h'(k_1(x))k_1'(x)(y), \bar{a} \rangle a$. Therefore,

$$\langle (hk_1)'(x)(y), \bar{a} \rangle a = \langle h'(k_1(x))k_1'(x)(y), \bar{a} \rangle a.$$

Since $\bar{a}$ is arbitrary, (*) follows, Thus. $h \in \mathscr{D}_H$ and hence ϕ is inner.

Remark. With the product defined above and the addition $f + g$ defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{for every} \quad x \in E,$$

$\mathscr{D}_H$ is a near-ring. The fact that every near-ring automorphism of $\mathscr{D}_H$ is inner can be proved in the same way as in [9]. In this case, h is in $\mathscr{L}$.

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