ON INJECTIVES IN SOME VARIETIES OF OCKHAM ALGEBRAS

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0. Introduction. The study of bounded distributive lattices endowed with an additional dual homomorphic operation began with a paper by J. Berman [**3**]. Subsequently these algebras were called distributive Ockham lattices and an order-topological duality theory for them was developed by A. Urquhart [**12**]. In [**9**], M. S. Goldberg extended this theory and described the injective algebras in the subvarieties of the variety $\mathcal{O}$ of distributive Ockham algebras which are generated by a single subdirectly irreducible algebra. The aim here is to investigate some elementary properties of injective algebras in join reducible members of the lattice of subvarieties of $K_{n,1}$ and to give a complete description of injective algebras in the subvarieties of the Ockham subvariety defined by the identity $x \wedge f^{2n}(x) = x$.

1. Preliminaries. A *distributive Ockham algebra* is an algebra $(A, \wedge, \vee, f, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and f is a unary operation defined on A such that, for all $x, y \in A$, $f(x \wedge y) = f(x) \vee f(y)$, $f(x \vee y) = f(x) \wedge f(y)$, $f(0) = 1$, $f(1) = 0$; i.e., f is a dual endomorphism of the lattice $(A, \wedge, \vee, 0, 1)$. The class of all distributive Ockham algebras is a variety henceforth denoted by $\mathcal{O}$. Let $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$, for all $n \geq 0$. For $n \geq 1$, $m \geq 0$ the subvariety of $\mathcal{O}$ defined by the identity $f^{2n+m}(x) = f^m(x)$ is denoted by $K_{n,m}$ and the subvariety of $\mathcal{O}$ defined by the identity $f^{2n}(x) \wedge x = x$ is denoted by $K^{\leq}_{n,0}$. For each $n \geq 1$, the proper inclusions $K_{n,0} \subset K^{\leq}_{n,0} \subset K_{n,1}$ hold; see [**10**].

Let K be a class of similar algebras. An algebra $I \in K$ is said to be (*weak*) *injective in* K if, for any algebra $A \in K$, any (onto) homomorphism from any subalgebra of A to I can be extended to a homomorphism from A to I. We say that I is a *retract* of $A \in K$ if there exist homomorphisms $r: A \to I$ and $s: I \to A$ such that $r \circ s = \mathrm{id}_I$; also I is an *absolute subretract* in K if its is a retract of each of its extensions in K. As usual, $\mathrm{Si}(K)$ denotes the set consisting of precisely one algebra from each of the isomorphism classes of the subdirectly irreducible (s.i.) algebras in K and $H(K)$ denotes the class of all homomorphic images of members of K. Let $S(K)$, $P(K)$ and $P_s(K)$ be the classes consisting of all isomorphic copies of subalgebras, direct products and subdirect products of members of K, respectively. Also let $V(K)$ denote the variety generated by K.

We recall that every retract of injective algebra in K is injective in K ([**1**]).

Let K be a class of algebras and for each $A \in K$, let $\Theta_A(a, b)$ denote the principal congruence of A collapsing a pair $a, b \in A$. The trivial and the universal congruences are denoted by Δ and ∇, respectively. A simplicity formula for K is a $\exists\forall$ conjunction of equations

$$\sigma(u, v) = (\exists x)(\forall y)\left\{ \mathop{\&}_{i=1}^{n} p_i(x, y, u, v) = q_i(x, y, u, v) \right\}$$

such that, for each $A \in K$, $\sigma(u, v)$ holds in $A \Leftrightarrow \Theta_A(u, v) \in \{\Delta, \nabla\}$.

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B. A. Davey and H. Werner [7, Theorem 1.3], described the injective algebras by using Boolean powers in those congruence-distributive varieties for which there exists a simplicity formula for the maximal s.i. algebras.

Since Goldberg [9, p. 200], has shown that there exists a simplicity formula for the class of all finite subdirectly irreducible algebras in $\mathcal{O}$, we conclude from Theorem 1.3 that the injectives in each member V of the lattice of subvarieties of $K_{n,1}$ are completely determined once the injective s.i. algebras in V are known.

2. Injectives in the subvarieties of $K_{n,1}$. For integers $n > m \geq 0$, $S_{n,m}$ will denote the Ockham space $(\mathbf{n}, \gamma_m)$ consisting of the directly ordered set $\mathbf{n} = \{0, \ldots, n-1\}$ and the map $\gamma_m : \mathbf{n} \to \mathbf{n}$ defined by $\gamma_m(p) = p + 1$ whenever $0 \geq p < n - 1$, and $\gamma_m(n-1) = m$. Let $L_{n,m}$ be the dual algebra of the Ockham space $S_{n,m}$. In [12, Theorem 13, b] it is shown that $K_{n,1} = SP(L_{2n+1,1})$ and $K_{n,0} = SP(L_{2n,0})$ and in [9, 2.9] it is shown that $L_{n,m}$ is s.i.

Proposition 2.1. *If $X \in Si(K_{n,1})$ and $V = V(\{X\})$, then a non-trivial algebra $A \in V$ is s.i. if and only if $A \in S(\{X\})$. In particular, the non-trival s.i. algebras of $K_{n,1}$ are, up to isomorphism, the subalgebras of $L_{2n+1,1}$.*

Proof. We observe that X is finite, [3, Theorem 7]. Using the result in [9, 2.5], we have $\mathrm{Si}(V) = HS(\{X\})$. Let Z be a homomorphic image of a subalgebra Y of X. From [10, Lemma 3] we have $Z \cong Y/\Theta$, for some congruence $\Theta \in \{\Delta, \mathrm{Ker} f, \nabla\}$. Hence Z must be isomorphic either to Y or $f(Y)$ or the trivial algebra. □

In [9], M. S. Goldberg described the injective algebras in any subvariety of $\mathcal{O}$ generated by a single finite s.i. algebra in $\mathcal{O}$. That part of his theorem which is of particular relevance in this note may be stated as in [2, Theorem 3]. Thus we conclude that, under the conditions of 2.1 the algebra X is injective in V.

The notion of a reflective subcategory of a category can be found in [1, Def. 18.1, p. 27].

Let $\mathcal{K}_{n,1}$ be the equational category whose class of objects is $K_{n,1}$ and let $\mathcal{K}_{n,0}$ be the subcategory of $\mathcal{K}_{n,1}$ whose class of objects is $K_{n,0}$.

Proposition 2.2. *There exists a reflector $R : \mathcal{K}_{n,1} \to \mathcal{K}_{n,0}$ which preserves monomorphisms.*

Proof. For each $A \in \mathrm{Ob}\, \mathcal{K}_{n,1}$, let $R(A) = f(A) = \{f^{2n}(x) \mid x \in A\}$ and $\Phi_R(A) : A \to R(A)$, $x \to f^{2n}(x)$. Obviously, $\Phi_R(A)$ is a $K_{n,1}$-morphism. Let $h : A \to B$ be a $K_{n,1}$-morphism, where $A \in \mathrm{Ob}\, \mathcal{K}_{n,1}$ and $B \in \mathrm{Ob}\, \mathcal{K}_{n,0}$. It is easily seen that there is a unique $\mathcal{K}_{n,0}$-morphism $\bar{h} = h|_{R(A)} : R(A) \to B$ such that $\bar{h} \circ \Phi_R(A) = h$. By [1, Theorem 2, p. 28], the assignment $A \to R(A)$ can be extended to a reflector $R : \mathcal{K}_{n,1} \to \mathcal{K}_{n,0}$ where, for each $h : A \to A'$, $R(h)$ is the only $\mathcal{K}_{n,0}$-morphism such that $R(h) \circ \Phi_R(A) = \Phi_R(A') \circ h$. Now, suppose that $h : A \to A'$ is a $\mathcal{K}_{n,1}$-monomorphism and hence one–one ($\mathcal{K}_{n,1}$ is an equational category) and let $a, b \in R(A)$ such that $R(h)(a) = R(h)(b)$. Then

$$\begin{aligned} R(h)(a) = R(h)(b) &\Leftrightarrow f^{2n}(R(h)(a)) = f^{2n}(R(h)(b)) \Leftrightarrow R(h)(f^{2n}(a)) = R(h)(f^{2n}(b)) \\ &\Leftrightarrow R(h) \circ \Phi_R(A)(a) = R(h) \circ \Phi_R(A)(b) \Leftrightarrow \Phi_R(A')(h(a)) = \Phi_R(A')(h(b) \\ &\Leftrightarrow f^{2n}(h(a)) = f^{2n}(h(b)) \\ &\Leftrightarrow h(f^{2n}(a)) = h(f^{2n}(b)) \Leftrightarrow h(a) = h(b). \end{aligned}$$

Hence $a = b$ and thus $R(h)$ is one–one. □

By [**1**, Theorem 6, p. 30] we can deduce the following result.

PROPOSITION 2.3. *If I is injective in $K_{n,0}$, then I is injective in $K_{n,1}$.*

REMARK. For each $A \in K_{n,1}$, $R(A) = f(A)$ is a retract of A, since $\Phi_R(A)$ is a retraction.

Henceforth X_i, $i = 1, \ldots, m$, $(m \geq 1)$ will denote subdirectly irreducible algebras in $K_{n,1}$ such that, for all $i, j \in \{1, \ldots, m\}$, $i \neq j$, $X_i \notin S(\{X\})$; moreover, let $V_i = V(\{X_i\})$ and $V = V(\{X_1, \ldots, X_m\}) = V_1 \vee \ldots \vee V_m$. As a consequence of Jónsson's lemma, namely that $\mathrm{Si}(V_1 \vee \ldots \vee V_m) = \mathrm{Si}(V_1) \cup \ldots \cup \mathrm{Si}(V_m)$, we have that a non trivial algebra $A \in V$ is s.i. if and only if $A \in \bigcup_{i=1}^{m} S(\{X_i\})$. Hence, for each $1 \leq i \leq m$, X_i is a maximal subdirectly irreducible algebra both in V_i and in V.

PROPOSITION 2.4. *Assume that $A, I \in \mathrm{Si}(V)$ are non-trivial algebras. If I is injective in V, then*

(i) *either $A \in S(\{I\})$ or $f(A) \in S(\{I\})$;*

(ii) *if X_i is simple, for some $i \in \{1, \ldots, m\}$, then $I \cong X_i$;*

(iii) *the number of fixed points of f in A is less than or equal to the number of fixed points of f in I.*

Proof. Let B be a two element Boolean algebra. Note that B is a subalgebra both of A and I. Thus there exist a homomorphism $h: A \to I$ such that $h \circ \mathrm{id}_{B,A} = \mathrm{id}_{B,I}$, where $\mathrm{id}_{B,I}$ and $\mathrm{id}_{B,A}$ are the inclusion homomorphisms.

(i) By [**10**, Lemma 3], $\ker h \in \{\Delta, \mathrm{Ker} f\}$. Hence either $A \in S(\{I\})$ or $f(A) \in S(\{I\})$.

(ii) If X_i is simple then $X_i = f(X_i)$. Therefore $X_i \cong I$, since X_i is a maximal subdirectly irreducible algebra in V.

(iii) This follows immediately from (i). □

COROLLARY 2.5. *If X_i and X_j are simple algebras, with $i, j \in \{1, \ldots, m\}$, $i \neq j$, then the trivial algebra is the only s.i. algebra in V which is injective in V.*

PROPOSITION 2.6. *Suppose that $L_{2n,0} \cong X_i$ for some $i \in \{1, \ldots, m\}$. Up to isomorphism $L_{2n,0}$ is the only non-trivial s.i. algebra in V which is injective in V.*

Proof. We recall that $K_{n,0} = V(\{L_{2n,0}\})$ and $L_{2n,0}$ is a simple algebra [**9**, 2.9(i)]. Furthermore if A is a simple algebra in V, then $A = f(A)$ and $A \in K_{n,0}$. As $L_{2n,0}$ is injective in $K_{n,0}$ we conclude by 2.3 that $L_{2n,0}$ is injective in $K_{n,1}$. Hence $L_{2n,0}$ is injective in V. According to 2.4(ii) the proof is complete. □

In [**6**, Lemma 2.9] it is shown that if I is a s.i. algebra which is weak injective in a congruence distributive variety generated by a finite set Y of finite algebras then $I \in H(Y)$.

PROPOSITION 2.7. (i) *If $I \in \mathrm{Si}(V)$ is a non-trivial algebra which is injective in V, then there exists $j \in \{1, \ldots, m\}$ such that either $I \cong X_j$ or $I \cong f(X_j)$.*

(ii) *X_j and $f(X_j)$ are the only non-trivial algebras in $\mathrm{Si}(V_j)$ which are injective in V_j.*

Proof. Part (i) follows from [**6**, Lemma 2.9], [**10**, Lemma 3] and the fact that every injective in V is a weak injective in V. As X_j is injective in V_j, $f(X_j)$ is injective in V_j, since $f(X_j)$ is a retract of X_j. □

PROPOSITION 2.8. *If Y is a non-simple algebra such that $Y \in S(\{X_i\}) \cap S(\{X_j\})$, for some $i, j \in \{1, \ldots, m\}$, $i \neq j$, then X_i and X_j are not injective in V.*

Proof. Let $Y \in S(\{X_i\}) \cap S(\{X_j\})$ and suppose that Y is not simple. Then by [**10**, Lemma 3] there exists $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and $f(y_1) = f(y_2)$. Let $\{l, k\} = \{i, j\}$ and $h: X_l \to X_k$ be a homomorphism. As X_l is a maximal subdirectly irreducible algebra and $X_l \not\cong X_k$, we have $\operatorname{Ker} h = \operatorname{Ker} f$ and hence $h(y_1) = h(y_2)$. Thus the inclusion homomorphism $\mathrm{id}_{Y,X_k}: Y \to X_k$ cannot be extended to a homomorphism from X_l to X_k; hence X_k is not injective in V. □

3. Injectives in the subvarieties of $K_{2,0}^{\leq}$. The subdirectly irreducible algebras in $K_{2,0}^{\leq}$ were described by M. Ramalho and M. Sequeira [**10**]. We adopt their notation for these algebras, namely $\{T, B, S, K\ K_1, K_2, K_3, M, M_1, A, A_1, A_2, A_3, A_4, A_5, C, C_1\}$ is a set of representatives of the isomorphism classes of the s.i. algebras in $K_{2,0}^{\leq}$. Up to isomorphism the algebras T, B, K, M, A and C are the simple algebras in $K_{2,0}^{\leq}$. We observe that the algebra C is isomorphic to the algebra $L_{4,0}$ introduced in section 2. We have $K_{2,0}^{\leq} = SP(\{C_1\})$, so that C_1 and C are the only non-trivial s.i. algebras in $K_{2,0}^{\leq}$ which are injective in $K_{2,0}^{\leq}$, according to 2.7(ii).

By 2.6, C is the only non-trivial s.i. algebra in $K_{2,0}^{\leq}$ which is injective in any proper subvariety of $K_{2,0}^{\leq}$ containing C.

The lattice Λ ($K_{2,0}^{\leq}$) of subvarieties of $K_{2,0}^{\leq}$ was studied by M. Sequeira [**11**]. The injective algebras in each subvariety of the variety $\mathbf{M_1} = V(\{M_1\})$ were described by R. Beazer [**2**].

For each subvariety V of the variety $K_{2,0}^{\leq}$ the set of all non-trivial s.i. algebras in V which are injective in V is denoted by $\mathrm{Inj}(\mathrm{Si}(V))$.

PROPOSITION 3.1. *The trivial algebra is the only subdirectly irreducible algebra which is injective in each of the following subvarieties of $K_{2,0}^{\leq}$.*

(i) $\mathbf{M \vee A}$, $\mathbf{S \vee M \vee A}$, $\mathbf{K_2 \vee M \vee A}$, $\mathbf{K_1 \vee M \vee A}$, $\mathbf{S \vee K_1 \vee M \vee A}$, $\mathbf{K_1 \vee K_2 \vee M \vee A}$, $\mathbf{K_3 \vee M \vee A}$, $\mathbf{K_2 \vee K_3 \vee M \vee A}$.

(ii) $\mathbf{M \vee A_1}$, $\mathbf{M \vee A_2}$, $\mathbf{M \vee A_3}$, $\mathbf{M \vee A_4}$, $\mathbf{M \vee A_5}$, $\mathbf{M \vee A_2 \vee A_3}$, $\mathbf{M \vee A_4 \vee A_5}$, $\mathbf{S \vee M \vee A_1}$, $\mathbf{S \vee M \vee A_2}$, $\mathbf{S \vee M \vee A_3}$, $\mathbf{S \vee M \vee A_2 \vee A_3}$, $\mathbf{K_2 \vee M \vee A_1}$, $\mathbf{K_2 \vee M \vee A_2}$, $\mathbf{K_2 \vee M \vee A_3}$, $\mathbf{K_2 \vee M \vee A_2 \vee A_3}$, $\mathbf{K_1 \vee M \vee A_4}$, $\mathbf{M \vee A_1 \vee A_4}$, $\mathbf{M \vee A_2 \vee A_3}$, $\mathbf{M \vee A_2 \vee A_4}$, $\mathbf{M \vee A_3 \vee A_4}$, $\mathbf{M \vee A_2 \vee A_3 \vee A_4}$, $\mathbf{K_3 \vee M \vee A_1}$, $\mathbf{K_3 \vee M \vee A_2}$, $\mathbf{K_3 \vee M \vee A_3}$, $\mathbf{K_3 \vee M \vee A_2 \vee A_3}$, $\mathbf{K_2 \vee K_3 \vee M \vee A_1}$, $\mathbf{K_2 \vee K_3 \vee M \vee A_2}$, $\mathbf{K_2 \vee K_3 \vee M \vee A_3}$, $\mathbf{K_2 \vee K_3 \vee M \vee A_2 \vee A_3}$, $\mathbf{K_2 \vee M \vee A_5}$, $\mathbf{K_3 \vee M \vee A_4}$, $\mathbf{K_3 \vee M \vee A_1 \vee A_4}$, $\mathbf{K_3 \vee M \vee A_2 \vee A_4}$, $\mathbf{K_3 \vee M \vee A_3 \vee A_4}$, $\mathbf{K_3 \vee M \vee A_2 \vee A_3 \vee A_4}$.

(iii) $\mathbf{M_1 \vee A}$.

Proof. Part (i) follows from 2.5, since M and A are simple algebras. Let V be one of the subvarieties listed in (ii). Let $I \in \mathrm{Si}(V)$ be a non-trivial algebra and suppose that I is injective in V. From 2.4(ii), we have that $I \cong M$ and, by 2.4(i), $f(A_i) = A$ is a subalgebra of M, for each $i \in \{1, 2, 3, 4, 5\}$. Part (iii) follows from 2.4(ii) and 2.4(iii). □

COROLLARY 3.2. *There are no non-trivial s.i. injective algebras in each of the following subvarieties of* $K_{2,0}^{\leq}$: $\mathbf{M_1} \vee \mathbf{A_1}$, $\mathbf{M_1} \vee \mathbf{A_2}$, $\mathbf{M_1} \vee \mathbf{A_3}$, $\mathbf{M_1} \vee \mathbf{A_4}$, $\mathbf{M_1} \vee \mathbf{A_5}$, $\mathbf{M_1} \vee \mathbf{A_2} \vee \mathbf{A_3}$, $\mathbf{M_1} \vee \mathbf{A_1} \vee \mathbf{A_4}$, $\mathbf{M_1} \vee \mathbf{A_2} \vee \mathbf{A_4}$, $\mathbf{M_1} \vee \mathbf{A_3} \vee \mathbf{A_4}$, $\mathbf{M_1} \vee \mathbf{A_2} \vee \mathbf{A_3} \vee \mathbf{A_4}$, $\mathbf{M_1} \vee \mathbf{A_4} \vee \mathbf{A_5}$.

Proof. We consider the subvariety $\mathbf{M_1} \vee \mathbf{A_1}$. From 2.7(i) we have

$$\mathrm{Inj}(\mathrm{Si}(\mathbf{M_1} \vee \mathbf{A_1})) \subseteq \{M_1, M = f(M_1), A_1, A = f(A_1)\}.$$

Suppose that A_1 is injective in $\mathbf{M_1} \vee \mathbf{A_1}$. Since A is a retract of A_1, A is injective in $\mathbf{M_1} \vee \mathbf{A_1}$. Therefore A is injective in $\mathbf{M} \vee \mathbf{A_1}$, a contradiction. Similarly M_1 is not injective in $\mathbf{M_1} \vee \mathbf{A_1}$. Clearly, M and A are not injective in $\mathbf{M_1} \vee \mathbf{A_1}$ since they are not injective in $\mathbf{M} \vee \mathbf{A}$. The proof is similar for any of the other varieties. □

We recall from [**7**, 12.(ii)] the following result.

PROPOSITION 3.3. *Let X be a finite set of finite algebras and assume that $K = SP(X)$ is congruence-distributive. If $I \in X$ and every subalgebra of I is either subdirectly irreducible or weak injective in K, then I is injective in K if and only if I is injective in X.*

PROPOSITION 3.4. *The algebra A is the only non-trivial s.i. algebra which is injective in* $\mathbf{S} \vee \mathbf{A}$, $\mathbf{K_2} \vee \mathbf{A}$, $\mathbf{K_1} \vee \mathbf{A}$, $\mathbf{K_3} \vee \mathbf{A}$, $\mathbf{S} \vee \mathbf{K_1} \vee \mathbf{A}$, $\mathbf{K_1} \vee \mathbf{K_2} \vee \mathbf{A}$ *and* $\mathbf{K_2} \vee \mathbf{K_3} \vee \mathbf{A}$.

Proof. Let V be one of the subvarieties listed. From 2.4(ii), we know that, if $I \in \mathrm{Si}(V)$ is non-trivial and injective in V, then $I \cong A$. According to 3.3 it suffices to show that A is injective in $\mathrm{Si}(V)$. For each $X \in \mathrm{Si}(V)$ with $X \neq A$, there is exactly one homomorphism h from X to A, which is defined in the following way: if $X \in \{K, B\}$, h is the inclusion; if $X = S$, $h(a) = 1$; if $X = K_2$, $h(a) = k$ and $h(b) = 1$; if $X = K_3$, $h(a) = h(c) = k$, $h(b) = 1$; if $X = K_1$, $h(a) = h(b) = k$. Thus it is easily verified that A is injective in $\mathrm{Si}(V)$. □

PROPOSITION 3.5. *The algebra A is the only non-trivial s.i. algebra which is injective in the following subvarieties of* $K_{2,0}^{\leq}$:

(i) $\mathbf{A_2} \vee \mathbf{A_3}$, $\mathbf{A_4} \vee \mathbf{A_5}$.

(ii) $\mathbf{S} \vee \mathbf{A_2} \vee \mathbf{A_3}$, $\mathbf{K_2} \vee \mathbf{A_2} \vee \mathbf{A_3}$, $\mathbf{K_3} \vee \mathbf{A_2} \vee \mathbf{A_3}$, $\mathbf{K_1} \vee \mathbf{K_2} \vee \mathbf{A_2} \vee \mathbf{A_3}$, $\mathbf{K_2} \vee \mathbf{K_3} \vee \mathbf{A_2} \vee \mathbf{A_3}$.

Proof. Part (i) follows from the 2.7(i), 2.8 and the fact that A is injective in $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$ and $\mathbf{A_5}$. We consider $V = \mathbf{S} \vee \mathbf{A_2} \vee \mathbf{A_3}$. From 2.7(i), we have that $\mathrm{Inj}(\mathrm{Si}(V)) \subseteq \{B, S, A_2, A_3, A\}$. By 2.8, A_2 and A_3 are not injective in V. If B was injective in V, then $A = f(A_2)$ would be a subalgebra of B. Hence B is not injective in V. Similarly, S is not injective in V. For the other subvarieties, the proof is analogous. □

PROPOSITION 3.6. *The non-trivial s.i. algebras which are injective in V are, respectively,*

(i) A_1 *and* A *if* $V \in \{\mathbf{S} \vee \mathbf{A_1}, \mathbf{K_2} \vee \mathbf{A_1}, \mathbf{K_3} \vee \mathbf{A_1}, \mathbf{K_2} \vee \mathbf{K_3} \vee \mathbf{A_1}\}$,

(ii) A_2 *and* A *if* $V \in \{\mathbf{S} \vee \mathbf{A_2}, \mathbf{K_2} \vee \mathbf{A_2}, \mathbf{K_3} \vee \mathbf{A_2}, \mathbf{K_2} \vee \mathbf{K_3} \vee \mathbf{A_2}\}$,

(iii) A_3 *and* A *if* $V \in \{\mathbf{S} \vee \mathbf{A_3}, \mathbf{K_2} \vee \mathbf{A_3}, \mathbf{K_3} \vee \mathbf{A_3}, \mathbf{K_2} \vee \mathbf{K_3} \vee \mathbf{A_3}\}$,

(iv) A_4 *and* A *if* $V \in \{\mathbf{K_1} \vee \mathbf{A_4}, \mathbf{K_3} \vee \mathbf{A_4}\}$,

(v) A_5 *and* A *if* $V = \mathbf{K_2} \vee \mathbf{A_5}$.

Proof. We shall only prove (i). Similar reasoning can be used in (ii)–(v). That A_1 is injective in each of the subvarieties indicated follows from the facts that A_1 is injective in $\mathbf{A_1}$ and that, for $i \in \{1,2,3\}$, there is exactly one homomorphism h from each $X \in S(K_i)$ to A_1 and $h(X) \in S(\{A\})$ (defined as in 3.4). Now A is injective since A is a retract of A_1. By 2.4(i) we conclude that S and B are not injective in $\mathbf{S} \vee \mathbf{A_1}$. The same holds for K and K_2 in $\mathbf{K_2} \vee \mathbf{A_1}$; also for K and K_3 in $\mathbf{K_3} \vee \mathbf{A_1}$ and for K, K_2, K_3 in $\mathbf{K_2} \vee \mathbf{K_3} \vee \mathbf{A_1}$. According to 2.7(i) the proof is complete. □

Proposition 3.7. *The non-trivial s.i. algebras which are injective in V are, respectively,*

(i) A_4, A_1 *and* A *if* $V = \mathbf{A_1} \vee \mathbf{A_4}$,
(ii) A_4, A_2 *and* A *if* $V = \mathbf{A_2} \vee \mathbf{A_4}$,
(iii) A_4, A_3 *and* A *if* $V = \mathbf{A_3} \vee \mathbf{A_4}$.

Proof. We recall that an algebra $X \in \mathrm{Si}(K_{n,1})$ is injective in $V(\{X\})$ and that $f(X)$ is a retract of X. (i) From 2.7(i) we have that $\mathrm{Inj}(\mathrm{Si}(\mathbf{A_1} \vee \mathbf{A_4}) \subseteq \{A_1, A_4, A = f(A_1) = f(A_4)\}$. To prove that A_1 is injective in $\mathbf{A_1} \vee \mathbf{A_4}$ it suffices to show that any homomorphism from any subalgebra of A_4 to A_1 can be extended to a homomoprhism from A_4 to A_1. Up to isomorphism, the subalgebras of A_4 are B, S, K_2, K, A, A_4. For each $X \in \{B, S, K, K_2\}$, there is exactly one homomorphism h from X to A_1 (defined in 3.4), thus the homomorphism $\Phi: A_4 \to A_1$ defined by $\Phi(x) = f^4(x)$ extends h. We observe that $\Phi(A_4) = A$ and $\Phi \mid A = \mathrm{id}_A$. If $h: A \to A_1$ is a homomorphism than $h \circ \Phi$ extends h. Similarly, we conclude that A_4 is injective in $\mathbf{A_1} \vee \mathbf{A_4}$. (ii) The injectivity of A_2 in $\mathbf{A_2} \vee \mathbf{A_4}$ follows as the injectivity of A_1 in $\mathbf{A_1} \vee \mathbf{A_4}$. To show that A_4 is injective in $\mathbf{A_2} \vee \mathbf{A_4}$ it suffices to show that, for $X \in \{B, K, K_1, A, A_1\}$, any homomorphism $h: X \to A_4$ can be extended to a homomophism $\bar{h}: A_2 \to A_4$. If $X \in \{B, K, K_1, A\}$ $\bar{h}$ is defined as in (i). If $h: A_1 \to A_4$ then $\bar{h}$ is defined by $\bar{h}(y) = h(b)$. (iii) That A_3 is injective in $\mathbf{A_3} \vee \mathbf{A_4}$ follows as in (i). To show that A_4 is injective in $\mathbf{A_3} \vee \mathbf{A_4}$ it suffices to show that, for $X \in \{B, K, K_1, A, A_1\}$, any homomorphism $h: X \to A_4$, can be extended to a homomorphism $\bar{h}: A_3 \to A_4$. If $X \in \{B, K, K_1, A\}$, $\bar{h}$ is defined as in (i) and if $X = A_1$, $\bar{h}$ is defined by $\bar{h}(s) = h(d)$. □

Corollary 3.8. *The non-trivial s.i. algebras which are injective in V are*

(i) A_4, A_1 *and* A *if* $V = \mathbf{K_3} \vee \mathbf{A_1} \vee \mathbf{A_4}$,
(ii) A_4, A_2 *and* A *if* $V = \mathbf{K_3} \vee \mathbf{A_2} \vee \mathbf{A_4}$,
(iii) A_4, A_3 *and* A *if* $V = \mathbf{K_3} \vee \mathbf{A_3} \vee \mathbf{A_4}$.

Proof. (i) Follows from 3.7(i) and 3.6(i), (iv). Also (ii) follows from 3.7(ii) and 3.6(ii), (iv) while (iii) follows from 3.7(iii) and 3.6(iii), (iv). □

Proposition 3.9. *The algebras A_4, A are the only non-trivial s.i. algebras which are injective in $\mathbf{A_2} \vee \mathbf{A_3} \vee \mathbf{A_4}$ and $\mathbf{K_3} \vee \mathbf{A_2} \vee \mathbf{A_3} \vee \mathbf{A_4}$.*

Proof. From 3.7 we have that A_2, A_3 are not injective. That A_4 is injective in $\mathbf{A_2} \vee \mathbf{A_3} \vee \mathbf{A_4}$ and in $\mathbf{K_3} \vee \mathbf{A_2} \vee \mathbf{A_3} \vee \mathbf{A_4}$ follows from 3.7(ii), (iii) and 3.8. □

REFERENCES

1. R. Balbes and P. Dwinger, *Distributive lattices* (University of Missouri Press, 1974).

2. R. Beazer, Injectives in some small varieties of Ockham algebras, *Glasgow Math. J.* **25** (1984), 183–191.

3. J. Berman, Distributive lattices with an additional unary operation, *Aequationes Math.* **16** (1977), 165–171.

4. T. S. Blyth and J. C. Varlet, On a common abstraction of de Morgan algebras and Stone algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **94** (1983), 301–308.

5. R. Cignoli, Injective De Morgan and Kleene algebras, *Proc. Amer. Math. Soc.*, **47** (1975), 269–278.

6. B. A. Davey, Weak injectivity and congruence extension in congruence-distributive equational classes, *Canad. J. Math.* **XXIX** (3) (1977), 449–459.

7. B. A. Davey and H. Werner, Injectivity and Boolean powers, *Math. Z.* **166** (1979), 205–223.

8. A. Day, Injectivity in equational classes of algebras, *Canad. J. Math.* **XXIV**(2) (1972), 209–220.

9. M. S. Goldberg, Distributive Ockham algebras: free algebras and injectivity, *Bull. Austral. Math. Soc.*, **24** (1981), 161–203.

10. M. Ramalho and M. Sequeira, On generalized MS-algebras, *Portugal. Math.* **44**(3) (1987), 315–328.

11. M. Sequeira, Varieties of Ockham algebras satisfying $x \leqslant f^4(x)$, *Portugal. Math.* **45**(4) (1988), 369–391.

12. A. Urquhart, Distributive lattices with a dual homomorphic operation, *Studia Logica* **38** (1979), 201–209.

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