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AN INTERPOLATION BY SUCCESSIVE DERIVATIVES AT A FINITE SET

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For an n-times differentiable function $f(x)$ whose derivatives $f^{(j)}(x_j)$ at $x = x_j$, $j = 0, 1, \ldots, n$ are specified, we introduce a sequence of fundamental polynomials $\{\pi(x \,|\, x_0, x_1, \ldots, x_n)\}_{n=0}^{\infty}$ to interpolate $f(x)$ with a remainder as

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_j)}{j!} \pi(x \,|\, x_0, x_1, \ldots, x_j) + R_n(x \,|\, x_0, x_1, \ldots, x_n).$$

The remainder $R_n(x \,|\, x_0, x_1, \ldots, x_n)$ is given in an integral form and Lagrange's form.

In addition, by introducing orthogonality of Sobolev type we verify the best optimality of the approximations and interpret the fundamental polynomials $\{\pi(x \,|\, x_0, x_1, \ldots, x_n)\}_{n=0}^{\infty}$ as a kind of Sobolev orthogonal polynomial.

1. INTRODUCTION

Consider a function $f(x)$, whose values are specified at the points $x = x_0$, x_1, x_2, $\ldots$, x_n. Lagrange's interpolation formula gives a polynomial of degree n whose values are the same as $f(x)$ at $x = x_j$, $j = 0, 1, 2, \ldots, n$ and which approximates $f(x)$ with a remainder. More generally, Newton's interpolation formula also gives a polynomial with the same properties. In fact, Newton's method still works for the case where all the points x_0, x_1, $x_2, \ldots$, x_n are equal. In that case the interpolation polynomials are reduced to Taylor polynomials. Also, Chebyshev interpolation is the case where the points x_0, $x_1 \ldots$, x_n are zeros of the Chebyshev polynomial.

On the other hand, for a differentiable function $f(x)$ the Hermite interpolation gives polynomials $H(x)$ that satisfies not only $H(x) = f(x)$ but also $H'(x) = f'(x)$ at the points $x = x_0$, x_1, $x_2, \ldots$, x_n. Such a polynomial $H(x)$ is also basically a variant of the Lagrange interpolation polynomial. Besides, there are several interpolation polynomials introduced by Everett, Bessel, Stirling, Aitken and others, which are essentially equivalent to the Lagrange interpolation polynomial that uses the same tabular points,

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although the representations are different (see [1]). Thus, it will be quite interesting to find a new scheme to interpolate an n times differentiable function whose successitive derivatives $f^{(j)}(x_j)$ at $x = x_j$, $j = 0, 1, 2, \ldots, n$, are known. To the best of the author's knowledge such a scheme has not appeared in the literature.

The purpose of this paper is to introduce a sequence of fundamental polynomials $\{\pi(x \mid x_0, x_1, \ldots, x_n)\}_{n=0}^{\infty}$ which can be used to interpolate a function such that $f^{(j)}(x_j)$ at $x = x_j$, $j = 0, 1, 2, \ldots, n$ are known.

In Section 2 we show that every polynomial $P(x)$ whose derivatives $P^{(j)}(x_j)$ at $x = x_j$, $j = 0, 1, 2, \ldots, n$, are given can be written as

$$P(x) = \sum_{j=0}^{n} \frac{P^{(j)}(x_j)}{j!} \pi(x \mid x_0, x_1, \ldots, x_j).$$

In Section 3 it is shown that every function $f(x)$ whose derivatives $f^{(j)}(x_j)$ at $x = x_j$, $j = 0, 1, 2, \ldots, n$, are known can be approximated with a remainder as

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_j)}{j!} \pi(x \mid x_0, x_1, \ldots, x_j) + R_n(x \mid x_0, x_1, \ldots, x_n),$$

which is reduced to Taylor's formula whenever all the points $x_0, x_1, \ldots, x_n$ are equal. The remainder will be expressed in an integral form or Lagrange form and estimated properly. In particular, it is shown that every entire function can be written as an infinite series of this type.

In the last section we discuss an orthogonal property of the fundamental polynomials $\{\pi(x \mid x_0, x_1, \ldots, x_n)\}_{n=0}^{\infty}$. In fact, the best optimality of the approximation is verified and the fundamental polynomials are interpreted as a kind of Sobolev orthogonal polynomial.

2. Interpolation polynomials

Throughout this paper all polynomials are assumed to be real polynomials in one variable.

It is well known that if $P(x_0), P(x_1), \ldots, P(x_n)$ are known the polynomial $P(x)$ of degree n is uniquely determined by the Lagrange interpolation polynomial. In another direction, if $P^{(k)}(a)$, $k = 0, 1, 2, \ldots, n$, are known the polynomial $P(x)$ is uniquely determined by Taylor's polynomial (or Newton's interpolation polynomial).

In this section we introduce a heuristic method to find a polynomial $P(x)$ of degree n whenever $P^{(k)}(x_k)$, $k = 0, 1, 2, \ldots, n$, are specified.

First, we develope a sequence of fundamental polynomials. For a nonnegative integer n and a finite number of points $x_0, x_1, \ldots, x_n$ on the real line $\mathbb{R}$ we denote by $\pi(x \mid x_0, x_1, \ldots, x_n)$ the polynomial $P(x)$ of degree n with the specified values

$$P^{(j)}(x_j) = 0\ ,\ j = 0, 1, \ldots, n-1$$

and

$$P^{(n)}(x) = n!\ ,\ x \in \mathbb{R}.$$

Here, the points $x_0, x_1, \ldots, x_n$ need not be distinct.

Lemma 2.1. *For an integer $n \geq 1$ the polynomial $\pi(x \mid x_0, x_1, \ldots, x_n)$ can be uniquely expressed as an iterated integral*

$$\pi(x \mid x_0, x_1, \ldots, x_n) = \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_{n-1}}^{t_{n-1}} n!\, dt_n\, dt_{n-1} \ldots dt_1. \tag{2.1}$$

Proof: Since

$$\pi^{(k)}(t \mid x_0, x_1, \ldots, x_n) = \int_{x_k}^{t} \pi^{(k+1)}(s \mid x_0, x_1, \ldots, x_n)\, ds$$

for $k = 0, 1, \ldots, n-1$ we have

$$\begin{aligned}
\pi(x \mid x_0, x_1, \ldots, x_n) &= \int_{x_0}^{x} \pi'(t_1 \mid x_0, x_1, \ldots, x_n)\, dt_1 \\
&= \int_{x_0}^{x} \int_{x_1}^{t_1} \pi''(t_2 \mid x_0, x_1, \ldots, x_n)\, dt_2\, dt_1 \\
&= \cdots\cdots \\
&= \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_{n-1}}^{t_{n-1}} \pi^{(n)}(t_n \mid x_0, x_1, \ldots, x_n)\, dt_n\, dt_{n-1} \ldots dt_1 \\
&= \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_{n-1}}^{t_{n-1}} n!\, dt_n\, dt_{n-1} \ldots dt_1.
\end{aligned}$$

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Example.

(i) $\pi(x \mid x_0) = 1$ and $\pi(x \mid x_0, x_1) = \int_{x_0}^{x} dt = x - x_0$.

(ii) $\pi(x \mid x_0, x_1, x_2) = \int_{x_0}^{x} \int_{x_1}^{t_1} 2!\, dt_2\, dt_1 = (x - x_1)^2 - (x_0 - x_1)^2$.

(iii) $\pi(x \mid x_0, x_1, x_2, x_3) = \int_{x_0}^{x} \int_{x_1}^{t_1} \int_{x_2}^{t_2} 3!\, dt_3\, dt_2\, dt_1$
$= (x - x_2)^3 - (x_0 - x_2)^3 - 3(x_1 - x_2)^2(x - x_0)$.

(iv) $\pi(x \mid x_0, x_1, x_2, x_3, x_4) = (x^4 - x_0^4) - 4x_3(x^3 - x_0^3) - 6x_2(x_2 - 2x_3)(x^2 - x_0^2)$
$- 2(2x_1^3 - 6x_3x_1^2 - 6x_1x_2^2 + 12x_1x_2x_3)(x - x_0)$.

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The polynomials $\pi(x\,|\,x_0, x_1, \dots, x_n)$ can be written by means of a determinant. If we define a double sequence by

$$\gamma_{m,n} = \sum_{j=0}^{N(m,n)} \frac{m!n!}{(m-j)!(n-j)!j!^2} x_j^{m+n-2j}$$

where $N(m,n) = \min(m,n)$ for nonnegative integers m and n then in view of the theory of Sobolev orthogonal polynomials (see [**2**]) we have

$$\pi(x\,|\,x_0, x_1, \cdots, x_n) = \det \begin{vmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n} \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n-1,0} & \gamma_{n-1,1} & \cdots & \gamma_{n-1,n} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

Of course this result can be verified, *a posteriori*, by a direct calculation of the determinant.

REMARK 2.2. (i) In the above it is not necessary that the points $x_0, x_1, \dots, x_n$ are distinct. In particular, if $x_0 = x_1 = \dots = x_n = a$ then it is easy to see that

$$\pi(x\,|\,a, a, \dots, a) = (x-a)^n.$$

(ii) From (2.1) we obtain

$$\pi'(x\,|\,x_0, x_1, \dots, x_n) = n\pi(x\,|\,x_1, x_2, \dots, x_n),$$

and

$$\pi''(x\,|\,x_0, x_1, \dots, x_n) = n(n-1)\pi(x\,|\,x_2, x_3, \dots, x_n).$$

In general, we have, for $k = 0, 1, \dots, n$

$$\pi^{(k)}(x\,|\,x_0, x_1, \dots, x_n) = \frac{n!}{(n-k)!}\pi(x\,|\,x_k, x_{k+1}, \dots, x_n). \tag{2.2}$$

(iii) From (2.2) we also have

$$\begin{aligned} \pi(x\,|\,x_0, x_1, \dots, x_n) &= n\int_{x_0}^{x} \pi(t_1\,|\,x_1, x_2, \dots, x_n)\,dt_1 \\ &= n(n-1)\int_{x_0}^{x}\int_{x_1}^{t_1} \pi(t_2\,|\,x_2, x_3, \dots, x_n)\,dt_2\,dt_1 \\ &= \frac{n!}{(n-k)!}\int_{x_0}^{x}\int_{x_1}^{t_1}\cdots\int_{x_{k-1}}^{t_{k-1}} \pi(t_k\,|\,x_k, x_{k+1}, \dots, x_n)\,dt_k\,dt_{k-1}\dots dt_1 \end{aligned}$$

for $k = 0, 1, \dots, n$.

(iv) If we put $\underline{x} = \min(x, x_0, x_1, \dots, x_n)$ and $\overline{x} = \max(x, x_0, x_1, \dots, x_n)$ then we have

$$\left| \pi(x \mid x_0, x_1, \dots, x_n) \right| \leq \int_{\underline{x}}^{\overline{x}} \int_{\underline{x}}^{\overline{x}} \cdots \int_{\underline{x}}^{\overline{x}} n!\, dt_n\, dt_{n-1} \dots dt_1 = (\overline{x} - \underline{x})^n.$$

Now we give a representation of a polynomial $P(x)$ whose k-th derivatives at x_k $k = 0, 1, \dots, n$, are specified. This result will be used to interpolate functions which are sufficiently differentiable.

Theorem 2.3. *Let $x_0, x_1, \dots,$ and x_n be points on the real line $\mathbb{R}$ and let $P(x)$ be a polynomial of degree n such that the values $P^{(k)}(x_k)$, $k = 0, 1, 2, \dots, n$, are specified. Then the polynomial can be uniquely represented as*

$$P(x) = \sum_{k=0}^{n} \frac{P^{(k)}(x_k)}{k!} \pi(x \mid x_0, x_1, \dots, x_k). \tag{2.3}$$

Proof: The uniqueness is easy.

Now we use mathematical induction. If $n = 1$ then

$$P(x) = P'(x_1)(x - x_0) + P(x_0) = P'(x_1)\pi(x \mid x_0, x_1) + P(x_0)\pi(x \mid x_0).$$

Thus (2.3) holds for $n = 1$. Now we assume that (2.3) is true for all the polynomial of degree $n - 1$. Let $Q(x)$ be $P'(x)$. Then $Q^{(k)}(x_{k+1}) = P^{(k+1)}(x_{k+1})$ for $k = 0, 1, 2, \dots, n-1$.

Using the induction hypothesis we have

$$Q(x) = \sum_{k=0}^{n-1} \frac{Q^{(k)}(x_{k+1})}{k!} \pi(x \mid x_1, x_2, \dots, x_{k+1})$$

or equivalently,

$$P'(x) = \sum_{k=0}^{n-1} \frac{P^{(k+1)}(x_{k+1})}{k!} \pi(x \mid x_1, x_2, \dots, x_{k+1})$$

Then it follows from (iii) in Remark 2.2 that

$$\begin{aligned}
P(x) &= \int_{x_0}^{x} P'(t)\,dt \;+\; P(x_0) \\
&= \sum_{k=0}^{n-1} \frac{P^{(k+1)}(x_{k+1})}{k!} \int_{x_0}^{x} \pi(t \,|\, x_1, x_2, \ldots, x_{k+1})\,dt \;+\; P(x_0) \\
&= \sum_{k=0}^{n-1} n\, \frac{P^{(k+1)}(x_{k+1})}{(k+1)!}\, \pi(x \,|\, x_0, x_1, \ldots, x_{k+1}) \;+\; P(x_0) \\
&= \sum_{k=0}^{n} \frac{P^{(k)}(x_k)}{k!}\, \pi(x \,|\, x_0, x_1, \ldots, x_k)
\end{aligned}$$

This completes the proof. □

In view of (i) of Remark 2.2 we can see that if $x_0 = x_1 = \ldots = x_n = a$ then (2.3) is exactly the same as the Taylor polynomial at $x = a$ that is,

$$P(x) = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!}\,(x-a)^k.$$

3. An interpolation with remainder

In this section we give an interpolation of a differentiable function in terms of a sequence of the fundamental polynomials introduced in the previous section. Moreover, using an estimate for the remainder term, analytic functions will be approximated by these fundamental polynomials.

Now we state the main theorem. (This reduces to Taylor's theorem, if $x_0 = x_1 = \ldots = x_n = a$.)

Theorem 3.1. *Let $f(x)$ be an $(n+1)$-times differentiable function on the interval $(a\,,b)$ and $x_0, x_1, \ldots, x_n$ be points in $(a\,,b)$. Then $f(x)$ can be written as*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_k)}{k!}\pi(x \,|\, x_0, x_1, \ldots, x_k) + R_n(x \,|\, x_0, x_1, \ldots, x_n) \tag{3.1}$$

where

$$R_n(x \,|\, x_0, x_1, \ldots, x_n) = \int_{x_0}^{x}\int_{x_1}^{t_1}\cdots\int_{x_n}^{t_n} f^{(n+1)}(t_{n+1})\,dt_{n+1}\,dt_n \ldots dt_1. \tag{3.2}$$

Proof: In view of the fundamental theorem of calculus we have

$$f^{(k)}(x) = f^{(k)}(x_k) + \int_{x_k}^{x} f^{(k+1)}(t)\,dt$$

for $k = 0, 1, 2, \dots, n$.

Then successive substitutions give

$$\begin{aligned}
f(x) &= f(x_0) + \int_{x_0}^{x} f'(t_1)\,dt_1 \\
&= f(x_0) + \int_{x_0}^{x} \left\{ f'(x_1) + \int_{x_1}^{t_1} f''(t_2) \right\} dt_2 \\
&= f(x_0)\pi(x \mid x_0) + f'(x_1)\pi(x \mid x_0, x_1) + \int_{x_0}^{x} \int_{x_1}^{t_1} f''(t_2)\,dt_2\,dt_1 \\
&= f(x_0)\,\pi(x \mid x_0) + f'(x_1)\,\pi(x \mid x_0, x_1) \\
&\qquad + \int_{x_0}^{x} \int_{x_1}^{t_1} \left\{ f''(x_2) + \int_{x_2}^{t_2} f'''(t_3) \right\} dt_2\,dt_1 \\
&= f(x_0)\,\pi(x \mid x_0) + f'(x_1)\,\pi(x \mid x_0, x_1) + \frac{f''(x_2)}{2!}\,\pi(x|x_0, x_1, x_2) \\
&\qquad + \int_{x_0}^{x} \int_{x_1}^{t_1} \int_{x_2}^{t_2} f'''(t_3)\,dt_3\,dt_2\,dt_1.
\end{aligned}$$

To complete the proof we use mathematical induction.

We assume that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_k)}{k!} \pi(x \mid x_0, x_1, \dots, x_k) + R_{n-1}(x \mid x_0, x_1, \dots, x_{n-1})$$

where

$$R_{n-1}(x \mid x_0, x_1, \dots, x_{n-1}) = \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_{n-1}}^{t_{n-1}} f^{(n)}(t_n)\,dt_n\,dt_{n-1} \dots dt_1.$$

By substituting

$$f^{(n)}(t_n) = f^{(n)}(x_n) + \int_{x_n}^{t_n} f^{(n+1)}(t_{n+1})\,dt_{n+1}$$

we obtain

$$\begin{aligned}
&R_{n-1}(x \mid x_0, x_1, \dots, x_{n-1}) \\
&= \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_{n-1}}^{t_{n-1}} \left\{ f^{(n)}(x_n) + \int_{x_n}^{t_n} f^{(n+1)}(t_{n+1})\,dt_{n+1} \right\} dt_n\,dt_{n-1} \dots,\ dt_1 \\
&= \frac{f^{(n)}(x_n)}{n!} \pi(x|x_0, x_1, \dots, x_n) + \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_n}^{t_n} f^{(n+1)}(t_{n+1})\,dt_{n+1}\,dt_n \dots,\ dt_1.
\end{aligned}$$

This implies that

$$R_{n-1}(x \mid x_0, x_1, \ldots, x_{n-1}) = \frac{f^{(n)}(x_n)}{n!}\pi(x \mid x_0, x_1, \ldots, x_n) + R_n(x \mid x_0, x_1, \ldots, x_n)$$

which completes the proof. $\square$

In (3.1) above it is easy to see from (iv) of Remark 2.2 that the remainder term $R_n(x|x_0, x_1, \ldots, x_n)$ can be estimated roughly as

$$\left|R_n(x|x_0, x_1, \ldots, x_n)\right| \leqslant \frac{M(\overline{x} - \underline{x})^n}{n!} \tag{3.3}$$

where

$$\overline{x} = \max\{x, x_0, x_1, \ldots, x_n\},$$
$$\underline{x} = \min\{x, x_0, x_1, \ldots, x_n\},$$

and

$$M = \max\left\{|f^{(n+1)}(t)| \;\middle|\; \underline{x} \le t \le \overline{x}\right\}.$$

Now we give a remainder term in a different form which is similar to Lagrange's remainder in the Taylor theorem, under some restriction on the points $x_0, x_1, \ldots, x_n$.

THEOREM 3.2. *Suppose that the function $f(x)$ is $(n+1)$-times differentiable in (a ,b) and the points $x_0, x_1, \ldots, x_n$ are chosen so that either $x \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n$ or $x \geqslant x_0 \geqslant x_1 \geqslant \ldots \geqslant x_n$. Then we have*

$$R_n(x \mid x_0, x_1, \ldots, x_n) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi(x \mid x_0, x_1, \ldots, x_n, x_n) \tag{3.4}$$

for some ξ between x and x_n. Moreover, in this case we obtain

$$\left|R_n(x \mid x_0, x_1, \ldots, x_n)\right| \leqslant \frac{K|x - x_n|^{n+1}}{(n+1)!} \tag{3.5}$$

where $K = \max\left\{|f^{(n+1)}(t)| \;\middle|\; t \text{ varies between } x \text{ and } x_n\right\}$.

PROOF: If $x = x_0$ then both sides in (3.4) are zero, so we assume $x \neq x_0$. For a fixed $x \neq x_0$ let M be the unique solution of

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_k)}{k!}\pi(x \mid x_0, x_1, \ldots, x_k) + \frac{M}{(n+1)!}\pi(x \mid x_0, x_1, \ldots, x_n, x_n).$$

Let

$$g(t) = \sum_{k=0}^{n} \frac{f^{(k)}(x_k)}{k!}\pi(t \mid x_0, x_1, \ldots, x_k) + \frac{M}{(n+1)!}\pi(t \mid x_0, x_1, \ldots, x_n, x_n) - f(t).$$

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Then we have

$$g^{(k)}(x_k) = 0, \quad k = 0, 1, \dots, n \tag{3.6}$$

and

$$g^{(n+1)}(t) = M - f^{(n+1)}(t), \quad a < t < b. \tag{3.7}$$

Also, $g(x) = 0$ by the choice of M. Since $g(x_0) = 0$, Rolle's theorem implies that there exists c_1 between x_0 and x such that $g'(c_1) = 0$. Since $g'(x_1) = 0$ and $x_1 \neq c_1$ a second application of Rolle's theorem shows that there exists c_2 between c_1 and x_1 such that $g''(c_2) = 0$. This process continues until we obtain c_{n+1} between c_n and x_n such that $g^{(n+1)}(c_{n+1}) = 0$. So we have (3.4) from (3.7) with $\xi = c_{n+1}$. On the other hand, the last assertion follows easily from (3.3). This completes the proof. ∎

It is well known that if f is real analytic on the real line $\mathbb{R}$ then for every compact K of $\mathbb{R}$ there exist $r > 0$ and $C > 0$ such that

$$\sup_{x \in K} \left| f^{(n)}(x) \right| \leqslant C \frac{n!}{r^n}, \quad n = 0, 1, 2, \dots .$$

Under a slightly stronger condition, analytic functions can be approximated by the polynomials discussed above as follows:

COROLLARY 3.3. *Let $f(x)$ be a real analytic function on the real line satisfying*

$$\sup_{x \in K} \left| f^{(n)}(x) \right| \leqslant C H^n n!, \quad n = 0, 1, 2, \dots . \tag{3.8}$$

for some $H > 0$ and let $(x_n)_{n=0}^{\infty}$ be a bounded sequence of real numbers.

If $H \,|\, \overline{x} - \underline{x} \,| < 1$ then $f(x)$ can be expressed as an infinite series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_n)}{n!} \pi(x \,|\, x_0, x_1, \dots, x_n) \tag{3.9}$$

where $\overline{x} = \sup\{x, x_0, x_1, x_2, \dots\}$ and $\underline{x} = \inf\{x, x_0, x_1, x_2, \dots\}$.

PROOF: We have only to show that

$$\lim_{n \to \infty} R_n(x | x_0, x_1, \dots, x_n) = 0.$$

From (3.2) and the hypothesis (3.8) we have

$$\begin{aligned} \left| R_n(x \,|\, x_0, x_1, \dots, x_n) \right| &\leqslant \left| \int_{x_0}^{x} \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} f^{(n+1)}(t_{n+1}) \, dt_{n+1} \, dt_n \dots dt_1 \right| \\ &\leqslant C H^{n+1} (n+1)! = \left| \int_{x_0}^{x} \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} dt_{n+1} \, dt_n \dots dt_1 \right| \\ &\leqslant C H^{n+1} (\overline{x} - \underline{x})^{n+1}, \end{aligned}$$

which goes to 0 as $n \to \infty$. ∎

In the above the condition (3.8) is satisfied for every $R > 0$ when f is a real analytic function which extends to an entire holomorphic function. Hence in this case the above result holds automatically as follows :

Corollary 3.4. *Let $f(x)$ be a real analytic function which extends an entire holomorphic function and let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. Then we have*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_k)}{k!} \pi(x \mid x_0, x_1, \ldots, x_k), \quad x \in \mathbb{R}. \tag{3.10}$$

4. The Sobolev orthogonality

Here we shall discuss some interesting properties of the fundamental polynomials $\pi(x|x_0, x_1, \ldots, x_n)$ from the point of view of orthogonality. Throughout this section the sequence $(x_n)_{n=0}^{\infty}$ is fixed and bounded. By $\mathcal{H}(\mathbb{R})$ we denote the set of all real analytic functions on $\mathbb{R}$ which extend to entire functions on the complex plane. Then it is well known that f belongs to $\mathcal{H}(\mathbb{R})$ if and only if for every compact subset $K \subset \mathbb{R}$ and for every $h > 0$ there exists a constant C such that

$$\sup_{x \in K} \left| f^{(k)}(x) \right| \leqslant C h^k k!, \quad k = 0, 1, 2, \ldots. \tag{4.1}$$

We define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}(\mathbb{R}) \times \mathcal{H}(\mathbb{R})$ as follows: for each f and g in $\mathcal{H}(\mathbb{R})$

$$\langle f, g \rangle = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_j) g^{(j)}(x_j)}{j!^2}$$

Then $\langle \cdot, \cdot \rangle$ gives an inner product on $\mathcal{H}(\mathbb{R})$ and a norm $\|\cdot\|$ given by $\langle \cdot, \cdot \rangle^{1/2}$. The only nontrivial thing for the bilinear form is the fact that $\langle f, f \rangle = \|f\|^2 = 0$ implies $f \equiv 0$. But this follows easily from Corollary 3.4.

We denote $\pi_n(x) = \pi(x|x_0, x_1, \ldots, x_n)$ for $n \geqslant 0$, for simplicity. Then it is easy to see that

$$\langle \pi_n, \pi_m \rangle = \sum_{j=0}^{\infty} \frac{{\pi_n}^{(j)}(x_j) {\pi_m}^{(j)}(x_j)}{j!^2} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

since

$$\pi_n^{(j)}(x_j) = \begin{cases} 0, & j = 0, 1, 2, \ldots, n-1 \\ n!, & j = n \\ 0, & j \geqslant n+1 \end{cases}$$

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This means that the polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ are an orthonormal polynomial system with respect to the (Sobolev) inner product $\langle \cdot , \cdot \rangle$.

Now we prove the completeness of this orthonormal polynomial system $\{\pi_n(x)\}_{n=0}^{\infty}$ in $\mathcal{H}(\mathbb{R})$.

Theorem 4.1. *The orthonormal family $\{\pi_n(x)\}_{n=0}^{\infty}$ is complete in $\mathcal{H}(\mathbb{R})$.*

Proof: We have only to show that for every $f \in \mathcal{H}(\mathbb{R})$ a sequence of polynomials $\sum_{j=0}^{N}(f^{(j)}(x_j))/(j!)\pi_j(x)$ converges to f in the normed topology on $\mathcal{H}(\mathbb{R})$. From Corollary 3.4 we have already that $\sum_{j=0}^{N}(f^{(j)}(x_j))/(j!)\pi_j(x)$ converges pointwisely to f. Moreover, Theorem 3.1 means that for every integer $N \geq 0$

$$f(x) = \sum_{j=0}^{N} \frac{f^{(j)}(x_j)}{j!}\pi_j(x) + R_N(x \mid x_0, x_1, \ldots , x_n).$$

From these facts we have

$$R_N^{(j)}(x_j \mid x_0, x_1, \ldots , x_n) = 0\,, \quad j = 0, 1, 2, \ldots , N$$

and

$$R_N^{(j)}(x \mid x_0, x_1, \ldots , x_n) = f^{(j)}(x)\,, \quad j = N+1, N+2, \ldots .$$

Since the sequence $(x_n)_{n=0}^{\infty}$ is bounded it follows that

$$\left\| f - \sum_{j=0}^{N} \frac{f^{(j)}(x_j)}{j!}\pi_j(x) \right\|^2 = \left\| R_N(x \mid x_0, x_1, \ldots , x_N) \right\|^2$$
$$= \sum_{j=N+1}^{\infty} \frac{\left|f^{(j)}(x_j)\right|^2}{j!^2} \leqslant C \sum_{j=N+1}^{\infty} h^{2j}.$$

The last inequality follows from (4.1). This implies that

$$\left\| f - \sum_{j=0}^{N} (f^{(j)}(x_j)) \big/ (j!)\pi_j(x) \right\|$$

converges to 0 as $N \to \infty$, by taking h so that $0 < h < 1$. ∎

Remark 4.2. In view of the above theorem we obtain some properties which are fundamental in the theory of Hilbert space.

(i) If $f \in \mathcal{H}(\mathbb{R})$ has an expression $f(x) = \sum_{j=0}^{\infty} a_j\pi_j(x)$, $a_j \in \mathbb{R}$ then we have

$$a_j = \frac{f^{(j)}(x_j)}{j!} = \langle f , \pi_j(x) \rangle$$

for $j = 0, 1, 2, \ldots$ and

$$\|f\|^2 = \sum_{j=0}^{\infty} \frac{\langle f, \pi_j(x)\rangle^2}{j!^2}.$$

(ii) By Bessel's inequality we can see that for each $f \in \mathcal{H}(\mathbb{R})$ the polynomial $\sum_{j=0}^{N} \big(f^{(j)}(x_j)\big)/(j!)\pi_j(x)$ gives the best approximation to f in the polynomials of degree at most N.

For the inner product $\langle\cdot,\cdot\rangle$, we call the double sequence defined by $\phi_{m,n} = \langle x^m, x^n\rangle$ for $m, n = 0, 1, 2, \ldots$, a moment sequence of the inner product $\langle\cdot,\cdot\rangle$. Moreover, we say that the inner product $\langle\cdot,\cdot\rangle$ is positive definite if

$$\Delta_n = \det[\phi_{i,j}]_{i,j=0}^{n} > 0$$

for each $n \geqslant 0$.

Using the theory of Sobolev orthogonal polynomials (see [**2**] for details) we can see that

(i) $\langle\cdot,\cdot\rangle$ is positive definite, since

$$\langle \pi_n, \pi_m\rangle = \delta_{m,n} \tag{4.2}$$

for any $m, n \geqslant 0$.

(ii) $\Delta_n = 1$ for any $n \geqslant 0$.

(iii) The polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ are the only polynomials satisfying (4.2) up to nonzero constant multiples.

(iv) The polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ form a Sobolev orthogonal polynomial system relative to $\langle\cdot,\cdot\rangle$.

References

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