ORTHOGONAL DECOMPOSITIONS OF MULTIVARIATE WEAKLY STATIONARY STOCHASTIC PROCESSES

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1. Introduction and summary. In this paper we shall study the relations between the ranks of q-variate, discrete-parameter, weakly stationary stochastic processes $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ satisfying the condition

1.1 $$\mathbf{x}_n = \mathbf{y}_n + \mathbf{z}_n, \quad \mathbf{y}_m \perp \mathbf{z}_n, \qquad -\infty < m, n < \infty,$$

and derive from them a characterization for the Wold decomposition and conditions for the concordance of the Wold and the Lebesgue–Cramér decompositions. We shall also extend these results to the continuous-parameter case. In order to describe our results, we must first recall the definitions and standard properties of such processes (cf. **(13; 6)**).

Let X be a complex Hilbert space and X^q the Cartesian product consisting of all q-dimensional (column) vectors with components in X, where q is a positive integer. As usual we shall endow X^q with a Gram-matricial structure:

(1.2) $$(\mathbf{x}, \mathbf{y}) = [(x^i, y^j)], \qquad \mathbf{x} = (x^i), \qquad \mathbf{y} = (y^i), \qquad i = 1, 2, \ldots, q.$$

We say that

(1.3) $$\mathbf{x} \perp \mathbf{y} \text{ if and only if } (\mathbf{x}, \mathbf{y}) = \mathbf{0}.$$

$|\mathbf{x}| = \sqrt{\{\text{trace } (\mathbf{x}, \mathbf{x})\}}$ is the usual norm in X^q, and it provides the appropriate topology for X^q.[1]

Linear combinations in X^q are taken with $q \times q$ matrix (not just complex) coefficients, and subspaces of X^q and linear operators on X^q are to be understood with this sense of linear combination. We then have

1.4. LEMMA. *$\mathbf{M}$ is a (closed) subspace of X^q if and only if $\mathbf{M} = M^q$ where M is a (closed) subspace of X; moreover, M is the set of all components of vectors in $\mathbf{M}$.*

1.5. LEMMA. *$\mathbf{T}$ is a (bounded) linear operator on X^q if and only if $\mathbf{T}(\mathbf{x}) = (T(x^i)), i = 1, 2, \ldots, q$ where T is a (bounded) linear operator on X.*

If $\mathbf{M}$, M, $\mathbf{T}$, and T are as in 1.4 and 1.5, then $\mathbf{M}$ and $\mathbf{T}$ are called the *inflations* of M and T respectively, and M and T are called the *uninflated versions* of $\mathbf{M}$ and $\mathbf{T}$ respectively. If M is a closed subspace of X and if P is the orthogonal projection onto M, then the inflation $\mathbf{P}$ of P acts as an orthogonal (see 1.3) projection onto $\mathbf{M}$, the inflation of M. We shall then write $(\mathbf{x}|\mathbf{M}) = \mathbf{P}(\mathbf{x})$.

A *q-variate, discrete-parameter, weakly stationary stochastic process* (S.P.), $\mathbf{x}$, is

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[1]Vectors and subspaces of X^q and $q \times q$ matrices will be written in boldface, while vectors and subspaces in X will not be written in boldface.

a sequence $\{\mathbf{x}_n, -\infty < n < \infty\}$ of vectors $\mathbf{x}_n$ in X^q such that $(\mathbf{x}_m, \mathbf{x}_n) = \mathbf{\Gamma}(m - n)$ depends only on $m - n$ and not on m and n separately. This is equivalent to saying that $\mathbf{x}_n = \mathbf{V}^n(\mathbf{x}_0)$ where $\mathbf{V}$ is the inflation of a unitary operator V on X. $\mathbf{V}$ or V will be called the *shift operator* of the process $\mathbf{x}$. We are interested in the following subspaces:

$\mathbf{M}_n(\mathbf{x})$ = the smallest closed subspace of X^q containing $\mathbf{x}_k$ for all $k \leqslant n$.

$\mathbf{M}_\infty(\mathbf{x})$ = the smallest closed subspace of X^q containing $\mathbf{x}_k$ for all k.

$\mathbf{M}_{-\infty}(\mathbf{x}) = \bigcap_n \mathbf{M}_n$.

If $\mathbf{x}$ and $\mathbf{y}$ are two S.P. with the same shift operator, we shall say that $\mathbf{y}$ is *subordinate* to $\mathbf{x}$ if $\mathbf{M}_\infty(\mathbf{y}) \subseteq \mathbf{M}_\infty(\mathbf{x})$ and that $\mathbf{y}$ is *dominated* by $\mathbf{x}$ if $\mathbf{M}_n(\mathbf{y}) \subseteq \mathbf{M}_n(\mathbf{x})$ for some finite, and hence all, n. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are S.P. satisfying (1.1), in which case a common shift operator may always be chosen for them, we say that $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is an *orthogonal decomposition* of $\mathbf{x}$. If $\mathbf{y}$ is subordinate to (dominated by) $\mathbf{x}$, we say that $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a *subordinate* (*dominated*) *decomposition*. Thus $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate (dominated) decomposition if and only if $\mathbf{M}_\infty(\mathbf{x}) = \mathbf{M}_\infty(\mathbf{y}) + \mathbf{M}_\infty(\mathbf{z})$ ($\mathbf{M}_n(\mathbf{x}) = \mathbf{M}_n(\mathbf{y}) + \mathbf{M}_n(\mathbf{z})$).

$\mathfrak{g}_n(\mathbf{x}) = (\mathbf{x}_n | \mathbf{M}_{n-1}(\mathbf{x})^\perp \cap \mathbf{M}_n(\mathbf{x})) = \mathbf{V}^n(\mathfrak{g}_0(\mathbf{x}))$ is called the nth *innovation vector* of $\mathbf{x}$. (If $\mathbf{M}$ is a subspace, $\mathbf{M}^\perp$ denotes its orthogonal complement in the sense of 3.1.) We shall be especially interested in the *rank* of $\mathbf{x}$, $r(\mathbf{x})$, which is defined to be the rank of the Gram matrix $(\mathfrak{g}_0(\mathbf{x}), \mathfrak{g}_0(\mathbf{x}))$. $\mathbf{r}(\mathbf{x})$ is easily seen to be the dimension of $W = M_{-1}(\mathbf{x})^\perp \cap M_0(\mathbf{x})$. W is a wandering subspace for V, the shift operator of $\mathbf{x}$, i.e. $V^m(W) \perp V^n(W)$ for all integers $m \neq n$.

If $\mathbf{x}$ is a S.P., then $\mathbf{v}(\mathbf{x})$ defined by $\mathbf{v}_n(\mathbf{x}) = (\mathbf{x}_n | \mathbf{M}_{-\infty}(\mathbf{x}))$ is an S.P. called the *deterministic part* of $\mathbf{x}$. $\mathbf{u}(\mathbf{x})$ defined by $\mathbf{u}_n(\mathbf{x}) = \mathbf{x}_n - \mathbf{v}_n(\mathbf{x})$ is an S.P. called the *purely non-deterministic* part of $\mathbf{x}$. $\mathbf{x} = \mathbf{u}(\mathbf{x}) + \mathbf{v}(\mathbf{x})$ is a dominated orthogonal decomposition of $\mathbf{x}$ and is called the *Wold decomposition* of $\mathbf{x}$. $\mathbf{x}$ is said to be *deterministic* or *purely non-deterministic* according as $\mathbf{x}$ equals $\mathbf{v}(\mathbf{x})$ or $\mathbf{u}(\mathbf{x})$.

Associated with every S.P. is a hermitian matrix-valued function $\mathbf{F}$ on $(0, 2\pi)$ called the spectral distribution. Its derivative $\mathbf{F}'$ (i.e. the derivative of its absolutely continuous part), a non-negative hermitian matrix-valued function, is called the spectral density. Results and arguments which deal directly with X^q are said to be in the *time domain*, while those involving $\mathbf{F}$ are in the *spectral domain*.

In terms of this framework the new results we have obtained may be described as follows.

In §§2 and 3 we show that an orthogonal decomposition $\mathbf{x} = \mathbf{y} + \mathbf{z}$ of an S.P. $\mathbf{x}$ is the Wold decomposition of $\mathbf{x}$ if and only if $\mathbf{y}$ is purely non-deterministic, $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate decomposition, and $r(\mathbf{y}) = r(\mathbf{x})$ (3.1). This theorem is proved by studying the relation between the ranks of the processes in an orthogonal decomposition. We show that if $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is an orthogonal decomposition, a subordinate orthogonal decomposition, or a dominated orthogonal decomposition of $\mathbf{x}$, then $r(\mathbf{x}) \geqslant \max[r(\mathbf{y}), r(\mathbf{z})]$, $r(\mathbf{x}) \geqslant r(\mathbf{y}) + r(\mathbf{z})$, or $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$ respectively (2.1, 2.6, 2.8). We also give examples to show that the last two conditions are not necessary (7.2, 7.3).

In the spectral theory (§§4, 5) we show that if $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate orthogonal decomposition of $\mathbf{x}$ and if $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ are the spectral distributions of the S.P. $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ respectively, then rank $\mathbf{F}'_x$ = rank $\mathbf{F}'_y$ + rank $\mathbf{F}'_z$ almost everywhere (Leb.) (4.5). With the aid of this result and known relations between the ranks of S.P. and the ranks of their spectral densities, we give a necessary and sufficient condition for the concordance of the Wold and the Lebesgue–Cramér decompositions (5.2). This result subsumes results obtained by Rozanov (**12**, Theorem 1), Masani (**5**, Corollary 2.8 and Theorem 4.5), Wiener and Masani (**13**, Theorem 7.11), and Kolmogorov (**4**, Theorem 23) as will be indicated in Corollaries 5.3–5.6.

In §6 we extend our results on discrete S.P. to continuous-parameter S.P. With a mean-continuous, continuous-parameter S.P. $\{\mathbf{x}(t),\ t \text{ real}\}$ with shift group $\{U_t, t \text{ real}\}$ we shall associate the discrete S.P. $\mathbf{x}_n = \mathbf{V}^n(\mathbf{x}(0))$ where $\mathbf{V}$ is the Cayley transform of H, iH being the infinitesimal generator of the shift group. We shall then be able to obtain the Wold decomposition of a continuous parameter S.P. into the sum of a deterministic S.P. and a one-sided moving average S.P. This generalizes the results of (**8**), where only the case $q = 1$ was considered. In the later part of §6 the spectral analysis of continuous-parameter processes is treated. A similar procedure for going from discrete to continuous-parameter results has been given by Gladyshev (**2**). His method differs from ours in that he uses spectral techniques throughout while our time-domain results are handled with time-domain techniques.

The results in §§2 and 3 and the first part of §6 pertain to the time-domain, and our proofs of these results are spectral-free. It is obviously desirable in the interest of coherence and simplicity to avoid spectral considerations in proving such results. With the time-domain analysis so completed, one can develop the spectral theory in an equally coherent manner. Quite apart from this, spectral-free treatments of the time-domain have been found to extend to situations in which spectral analysis fails. For example, Masani (**7**) was able to generalize the time-domain arguments used in (**8**) to obtain valuable results concerning the structure of continuous-parameter semigroups of isometries whereas the techniques of (**2**) would not apply. Thus it seems worth while to give spectral-free proofs of time-domain theorems.

Most of the results here are contained in the author's doctorial dissertation at Indiana University.[2] After the writer had submitted this dissertation, in September 1963, he learned that the results in §§3–5 had been essentially duplicated by Jang Ze-pei (**14**, Part I). Jang Ze-pei considers processes with absolutely continuous spectral distributions. As he points out, most results about S.P. with absolutely continuous spectral distribution can be translated into results about general processes. Thus, for example, the statement that a certain process with absolutely continuous spectral distribution is purely non-

[2]The author is happy to acknowledge the help which Professor Masani, the director of this research, has so graciously given.

deterministic (regular) corresponds to the concordance of the Wold and Lebesgue–Cramér decompositions for general processes. We mention that Jang Ze-pei's proof of the result, which corresponds to our Theorem 4.7, uses a rather cumbersome diagonalization of the spectral densities, which we have avoided. Also our proof of Theorem 3.1 is spectral-free and without the assumption of an absolutely continuous spectral distribution. Jang Ze-pei (**14**, Parts I and II) also gives spectral criteria for a process to be deterministic. We have not considered that question here.

2. The ranks of orthogonal decompositions. In this section we shall investigate the relationships of the ranks of the processes in the orthogonal decomposition $\mathbf{x} = \mathbf{y} + \mathbf{z}$. We start with a general theorem.

2.1. THEOREM. *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is an orthogonal decomposition of the S.P.* $\mathbf{x}$, *then* $r(\mathbf{x}) \geqslant \max\{r(\mathbf{y}), r(\mathbf{z})\}$.

Theorem 2.1 is an immediate consequence of the following more general result.

2.2. THEOREM. *Let* $\mathbf{x}$ *be an S.P. with shift operator* $\mathbf{V}$, *and let* $\mathbf{S}$ *be a subspace of* X^q *such that* $\mathbf{V}(\mathbf{S}) = \mathbf{S}$. *Then:*

(a) $\mathbf{y}_n = (\mathbf{x}_n|\mathbf{S})$ *is an S.P. with the same shift operator* $\mathbf{V}$.

(b) $\mathbf{g}_0(\mathbf{y}) = (\mathbf{g}_0(\mathbf{x})|\mathbf{M}_{-1}(\mathbf{y})^\perp \cap \mathbf{M}_0(\mathbf{y}))$.

(c) $r(\mathbf{y}) \leqslant r(\mathbf{x})$.

Proof. (a) Since $\mathbf{V}(\mathbf{S}) = \mathbf{S}$ and $\mathbf{V}$ is unitary, $\mathbf{V}(\cdot|\mathbf{S}) = (\mathbf{V}(\cdot)|\mathbf{S})$. Hence $V^n(y_0) = y_n$ as desired.

(b) Since $(\mathbf{x}_n|\mathbf{S})$ is in $\mathbf{M}_n(\mathbf{y}) \subseteq \mathbf{S}$, we have

$$\begin{aligned}[(\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{x}))|\mathbf{M}_0(\mathbf{y})] &= ([(\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{x}))|\mathbf{S}]|\mathbf{M}_0(\mathbf{y})) \\ &= ([(\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{x}))|\mathbf{S}]|\mathbf{M}_{-1}(\mathbf{y})) \\ &= [(\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{x}))|\mathbf{M}_{-1}(\mathbf{y})].\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{g}_0(\mathbf{y}) &= (\mathbf{y}_0|\mathbf{M}_{-1}(\mathbf{y})^\perp \cap \mathbf{M}_0(\mathbf{y})) \\ &= (\mathbf{x}_0|\mathbf{M}_0(\mathbf{y})) - [(\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{x}))|\mathbf{M}_0(\mathbf{y})] \\ &\quad + [(\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{x}))|\mathbf{M}_{-1}(\mathbf{y})] - (\mathbf{x}_0|\mathbf{M}_{-1}(\mathbf{y})) \\ &= (\mathbf{g}_0(\mathbf{x})|\mathbf{M}_{-1}(\mathbf{y})^\perp \cap \mathbf{M}_0(\mathbf{y})) \qquad \text{as desired.}\end{aligned}$$

(c) $\dim\{g_0^1(\mathbf{y}), \ldots, g_0^q(\mathbf{y})\} \leqslant \dim\{g_0^1(\mathbf{x}), \ldots, g_0^q(\mathbf{x})\}$ since $\{g_0^1(\mathbf{y}), \ldots, g_0^q(\mathbf{y})\}$ is the image of $\{g_0^1(\mathbf{x}), \ldots, g_0^q(\mathbf{x})\}$ under the linear transformation $(.|M_{-1}(\mathbf{y})^\perp \cap M_0(\mathbf{y}))$. Thus

$$r(\mathbf{y}) = \dim\{g_0^1(\mathbf{y}), \ldots, g_0^q(\mathbf{y})\} \leqslant \dim\{g_0^1(\mathbf{x}), \ldots, g_0^q(\mathbf{x})\} = r(\mathbf{x}).$$

Next we shall give two lemmas which will be useful in studying subordinate decompositions.

2.3. LEMMA. *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is an orthogonal decomposition of the S.P.* $\mathbf{x}$, *then:*

(a) $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{z}) + [\mathbf{v}(\mathbf{y}) + \mathbf{v}(\mathbf{z}) - \mathbf{v}(\mathbf{x})]$ *is an orthogonal decomposition of* $\mathbf{u}(\mathbf{x})$.

(b) *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a subordinate decomposition, then so is the decomposition in* (a).

(c) *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a dominated decomposition, then so is the decomposition in* (a).

Proof. (a) Clearly $\mathbf{u}_n(\mathbf{x}) = \mathbf{u}_n(\mathbf{y}) + \mathbf{u}_n(\mathbf{z}) + [\mathbf{v}_n(\mathbf{y}) + \mathbf{v}_n(\mathbf{z}) - \mathbf{v}_n(\mathbf{x})]$, and we have

$$\mathbf{M}_\infty(\mathbf{v}(\mathbf{x})) = \mathbf{M}_{-\infty}(\mathbf{x}) \subseteq \bigcap_{n\leqslant 0} \{\mathbf{M}_n(\mathbf{u}(\mathbf{y})) + \mathbf{M}_n(\mathbf{u}(\mathbf{z})) + \mathbf{M}_n(\mathbf{v}(\mathbf{y})) + \mathbf{M}_n(\mathbf{v}(\mathbf{z}))\}.$$

Since the four subspaces inside $\{\ldots\}$ are mutually orthogonal and decreasing as $n \to -\infty$, the intersection of the sum is the sum of the intersections. Thus

$$\mathbf{M}_\infty(\mathbf{v}(\mathbf{x})) \subseteq \mathbf{M}_\infty(\mathbf{v}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{v}(\mathbf{z})). \tag{1}$$

Therefore,

$$\begin{aligned}\mathbf{M}_\infty(\mathbf{v}(\mathbf{y}) + \mathbf{v}(\mathbf{z}) - \mathbf{v}(\mathbf{x})) &\subseteq \{\mathbf{M}_\infty(\mathbf{v}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{v}(\mathbf{z}))\} \\ &\subseteq \{\mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{u}(\mathbf{z}))\}^\perp.\end{aligned}$$

Since $\mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) \perp \mathbf{M}_\infty(\mathbf{u}(\mathbf{z}))$, (a) is proved.

(b) By (1), we have

$$\mathbf{M}_\infty(\mathbf{v}(\mathbf{x})) \perp \{\mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{u}(\mathbf{z}))\}.$$

Hence, since $\mathbf{y}$ and $\mathbf{z}$ are subordinate to $\mathbf{x}$,

$$\mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{u}(\mathbf{z})) \subseteq \mathbf{M}_\infty(\mathbf{x}) \cap \mathbf{M}_\infty(\mathbf{v}(\mathbf{x}))^\perp = \mathbf{M}_\infty(\mathbf{u}(\mathbf{x})).$$

Thus the decomposition in (a) is a subordinate decomposition.

(c) is proved in the same way as (b).

2.4. LEMMA. *Let V be a unitary operator on a Hilbert space H. Let X and Y be wandering subspaces for V such that:*

(i) $\sum_{k=-\infty}^{\infty} V^k(X) \subseteq \sum_{k=-\infty}^{\infty} V^k(Y)$,

(ii) $\dim Y < \infty$,

then

(a) $\dim X \leqslant \dim Y$.

(b) $\dim X = \dim Y$ *if and only if*

$$\sum_{k=-\infty}^{\infty} V^k(X) = \sum_{k=-\infty}^{\infty} V^k(Y).$$

Part (a) of Lemma 2.4 is due to Halmos (**3′**, Lemma 4). For a proof of Lemma 2.4 the reader is referred to (**9**, Theorem 1).

2.5. THEOREM. *Let* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *be an orthogonal decomposition of the S.P.* $\mathbf{x}$, *and suppose that* $\mathbf{y}$ *and* $\mathbf{z}$ *are purely non-deterministic S.P. Then* $\mathbf{x}$ *is also purely* *non-deterministic and* $r(\mathbf{x}) \leqslant r(\mathbf{y}) + r(\mathbf{z})$. $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$ *if and only if* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a subordinate decomposition.*

Proof. Clearly we have $\mathbf{M}_n(\mathbf{x}) \subseteq \mathbf{M}_n(\mathbf{y}) + \mathbf{M}_n(\mathbf{z})$ for all integers n. Taking intersections on both sides and noticing that here the intersection of the sum is the sum of the intersections, we see that $\mathbf{M}_{-\infty}(\mathbf{x}) \subseteq \{\mathbf{0}\}$ and hence $\mathbf{x}$ is purely non-deterministic.

Since $M_\infty(\mathbf{x}) \subseteq M_\infty(\mathbf{y}) + M_\infty(\mathbf{z})$ and since $M_\infty(\mathbf{x})$ and $M_\infty(\mathbf{y}) + M_\infty(\mathbf{z})$ are generated by the wandering subspaces $M_{-1}(\mathbf{x})^\perp \cap M_0(\mathbf{x})$ and $\{M_{-1}(\mathbf{y})^\perp \cap M_0(\mathbf{y})\} + \{M_{-1}(\mathbf{z})^\perp \cap M_0(\mathbf{z})\}$ respectively, Lemma 2.4 (a) yields

$$\begin{aligned} r(\mathbf{x}) &= \dim \{M_{-1}(\mathbf{x})^\perp \cap M_0(\mathbf{x})\} \\ &\leqslant \dim \{\{M_{-1}(\mathbf{y})^\perp \cap M_0(\mathbf{y})\} + \{M_{-1}(\mathbf{z})^\perp \cap M_0(\mathbf{z})\}\} \\ &= \dim \{M_{-1}(\mathbf{y})^\perp \cap M_0(\mathbf{y})\} + \dim \{M_{-1}(\mathbf{z})^\perp \cap M_0(\mathbf{z})\} = r(\mathbf{y}) + r(\mathbf{z}). \end{aligned}$$

In the same manner Lemma 2.4 (b) yields $\mathbf{M}_\infty(\mathbf{x}) = \mathbf{M}_\infty(\mathbf{y}) + \mathbf{M}_\infty(\mathbf{z})$ if and only if $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$.

One might have hoped that the subordinate decomposition $\mathbf{x} = \mathbf{y} + \mathbf{z}$ given in Theorem 2.5 was actually a dominated decomposition. Example 7.4 is a decomposition $\mathbf{x} = \mathbf{y} + \mathbf{z}$ satisfying the conditions of Theorem 2.5 which is a subordinate decomposition but not a dominated one.

2.6. THEOREM. *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a subordinate orthogonal decomposition of the S.P.* $\mathbf{x}$, *then* $r(\mathbf{x}) \geqslant r(\mathbf{y}) + r(\mathbf{z})$.

Proof. Lemma 2.3 (b) yields

$$\mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{u}(\mathbf{z})) \subseteq \mathbf{M}_\infty(\mathbf{u}(\mathbf{x})). \tag{1}$$

By the same argument as in the proof of Theorem 2.5, we obtain

$$\begin{aligned} r(\mathbf{x}) &= \dim \{M_{-1}(\mathbf{x})^\top \cap M_0(\mathbf{x})\} \\ &\geqslant \dim \{\{M_{-1}(\mathbf{y})^\perp \cap M_0(\mathbf{y})\} + \{M_{-1}(\mathbf{z})^\perp \cap M_0(\mathbf{z})\}\} \\ &= \dim \{M_{-1}(\mathbf{y})^\perp \cap M_0(\mathbf{y})\} + \dim \{M_{-1}(\mathbf{z})^\perp \cap M_0(\mathbf{z})\} = r(\mathbf{y}) + r(\mathbf{z}). \end{aligned}$$

That the converse of Theorem 2.6 is false is shown in Example 7.2. As Theorem 2.5 suggests, there are orthogonal decompositions $\mathbf{x} = \mathbf{y} + \mathbf{z}$ such that $r(\mathbf{x}) < r(\mathbf{y}) + r(\mathbf{z})$ (cf. Example 7.2).

2.7. COROLLARY. *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a subordinate orthogonal decomposition of the S.P.* $\mathbf{x}$, *then* $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$ *if and only if* $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{z})$.

Proof. From equation (1) in the proof of Theorem 2.6 and from Lemma 2.4 (a) we see that $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$ if and only if $\mathbf{M}_\infty(\mathbf{u}(\mathbf{x})) = \mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{u}(\mathbf{z}))$. By Lemma 2.3 (b) $\mathbf{M}_\infty(\mathbf{u}(\mathbf{x})) = \mathbf{M}_\infty(\mathbf{u}(\mathbf{y})) + \mathbf{M}_\infty(\mathbf{u}(\mathbf{z}))$ if and only if $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y}) + \mathbf{v}(\mathbf{z})$, which, of course, is true if and only if $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{z})$.

Example 7.3 shows that there are subordinate decompositions with $r(\mathbf{x}) > r(\mathbf{y}) + r(\mathbf{z})$.

2.8. THEOREM. *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a dominated orthogonal decomposition of the S.P.* $\mathbf{x}$, *then* $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$.

Proof. Since $\mathbf{M}_n(\mathbf{x}) = \mathbf{M}_n(\mathbf{y}) + \mathbf{M}_n(\mathbf{z})$ for all n, we have

$$\begin{aligned} \mathbf{M}_{-1}(\mathbf{x}) + \{\mathbf{M}_{-1}(\mathbf{x})^{\perp} \cap \mathbf{M}_0(\mathbf{x})\} &= \mathbf{M}_0(\mathbf{x}) \\ &= \mathbf{M}_0(\mathbf{y}) + \mathbf{M}_0(\mathbf{z}) \\ &= \mathbf{M}_{-1}(\mathbf{y}) + \{\mathbf{M}_{-1}(\mathbf{y})^{\perp} \cap \mathbf{M}_0(\mathbf{y})\} \\ &\quad + \mathbf{M}_{-1}(\mathbf{z}) + \{\mathbf{M}_{-1}(\mathbf{z})^{\perp} \cap \mathbf{M}_0(\mathbf{z})\} \\ &= \mathbf{M}_{-1}(\mathbf{x}) + \{\mathbf{M}_{-1}(\mathbf{y})^{\perp} \cap \mathbf{M}_0(\mathbf{y})\} \\ &\quad + \{\mathbf{M}_{-1}(\mathbf{z})^{\perp} \cap \mathbf{M}_0(\mathbf{z})\}. \end{aligned}$$

Since the terms of the sums are orthogonal, we obtain

$$\mathbf{M}_{-1}(\mathbf{x})^{\perp} \cap \mathbf{M}_0(\mathbf{x}) = \{\mathbf{M}_{-1}(\mathbf{y})^{\perp} \cap \mathbf{M}_0(\mathbf{y})\} + \{\mathbf{M}_{-1}(\mathbf{z})^{\perp} \cap \mathbf{M}_0(\mathbf{z})\}.$$

Hence $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$.

That the converse of Theorem 2.8 is false is shown in Examples 7.2 and 7.4.

2.9. COROLLARY. *If* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a dominated orthogonal decomposition of the S.P.* $\mathbf{x}$, *then* $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{z})$ *and* $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y}) + \mathbf{v}(\mathbf{z})$.

Corollary 2.9 follows immediately from Theorem 2.8 and Corollary 2.7.

3. Characterization of the Wold decomposition. In this section we shall use the results of the last section to characterize the Wold decomposition.

3.1. THEOREM. *Let* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *be an orthogonal decomposition of the S.P.* $\mathbf{x}$. $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is the Wold decomposition of* $\mathbf{x}$ *if and only if* $\mathbf{y}$ *is purely non-deterministic,* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a subordinate decomposition, and* $r(\mathbf{x}) = r(\mathbf{y})$.

Proof. That the Wold decomposition has the desired properties is well known. Suppose, therefore, that $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate orthogonal decomposition of the S.P. $\mathbf{x}$ where $\mathbf{y}$ is purely non-deterministic and $r(\mathbf{x}) = r(\mathbf{y})$. Then by Theorem 2.6 $\mathbf{z}$ is deterministic and $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$. Thus by Corollary 2.7 $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{z}) = \mathbf{y}$ and $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y}) + \mathbf{v}(\mathbf{z}) = \mathbf{z}$.

The next theorem is well known (cf. e.g. (**2**, Theorem 1)).

3.2. THEOREM. *Let* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *be an orthogonal decomposition of the S.P.* $\mathbf{x}$. $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is the Wold decomposition of* $\mathbf{x}$ *if and only if* $\mathbf{y}$ *is purely non-deterministic,* $\mathbf{z}$ *is deterministic, and* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *is a dominated decomposition.*

Proof. As in Theorem 3.1 we shall only prove the "if part." Since $\mathbf{z} = \mathbf{y} + \mathbf{z}$ is a dominated decomposition and $\mathbf{z}$ is deterministic, Theorem 2.8 yields $r(\mathbf{x}) = r(\mathbf{y})$. Thus the conditions of Theorem 3.1 are fulfilled.

If in Theorem 3.2 "dominated" is replaced by "subordinate," the resulting statement is true for univariate processes (for $q = 1$, $r(\mathbf{x}) = r(\mathbf{y}) = 1$), but is false in general as is shown by Example 7.5.

4. Spectral analysis of subordinate orthogonal decompositions. In this section we shall first discuss a theorem proved recently by M. Rosenberg **(10)** on the isomorphism between the spectral domain $\mathbf{L}_{2,F}(0, 2\pi)$ and the time domain $\mathbf{M}_\infty(\mathbf{x})$ of an S.P. $\mathbf{x}$. With the aid of this isomorphism we shall study the relationship between the spectral distributions $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ of the S.P. $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ where $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate orthogonal decomposition.

Let $\mathbf{x}$ be an S.P. with shift operator V. Let

$$\mathbf{V} = \int_0^{2\pi} e^{i\theta}\, d\mathbf{E}_\theta$$

be the spectral representation of V and define $\mathbf{F}(\theta) = 2\pi(\mathbf{E}_\theta, \mathbf{x}_0, \mathbf{x}_0)$. The matrix-valued function $\mathbf{F}$ is called the spectral distribution function of $\mathbf{x}$. M. Rosenberg **(10**, §3**)** has defined the integral

$$(\mathbf{\Phi}, \mathbf{\Psi})_F = \int_0^{2\pi} \mathbf{\Phi}\, d\mathbf{F}\mathbf{\Psi}^*$$

of $q \times q$ matrix-valued functions $\mathbf{\Phi}$ and $\mathbf{\Psi}$ with respect to $\mathbf{F}$. With his definition the set of all $q \times q$ matrix-valued functions $\mathbf{\Phi}$ such that $\mathbf{\Phi}$ is measurable and $(\mathbf{\Phi}, \mathbf{\Phi})_F$ is finite is a (complete) Hilbert space $\mathbf{L}_{2,F}(0, 2\pi)$ with inner product $((\mathbf{\Phi}, \mathbf{\Psi}))_F = \operatorname{tr}\, (\mathbf{\Phi}, \mathbf{\Psi})_F$ (cf. **(10**, 3.9**)**). He is then able to define the stochastic integral

$$\boldsymbol{\phi} = \int_0^{2\pi} \mathbf{\Phi}(\theta)\, d\mathbf{E}_\theta(\mathbf{x}_0)$$

of a function $\mathbf{\Phi}$ in $\mathbf{L}_{2,F}(0, 2\pi)$ with respect to the process of orthogonal increments $\{\mathbf{E}_\theta(\mathbf{x}_0), 0 \leqslant \theta \leqslant 2\pi\}$. $\boldsymbol{\phi}$ is a vector in X^q, and he then proves (cf. **(10**, 4.6**)**):

4.1. Isomorphism Theorem. *Let* $\mathbf{x}$, E_θ, *and* $\mathbf{F}$ *be as above. Then the correspondence between* $\mathbf{\Phi}$ *and*

$$\boldsymbol{\phi} = \int_0^{2\pi} \mathbf{\Phi}(\theta)\, d\mathbf{E}_\theta(\mathbf{x}_0)$$

is an isomorphism from the Hilbert space $\mathbf{L}_{2,F}(0, 2\pi)$ *onto the Hilbert space* $\mathbf{M}_\infty(\mathbf{x})$. *Moreover* $(\mathbf{\Phi}, \mathbf{\Psi})_F = (\boldsymbol{\phi}, \boldsymbol{\psi})$ *whenever* $\mathbf{\Phi}$ *and* $\mathbf{\Psi}$ *correspond to* $\boldsymbol{\phi}$ *and* $\boldsymbol{\psi}$ *respectively.*

We shall also use the following result which can be derived from Theorem 4.1 (cf. **(11**, 7.24**)**).

4.2. Theorem. *Let* $\mathbf{x}$ *be an S.P. with shift operator* V *and spectral distribution* $\mathbf{F}_x$.

(a) *If* $\mathbf{y}_0$ *is in* $\mathbf{M}_\infty(\mathbf{x})$ *and if* $\mathbf{y}_0$ *corresponds to* $\mathbf{\Phi}$ *under the isomorphism between* $\mathbf{M}_\infty(\mathbf{x})$ *and* $\mathbf{L}_{2,Fx}(0, 2\pi)$, *then the S.P.* $\mathbf{y}_n = \mathbf{V}^n(\mathbf{y}_0)$ *has a spectral distribution* $\mathbf{F}_y$ *satisfying*

$$\mathbf{F}'_y(\theta) = \mathbf{\Phi}(\theta)\mathbf{F}'_x(\theta)\mathbf{\Phi}(\theta)^* \text{ a.e.(Leb.)},$$

where $\mathbf{F}'$ *indicates the derivative of* $\mathbf{F}$ *with respect to Lebesgue measure on* $(0, 2\pi)$.

(b) *Let* $\mathbf{y}_0$ *and* $\mathbf{z}_0$ *be in* $\mathbf{M}_\infty(\mathbf{x})$ *and correspond to* $\mathbf{\Phi}_y$ *and* $\mathbf{\Phi}_z$ *respectively under the isomorphism, and let* $\mathbf{y}_n = \mathbf{V}^n(\mathbf{y}_0)$ *and* $\mathbf{z}_n = \mathbf{V}^n(\mathbf{z}_0)$. *Then* $\mathbf{y}_m \perp \mathbf{z}_n$ *for all m and n implies that*

$$\mathbf{\Phi}_y(\theta)\mathbf{F'}_x(\theta)\mathbf{\Psi}_z(\theta)^* = \mathbf{0} \text{ a.e.(Leb.)}.$$

If $\mathbf{A}$ is a $q \times q$ matrix, let $R(\mathbf{A}) = \{\mathbf{b}\colon \mathbf{b} = \mathbf{A}\mathbf{a} \text{ for some } \mathbf{a} \text{ in } C^q\}$ be the range of $\mathbf{A}$.

4.3. Lemma. *Let* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *be an orthogonal decomposition of the S.P.* $\mathbf{x}$. *Then:*
(a) $\mathbf{F}_x(\theta) = \mathbf{F}_y(\theta) + \mathbf{F}_z(\theta)$,
(b) $R(\mathbf{F'}_x(\theta)) = R(\mathbf{F'}_y(\theta)) + R(\mathbf{F'}_z(\theta))$ a.e.(Leb.).

Proof. (a) is well known (cf. e.g. **(13**, §7**)**).
(b) Differentiating (a) we get $\mathbf{F'}_x(\theta) = \mathbf{F'}_y(\theta) + \mathbf{F'}_z(\theta)$ a.e.(Leb.). Since $\mathbf{F'}_x(\theta)$, $\mathbf{F'}_y(\theta)$, and $\mathbf{F'}_z(\theta)$ are non-negative hermitian matrices (cf. e.g. **(13)**), (b) follows from elementary considerations.

4.4. Lemma. *Let* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *be a subordinate orthogonal decomposition of the S.P.* $\mathbf{x}$. *Then:*
(a) $R(\mathbf{F'}_y(\theta)) \cap R(\mathbf{F'}_z(\theta)) = \{\mathbf{0}\}$ a.e.(Leb.).
(b) *Let* $\mathbf{\Phi}_y$ *and* $\mathbf{\Phi}_z$ *correspond to* $\mathbf{y}_0$ *and* $\mathbf{z}_0$ *respectively under the isomorphism between* $\mathbf{L}_{2,Fx}(0, 2\pi)$ *and* $\mathbf{M}_\infty(\mathbf{x})$. *Let* $\tilde{\mathbf{\Phi}}_y(\theta)$ *and* $\tilde{\mathbf{\Phi}}_z(\theta)$ *denote the restrictions of the operators* $\mathbf{\Phi}_y(\theta)$ *and* $\mathbf{\Phi}_z(\theta)$ *respectively to* $R(\mathbf{F'}_x(\theta))$. *Then* $\tilde{\mathbf{\Phi}}_y(\theta)$ *is the projection of* $R(\mathbf{F'}_x(\theta))$ *onto* $R(\mathbf{F'}_y(\theta)$ *along* $R(\mathbf{F'}_z(\theta))$ a.e.(Leb.), *and* $\tilde{\mathbf{\Phi}}_z(\theta)$ *is the projection from* $R(\mathbf{F'}_x(\theta))$ *onto* $R(\mathbf{F'}_z(\theta))$ *along* $R(\mathbf{F'}_y(\theta))$ a.e.(Leb.) (cf. Halmos **(3**, §30**)** for terminology).

Proof. To prove the theorem we shall establish in succession the following results:
(1) $\mathbf{\Phi}_y(\theta) + \mathbf{\Phi}_z(\theta) = I$ on $R(\mathbf{F'}_x(\theta))$ a.e.(Leb.).
(2) $\mathbf{F'}_y(\theta) = \mathbf{\Phi}_y(\theta)\mathbf{F'}_x(\theta)\mathbf{\Phi}_y(\theta)^* = \mathbf{\Phi}_y(\theta)\mathbf{F'}_x(\theta) = \mathbf{F'}_x(\theta)\mathbf{\Phi}_y(\theta)^*$ a.e.(Leb.).
(3) $\tilde{\mathbf{\Phi}}_y(\theta)$ maps $R(\mathbf{F'}_x(\theta))$ onto $R(\mathbf{F'}_y(\theta))$ a.e.(Leb.).
(4) $\tilde{\mathbf{\Phi}}_z(\theta)$ maps $R(\mathbf{F'}_x(\theta))$ onto $R(\mathbf{F'}_z(\theta))$ a.e.(Leb.).
(5) $\tilde{\mathbf{\Phi}}_y(\theta)^2 = \tilde{\mathbf{\Phi}}_y(\theta)$ a.e.(Leb.).
(1) To prove (1) observe that $\mathbf{y}_0 + \mathbf{z}_0 = \mathbf{x}_0$ and that $\boldsymbol{I}$ corresponds to $\mathbf{x}_0$ under the isomorphism in question.
(2) By Theorem 4.2(a)

$$\mathbf{F'}_y(\theta) = \mathbf{\Phi}_y(\theta)\mathbf{F'}_x(\theta)\mathbf{\Phi}_y(\theta)^* \quad \text{a.e.(Leb.)},$$

and by part (b) of that theorem, we have

$$\mathbf{\Phi}_y(\theta)\mathbf{F'}_x(\theta)(\boldsymbol{I} - \mathbf{\Phi}_y(\theta))^* = \mathbf{0} \quad \text{a.e.(Leb.)}.$$

Hence the second equality in (2). The last equality in (2) is obtained by taking conjugate transposes on both sides of the second equality.
(3) Follows at once from the equation

$$\mathbf{F'}_y(\theta) = \mathbf{\Phi}_y(\theta)\mathbf{F'}_x(\theta) \quad \text{a.e.(Leb.)}.$$

(4) Follows from the equations

$$\begin{aligned}\mathbf{F}'_z(\theta) &= \mathbf{F}'_x(\theta) - \mathbf{F}'_y(\theta)\\ &= (\boldsymbol{I} - \mathbf{\Phi}_y(\theta))\mathbf{F}'_x(\theta) = \mathbf{\Phi}_z(\theta)\mathbf{F}'_x(\theta) \quad \text{a.e.(Leb.).}\end{aligned}$$

(5) Using (2) we get

$$\begin{aligned}\mathbf{\Phi}_y(\theta)^2\mathbf{F}'_x(\theta) &= \mathbf{\Phi}_y(\theta)(\mathbf{\Phi}_y(\theta)\mathbf{F}'_x(\theta))\\ &= \mathbf{\Phi}_y(\theta)\mathbf{F}'_x(\theta)\mathbf{\Phi}_y(\theta)^*\\ &= \mathbf{F}'_y(\theta) = \mathbf{\Phi}_y(\theta)\mathbf{F}'_x(\theta) \quad \text{a.e.(Leb.),}\end{aligned}$$

which yields (5).

Conditions (1)–(5) and Lemma 4.3(b) imply that $\tilde{\mathbf{\Phi}}_y(\theta)$ is the projection from $R(\mathbf{F}'_x(\theta))$ onto $R(\mathbf{F}'_y(\theta))$ along $R(\mathbf{F}'_z(\theta))$ a.e.(Leb.), and a similar statement is obtained by interchanging $\mathbf{y}$ and $\mathbf{z}$ (cf. Halmos **(3**, §30**)**). But these, in turn, imply (a).

Notice that it is not true that $R(\mathbf{F}'_y(\theta)) \perp R(\mathbf{F}'_z(\theta))$ a.e.(Leb.) (cf. the example given by Masani in **(5)**). We summarize the results of the last two lemmas as follows:

4.5. THEOREM. *Let* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *be a subordinate orthogonal decomposition of the S.P.* $\mathbf{x}$. *Then:*

(a) $R(\mathbf{F}'_x(\theta)) = R(\mathbf{F}'_y(\theta)) + R(\mathbf{F}'_z(\theta))$ a.e.(Leb.).

(b) $R(\mathbf{F}'_y(\theta)) \cap R(\mathbf{F}'_z(\theta)) = \{\mathbf{0}\}$ a.e.(Leb.).

(c) $\operatorname{rank} \mathbf{F}'_x(\theta) = \operatorname{rank} \mathbf{F}'_y(\theta) + \operatorname{rank} \mathbf{F}'_z(\theta)$ a.e.(Leb.).

5. The Wold decomposition and the Lebesgue–Cramér decomposition. The Wold decomposition of a process $\mathbf{x}$ into its purely non-deterministic part $\mathbf{u}$ and its deterministic part $\mathbf{v}$ satisfies (1.1), i.e. $\mathbf{x}_n = \mathbf{u}_n + \mathbf{v}_n$ and $\mathbf{u}_m \perp \mathbf{v}_n$, $-\infty < m$, $m < \infty$. In addition $\mathbf{u}$ is dominated by $\mathbf{x}$. Thus by Lemma 4.3 we have a decomposition $\mathbf{F}_x = \mathbf{F}_u + \mathbf{F}_v$ of the spectral distribution of $\mathbf{x}$. We also have the Lebesgue–Cramér decomposition $\mathbf{F}_x = \mathbf{F}_a + \mathbf{F}_b$, where $\mathbf{F}_a$ is the absolutely continuous part (with respect to Lebesgue measure) and $\mathbf{F}_b$ is the jump-singular part of $\mathbf{F}_x$ (cf. Cramér **(1)**). In this section we shall give a necessary and sufficient condition for the concordance of these two decompositions, i.e. for $\mathbf{F}_u = \mathbf{F}_a$ and $\mathbf{F}_v = \mathbf{F}_b$.

The following lemma is well known (cf. e.g. Masani **(6)**).

5.1. LEMMA. *Let* $\mathbf{x}$ *and* $\mathbf{F}_u$ *be as above, then*

(a) $\mathbf{F}_u$ *is absolutely continuous.*

(b) $\operatorname{rank} \mathbf{F}'_u(\theta) = r(\mathbf{x})$ a.e.(Leb.).

5.2. THEOREM. $\mathbf{F}_u = \mathbf{F}_a$ *and* $\mathbf{F}_v = \mathbf{F}_b$ *if and only if rank* $\mathbf{F}'_x(\theta) = r(\mathbf{x})$ a.e.(Leb.).

Proof. Suppose $\mathbf{F}_u = \mathbf{F}_a$. Using the fact that $\mathbf{F}'_b = \mathbf{0}$ a.e.(Leb.) and Theorem 4.5(c), we obtain

$$\text{rank } \mathbf{F}'_x(\theta) = \text{rank } \mathbf{F}'_a(\theta) = \text{rank } \mathbf{F}'_u(\theta) = r(\mathbf{x}) \quad \text{a.e.(Leb.)}.$$

Conversely, if rank $\mathbf{F}'_x(\theta) = r(\mathbf{x})$ a.e.(Leb.), then using 4.5(c) again

$$\begin{aligned} \text{rank } \mathbf{F}'_v(\theta) &= \text{rank } \mathbf{F}'_x(\theta) - \text{rank } \mathbf{F}'_u(\theta) \\ &= \text{rank } \mathbf{F}'_x(\theta) - r(\mathbf{x}) = \mathbf{0} \quad \text{a.e.(Leb.)}. \end{aligned}$$

Therefore, $\mathbf{F}'_v(\theta) = \mathbf{0}$ a.e.(Leb.), and hence $\mathbf{F}'_x(\theta) = \mathbf{F}'_u(\theta)$ a.e.(Leb.). Since $\mathbf{F}_u$ is absolutely continuous, $\mathbf{F}_u = \mathbf{F}_a$ as desired.

This theorem subsumes the results of various authors in special cases:

5.3. Corollary (Rozanov (**12**, Theorem 1)). *Suppose* $\mathbf{F}_x$ *is absolutely continuous and that* rank $\mathbf{F}'_x(\theta) = 1$ a.e.(Leb.). *Then* $\mathbf{x}$ *is either deterministic or purely non-deterministic.*

5.4. Corollary (Masani (**5**, Corollary 2.8 and Theorem 4.5)). *Suppose* $\mathbf{x}$ *is a bivariate process and that* $r(\mathbf{x}) = 1$. *Then* $\mathbf{F}_u = \mathbf{F}_a$ *and* $\mathbf{F}_v = \mathbf{F}_b$ *if and only if* det $\mathbf{F}'_x(\theta) = 0$ a.e.(Leb.).

5.5. Corollary (Wiener and Masani (**13**, Theorem 7.11)). *Suppose that* $\mathbf{x}$ *is a* q*-variate process and that* $r(\mathbf{x}) = q$. *Then* $\mathbf{F}_u = \mathbf{F}_a$ *and* $\mathbf{F}_v = \mathbf{F}_b$.

5.6. Corollary (Kolmogorov (**4**, Theorem 23)). *Suppose that* $\mathbf{x}$ *is a univariate non-deterministic* ($r(\mathbf{x}) > 0$) *process. Then* $\mathbf{F}_u = \mathbf{F}_a$ *and* $\mathbf{F}_v = \mathbf{F}_b$.

6. Continuous parameter processes. This section is devoted to describing a procedure for extending results concerning discrete-parameter S.P. to continuous-parameter ones. Gladyshev (**2**) has also given such a procedure. However, his procedure was entirely in the spectral domain while we treat time-domain results by purely time-domain techniques. As the results given here are direct generalizations of those in (**8**), we shall omit the proofs.

A *q-variate, continuous-parameter, weakly stationary stochastic process* on X^q is a function $\mathbf{x}$ from the real line R to X^q such that $(\mathbf{x}(s), \mathbf{x}(t)) = \mathbf{\Gamma}(s - t)$ depends only on $s - t$ and not on s and t separately. This is equivalent to saying that there exists a one-parameter group $\{U_t, t \text{ real}\}$ of unitary operators on X such that $\mathbf{x}(t) = \mathbf{U}_t(\mathbf{x}(0))$. Throughout we shall assume that $\{U_t, t \text{ real}\}$ is strongly continuous, i.e. for each x in X $\|U_{s+t}(x) - U_t(x)\| \to 0$ as $s \to 0$. This is the same as requiring that the process $\mathbf{x}$ be a continuous function from R to X^q.

Let $\{U_t, t \text{ real}\}$ be a strongly continuous one-parameter group of unitary operators on X. It is known that $\{U_t, t \text{ real}\}$ has an *infinitesimal generator*[3]

$$iH = \lim_{t\to 0} (1/t)\{U_t - I\} \quad \text{on } D, \tag{6.1}$$

[3]Limits in this section are to be understood in the strong sense, i.e. for every α in D $\|\{(1/t)(U_t - I) - iH\}(\alpha)\| \to 0$ as $t \to 0$.

where H is a self-adjoint operator with domain D, and D is a linear manifold everywhere dense in X.

Now let V be the *Cayley transform* of H:

$$V = c(H) = (H - iI)(H + iI)^{-1} \quad \text{on } X. \tag{6.2}$$

Then V is a unitary operator on X such that

$$\begin{cases} \text{(a)} \quad H = i(I + V)(I - V)^{-1} \quad \text{on } D, \\ \text{(b)} \quad U_t V^n = V^n U_t \quad \text{on } X, \qquad -\infty < n, t < \infty, n \text{ an integer.} \end{cases} \tag{6.3}$$

The relationships between U_t and V^n for arbitrary t and n are given by the following equations (cf. (**8**, 2.7 and 2.8)):

$$\begin{cases} U_{\pm t} = e^{-t}I + \lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} \dfrac{1}{k!}\left(\dfrac{-nt}{n+1}\right)^k \{(I + A_{\pm n})^k - I\}, \qquad t \geqslant 0, \\ \text{where } A_{\pm n} = \dfrac{2n}{n+1} \sum\limits_{j=1}^{\infty} \left(\dfrac{n-1}{n+1}\right)^{j-1} V^{\pm j}, \qquad n \geqslant 0, \end{cases} \tag{6.4}$$

and

$$\begin{cases} V^{\pm n} = I + 2\displaystyle\int_0^{\infty} L'_n(2t)e^{-t}U_{\pm t}\,dt, \qquad n \geqslant 0, \\ \text{where } L_n(t) = \sum\limits_{k=0}^{n} \dfrac{(-1)^k}{k!}\dbinom{n}{k} t^k, \qquad n \geqslant 0 \quad (n\text{th Laguerre polynomial}). \end{cases} \tag{6.5}$$

Using these equations, the following basic lemma may be derived (cf. (**8**, 2.9) for the case $q = 1$).

6.6. Lemma. *Let* $\mathbf{X}$ *be a subset of* X^q, *then:*
(a) $\sigma\{\mathbf{U}_t(\mathbf{X}), t \geqslant 0\} = \sigma\{\mathbf{V}^n(\mathbf{X}), n \geqslant 0\}$.[4]
(b) $\sigma\{\mathbf{U}_{-t}(\mathbf{X}), t \geqslant 0\} = \sigma\{\mathbf{V}^{-n}(\mathbf{X}), n \geqslant 0\}$.

Now let $\mathbf{x}$ be a q-variate, continuous-parameter, weakly stationary stochastic process with a strongly continuous shift group $\{U_t, t \text{ real}\}$, and let $\mathbf{x}'$ denote the discrete-parameter process determined by $\mathbf{x}'_n = \mathbf{V}^n(\mathbf{x}(0))$ where V is given by (6.1) and (6.2). $\mathbf{x}'$ is called the *discrete-parameter process associated with* $\mathbf{x}$. The subspaces determined by $\mathbf{x}'$ will be denoted by $\mathbf{M}'_{-\infty}$, $\mathbf{M}'_n$, and $\mathbf{M}'_{\infty}$. Using Lemma 6.6, we can establish the following theorem by a direct generalization of the proof given in (**8**, 4.4) for the case $q = 1$.

6.7. Theorem. (a) $\mathbf{M}_0 = \mathbf{M}'_0$. (b) $\mathbf{M}_{\infty} = \mathbf{M}'_{\infty}$. (c) $\mathbf{M}_{-\infty} = \mathbf{M}'_{-\infty}$.

The following corollary is an immediate consequence of Theorem 6.7 and the relevant definitions.

6.8. Corollary. (a) *The discrete-parameter process associated with the purely non-deterministic part (deterministic part) of* $\mathbf{x}$ *is the purely non-deterministic part (deterministic part) of* $\mathbf{x}'$.

(b) $\mathbf{y}$ *is subordinate to (dominated by)* $\mathbf{x}$ *if and only if* $\mathbf{y}'$ *is subordinate to (dominated by)* $\mathbf{x}'$.

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[4] $\sigma\{\ldots\}$ denotes the smallest closed subspace containing the vectors $\{\ldots\}$.

Corollary 6.8 makes it natural to define the *rank* of a continuous-parameter process $\mathbf{x}$ to be equal to the rank of its associated discrete-parameter process. With this definition of rank and with Corollary 6.8, the results of §§2, 3 carry over immediately to the continuous-parameter case.

We shall now show how continuous-parameter moving averages may be obtained (cf. (**8**, §6)). Let $\mathbf{h}$ be a q-variate, continuous-parameter, weakly stationary stochastic process with $\mathbf{h}'$ as its associated discrete-parameter process. Then $\mathbf{h}'$ is an orthogonal process, i.e. $(\mathbf{h}'_m, \mathbf{h}'_n) = \delta_{mn}\mathbf{K}$ where $\mathbf{K}$ is a $q \times q$ non-negative hermitian matrix, if and only if $(\mathbf{h}(t)|\mathbf{M}_s(\mathbf{h})) = e^{s-t}\mathbf{h}(s)$ for $s \leqslant t$. With such a process $\mathbf{h}$ we may associate a new (non-stationary) process

$$\xi(t) = \frac{1}{\sqrt{2}}\left\{\mathbf{h}(t) - \mathbf{h}(0) + \int_0^t \mathbf{h}(s)\,ds\right\}.$$

$\{\xi(t), t \text{ real}\}$ is a process with stationary and orthogonal increments and $|\{\xi(t) - \xi(s)|^2 = |t - s|\mathbf{K}$. The $\mathbf{h}$ process may be recovered from $\{\xi(t), t \text{ real}\}$ by

$$\mathbf{h}(t) = \sqrt{2}\int_{-\infty}^t e^{s-t}\,d\xi(s).$$

One can now show that $\mathbf{M}_t(\mathbf{h})$ is equal to the set of all stochastic integrals

$$\int_{-\infty}^t \mathbf{C}(s)\,d\xi(s)$$

where $\mathbf{C}$ is a measurable $q \times q$ matrix-valued function such that

$$\int_{-\infty}^t \mathbf{C}(s)\mathbf{K}\mathbf{C}(s)^*\,ds < \infty.$$

Finally one obtains: $\mathbf{x}$ is a (one-sided) moving average, i.e.

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} \mathbf{C}(s)\,d\xi(t-s) \qquad \left(\mathbf{x}(t) = \int_0^{\infty} \mathbf{C}(s)\,d\xi(t-s)\right)$$

if and only if its associated discrete parameter process is also, i.e.

$$\mathbf{x}'_n = \sum_{-\infty}^{\infty} \mathbf{C}_k\mathbf{h}'_{n-k} \qquad \left(\mathbf{x}'_n = \sum_0^{\infty} \mathbf{C}_k\,\mathbf{h}'_{n-k}\right).$$

The following version of the Wold decomposition now follows from the discrete case.

6.9. Theorem. *Let* $\mathbf{x}$ *be a* q*-variate, mean-continuous, continuous-parameter weakly stationary stochastic process. Then* $\mathbf{x}(t) = \mathbf{u}(t) + \mathbf{v}(t)$, *where*

(a) $\mathbf{v}$ *is a deterministic process and* $\mathbf{M}_t(\mathbf{v}) = \mathbf{M}_{-\infty}(\mathbf{x})$ *for all* t,

(b) $\mathbf{u}$ *is purely non-deterministic process and* $\mathbf{M}_t(\mathbf{u}) = \mathbf{M}_{-\infty}(\mathbf{x})^\perp \cap \mathbf{M}_t(\mathbf{x})$ *for all* t,

(c) $\mathbf{u}(t) = \int_0^{\infty} \mathbf{C}(s)\,d\xi(t-s)$ *where* $\{\xi(t), t \text{ real}\}$ *is a process with stationary and orthogonal increments and* $\mathbf{M}_t(\xi) = \mathbf{M}_t(\mathbf{u})$ *for all* t.

Having so completed the requisite time-domain analysis of our S.P., we can pursue its spectral analysis in a coherent way. If $\{U_t, t \text{ real}\}$ is a strongly continuous one-parameter group of unitary operators on X, then by Stones theorem,

$$U_t = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE_\lambda$$

where $\{E_\lambda, \lambda \text{ real}\}$ is a resolution of the identity on X. $\mathbf{F}(\lambda) = 2\pi(\mathbf{E}_\lambda(\mathbf{x}(0)), \mathbf{x}(0))$ is defined to be the *spectral distribution* of $\mathbf{x}$ where $\mathbf{x}(t) = \mathbf{U}_t(\mathbf{x}(0))$. If $\mathbf{G}$ is the spectral distribution of the associated discrete-parameter process, then $\mathbf{F}(\lambda) = \mathbf{G}(\theta)$ where $e^{i\theta} = c(\lambda) = (\lambda - i)/(\lambda + i)$ is the Cayley transform. We then have

6.10. THEOREM. *Let* $\mathbf{F}$ *be the spectral distribution of a continuous-parameter process and* $\mathbf{G}$ *the spectral distribution of its associated discrete-parameter process. Let* $\mathbf{F}_u$, $\mathbf{F}_v$, $\mathbf{F}_a$, *and* $\mathbf{F}_b$ *be the spectral distributions of the purely non-deterministic part and deterministic part, and the absolutely continuous and singular part of* x *respectively. Define* $\mathbf{G}_{u'}$, $\mathbf{G}_{v'}$, $\mathbf{G}_a$, *and* $\mathbf{G}_b$ *similarly. Then:* (a) $\mathbf{F}_u(\lambda) = \mathbf{G}_{u'}(c(\lambda))$, (b) $\mathbf{F}_v(\lambda) = \mathbf{G}_{v'}(c(\lambda))$, (c) $\mathbf{F}_a(\lambda) = \mathbf{G}_a(c(\lambda))$, (d) $\mathbf{F}_b(\lambda) = \mathbf{G}_b(c(\lambda))$.

The transition from the results given in §§4, 5 for discrete-parameter processes to the corresponding results for continuous-parameter ones is now immediate. Indeed, in view of Theorem 6.10, Corollary 6.8, and the definition of rank, any theorems involving only concepts of domination, subordination, purely non-deterministic parts, deterministic parts, ranks, or certain properties of the spectral distributions which have been proved for discrete- (continuous-) parameter processes will automatically be true for continuous- (discrete-) parameter ones.

7. Examples. In this section we shall give several examples to indicate the extent to which our results are the best possible. To this end we shall need the following result due to Kolmogorov (cf. (**4**, Theorem 23)).

7.1. LEMMA. (a) *Let* $\mathbf{x}$ *be a univariate* ($q = 1$) *S.P. with absolutely continuous spectral distribution* $\mathbf{F}$. *Then* $\mathbf{x}$ *is purely non-deterministic or deterministic according as* $\log \mathbf{F}'$ *is in* L_1 *or not.*

(b) *Let* f *be any non-negative, integrable function on* $(0, 2\pi)$. *There exist a univariate S.P. with absolutely continuous spectral distribution* $\mathbf{F}$ *given by* $\mathbf{F}' = f$ a.e.

7.2. *Example.* Let $\mathbf{y}$ and $\mathbf{z}$ be mutually orthogonal univariate S.P. with the same absolutely continuous spectral distribution $\mathbf{F}$. Theorem 4.5(c) shows that $\mathbf{x} = \mathbf{y} + \mathbf{z}$ cannot be a subordinate decomposition, while Lemma 4.3(a) and 7.1 imply that $r(\mathbf{x}) = r(\mathbf{y}) = r(\mathbf{z})$. Therefore, if $\log \mathbf{F}'$ is in L_1, $r(\mathbf{x}) < r(\mathbf{y}) + \mathrm{j}(\mathbf{z})$, and if $\log \mathbf{F}'$ is not in L_1, then $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$.

7.3. *Example.* Let A and B be disjoint, non-degenerate, intervals contained in $(0, 2\pi)$. Let $\mathbf{y}$ and $\mathbf{z}$ be mutually orthogonal univariate S.P. with absolutely continuous spectral distributions given by $\mathbf{F'}_y = 1_A$ and $\mathbf{F'}_z = 1_B$ respectively. Theorem 12 of **(4)** shows that $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate decomposition, while 7.1 implies that $r(\mathbf{y}) = r(\mathbf{z}) = 0$ and $r(\mathbf{x}) = 1$ if and only if $\overline{A \cup B} = (0, 2\pi)$. Thus, if $\overline{A \cup B} = (0, 2\pi)$, $r(\mathbf{x}) > r(\mathbf{y}) + r(\mathbf{z})$, and if $\overline{A \cup B} \neq (0, 2\pi)$, $r(\mathbf{x}) = r(\mathbf{y}) + r(\mathbf{z})$.

7.4. *Example.* Let $\mathbf{u}$ and $\mathbf{v}$ be mutually orthogonal, orthonormal univariate S.P. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be the bivariate $(q = 2)$ S.P. given by

$$\mathbf{x}_n = \begin{bmatrix} \mathbf{u}_n \\ \mathbf{v}_n \end{bmatrix}, \quad \mathbf{y}_n = \frac{1}{2}\begin{bmatrix} \mathbf{u}_n + \mathbf{v}_{n+1} \\ \mathbf{u}_{n-1} + \mathbf{v}_n \end{bmatrix}, \quad \mathbf{z}_n = \frac{1}{2}\begin{bmatrix} \mathbf{u}_n - \mathbf{v}_{n+1} \\ \mathbf{v}_n - \mathbf{u}_{n-1} \end{bmatrix}.$$

The relevant subspaces can easily be calculated directly to show that $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate orthogonal decomposition, which is not a dominated decomposition, and that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all purely non-deterministic. $r(x) = 2 = 1 + 1 = r(y) + r(z)$.

7.5. *Example.* Let $\mathbf{u}$ and $\mathbf{v}$ be as in 7.4 and let $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ be an orthogonal subordinate decomposition of $\mathbf{v}$ into deterministic processes as in 7.3. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be given by

$$\mathbf{x}_n = \begin{bmatrix} \mathbf{u}_n + \mathbf{v'}_n \\ \mathbf{u}_n - \mathbf{v''}_n \end{bmatrix}, \quad \mathbf{y}_n = \begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_n \end{bmatrix}, \quad \mathbf{z}_n = \begin{bmatrix} \mathbf{v'}_n \\ -\mathbf{v''}_n \end{bmatrix}.$$

Clearly $\mathbf{M}_\infty(\mathbf{z}) \subseteq \mathbf{M}_\infty(\mathbf{x})$, and thus $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is a subordinate orthogonal decomposition. It is also clear that $\mathbf{y}$ is purely non-deterministic with rank 1 and that $\mathbf{z}$ is deterministic. The spectral distribution $\mathbf{F}$ of $\mathbf{x}$ is absolutely continuous and is given by

$$F' = \begin{bmatrix} 1 + 1_A & 1 \\ 1 & 1 + 1_B \end{bmatrix}.$$

Since $\log(\det \mathbf{F}') = 0$ is in L_1, $\mathbf{x}$ is purely non-deterministic with rank 2 (cf. e.g. Wiener and Masani **(13**, Theorem 7.10**)**).

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