A REMARK ON PROJECTIVE MODULES

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Let $R$ denote the field of real numbers and let $A$ be the ring of regular functions on $R^n$, that is the localization of $R[T_1, \ldots, T_n]$ with respect to the set of all polynomials vanishing nowhere on $R^n$. Let $X$ be an algebraic subset of $R^n$ and let $I(X)$ be the ideal of $A$ of all functions vanishing on $X$. Assume that $X$ is compact and nonsingular and $k = \text{codim} X = 1, 2, 4$ or $8$. We prove here that if the $A/I(X)$-module $I(X)/I(X)$ can be generated by $k$ elements, then there exist a projective $A$-module $P$ of rank $k$ and a homomorphism from $P$ onto $I(X)$.

1. Introduction

Let $R$ denote the field of real numbers and let $A$ be the ring of all functions $f : R^n \to R$ such that $f = \phi/\psi$ for some polynomial functions $\phi, \psi : R^n \to R$ with $\psi$ vanishing nowhere. In other words, $A$ is (isomorphic to) the localization of the polynomial ring $R[T_1, \ldots, T_n]$ with respect to the set consisting of all polynomials vanishing nowhere on $R^n$. Given a subset $X$ of $R^n$, we denote by $I(X)$ the ideal of $A$ of all functions vanishing on $X$.

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In this note we prove the following.

**Theorem.** Let $X$ be a nonsingular algebraic subset of $\mathbb{R}^n$ of codimension $k$. Assume that the $A/I(X)$-module $I(X)/I(X)^2$ can be generated by $k$ elements. If $k = 1, 2, 4$ or $8$ and $X$ is compact, then there exist a finitely generated projective $A$-module $P$ of rank $k$ and a surjective homomorphism $h: P \rightarrow I(X)$.

For $k = 1$ or $2$ some better results are known. Indeed, since $A$ is a factorial ring (being a localization of $\mathbb{R}[T_1, \ldots, T_n]$), the ideal $I(X)$ is principal if $k = 1$, without the compactness assumption. If $k = 2$, then the ideal $I(X)$ is a complete intersection (see [4]) and one can even drop the compactness assumption for $\dim X = 1$ (see [8]). Moreover, the theorem holds true if $k = 2$ but $X$ is not necessarily compact (see for example, [7, Theorem 3.1]).

It is unknown whether all finitely generated projective $A$-modules of rank greater than one are free (proofs of the Serre conjecture concerning finitely generated projective modules over polynomial rings do not seem to extend to this case, see [10, 12]). Therefore the theorem does not allow us to conclude that the ideal $I(X)$ is a complete intersection for $k = 2, 4$ or $8$ (see the remark above for $k = 2$).

The author does not know whether the theorem remains true for $k = 4$ or $8$ if one drops the compactness assumption or replaces $\mathbb{R}$ by another, say real closed, field.

2. Proof of the Theorem

Our terminology and notions concerning real algebraic geometry are consistent with those of [2], [3] and [13]. In particular, $A$ is the ring of regular functions on $\mathbb{R}^n$ (see [3, Chapter 3] or [11]). Also recall that an algebraic vector bundle $\xi$ over an affine real algebraic variety $X$ is said to be strongly algebraic if there exists an algebraic bundle $\eta$ over $X$ such that $\xi \otimes \eta$ is algebraically isomorphic to a product vector bundle $X \times \mathbb{R}^m$ (see [2], [3, Chapter 12] and [13]).

**Example 1.** The real projective space $\mathbb{R}P^n$ with its standard structure of an abstract real algebraic variety is an affine variety (see [3, Theorem 3.4.4] or [1, p. 432]). Moreover, every $C^\infty \mathbb{R}$-vector bundle
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over $\mathbb{RP}^n$ is $C^\infty$ isomorphic to a strongly algebraic vector bundle (see [3, Example 12.3.7(c)]). Indeed, let $\xi$ be a $C^\infty R$-vector bundle over $\mathbb{RP}^n$. Then $\xi$ is stably equivalent to the canonical line bundle $\gamma_n$ over $\mathbb{RP}^n$ or to the direct sum of several copies of $\gamma_n$ [6, p. 223, Theorem 12.7]. Obviously, $\gamma_n$ is strongly algebraic and hence $\xi$ is stably equivalent to a strongly algebraic vector bundle. It follows that $\xi$ is $C^\infty$ isomorphic to a strongly algebraic vector bundle (see [2, p. 109]).

The next technical result is proved in [13, Proposition 2].

**Lemma 2.** Let $X$ be an affine nonsingular real algebraic variety and let $\xi$ be a strongly algebraic vector bundle over $X$. Assume that $X$ is compact in the Euclidean topology. If $s$ is a $C^\infty$ section of $\xi$ vanishing on a closed nonsingular algebraic subvariety $Y$ of $X$, then there exists an algebraic section $u$ of $\xi$ which is arbitrarily close to $s$ in the $C^\infty$ topology and vanishes on $Y$.

The last auxiliary result is the following.

**Lemma 3.** Let $A$ be a closed $C^\infty$ submanifold of a $C^\infty$ manifold $M$. Assume that the normal vector bundle of $A$ in $M$ is trivial. If $\operatorname{codim} A = 1, 2, 4$ or $8$, then there exist a $C^\infty R$-vector bundle $\gamma$ over $M$ and a $C^\infty$ section $s$ of $\gamma$ such that $\operatorname{rank} s = \operatorname{codim} A$, $s$ is transverse to the zero section of $\gamma$ and the set of zeros $s^{-1}(0)$ of $s$ is equal to $A$.

Proof. Let $k = \operatorname{codim} A$ and let $S^k$ be the unit $k$-dimensional sphere. Since the normal vector bundle of $A$ is trivial, there exist a $C^\infty$ map $f: M \to S^k$ and a regular value $y$ of $f$ such that $f^{-1}(y) = A$ (see [9]). If $k = 1, 2, 4$ or $8$, then one can find a $C^\infty R$-vector bundle $\gamma$ over $S^k$ and a $C^\infty$ section $u$ of $\gamma$ such that $\operatorname{rank} u = k$, $u$ is transverse to the zero section of $\gamma$ and $u^{-1}(0) = \{y\}$ (the construction of $\gamma$ and $u$ is easily available if one identifies $S^1$, $S^2$, $S^4$ and $S^8$ with the projective line over the reals, complexes, quaternions and Cayley...
numbers, respectively). It suffices to set $\xi = f^*\gamma$ and $s = f^*u$, where, as usual, $f^*\gamma$ denotes the pull-back vector bundle and $f^*u$ denotes the pull-back section.

Proof of the Theorem. We identify $\mathcal{R}^n$ with a subset of $\mathcal{R}F^\mathcal{N}$ via the map which sends $(x_1, \ldots, x_n)$ to $[1, x_1, \ldots, x_n]$. Let $Y$ be the Zariski closure of $X$ in $\mathcal{R}F^\mathcal{N}$. Then $Y = X \cup X'$, where $X'$ is contained in $\mathcal{R}F^\mathcal{N} \setminus \mathcal{R}^n$. Notice that $X$ is a $C^\infty$ submanifold of $\mathcal{R}F^\mathcal{N}$ and the normal vector bundle of $X$ is trivial. It follows from Lemma 3 that there exist a $C^\infty$ vector bundle $\xi$ over $\mathcal{R}F^\mathcal{N}$ and a $C^\infty$ section $s$ of $\xi$ such that rank $\xi = k$, $s$ is transverse to the zero section of $\xi$ and $s^{-1}(0) = X$. By Example 1, we can assume that $\xi$ is a strongly algebraic vector bundle.

Let $\text{Sing}(Y)$ be the set of singular points of $Y$. By the Hironaka theorem [5], there exist a nonsingular real algebraic variety $V$ and a real algebraic morphism $\pi : V \to \mathcal{R}F^\mathcal{N}$ such that $\pi$ isomorphically transforms $V \setminus \pi^{-1}(\text{Sing}(Y))$ onto $\mathcal{R}F^\mathcal{N} \setminus \text{Sing}(Y)$ and the Zariski closure $Z$ of $\pi^{-1}(Y \setminus \text{Sing}(Y))$ in $V$ is nonsingular. Moreover, since $\pi$ is the composition of finitely many algebraic blowing-ups, it is a proper map in the Euclidean topology (in particular, $V$ is compact) and $V$ is an affine real algebraic variety. Notice that $Z = Z_1 \cup Z_2$, where $Z_1 = \pi^{-1}(X)$ and $Z_2$ is a Zariski closed subset of $V$ disjoint from $Z_1$. Since $Z$ and $Z_2$ are both Zariski closed, $Z$ is nonsingular and $\dim Z = \dim Z_2$, it follows that $Z_1$ is Zariski closed in $V$ (see [1, Lemma 1.6]) and, of course, nonsingular.

Clearly, the pull-back vector bundle $\pi^*\xi$ over $V$ is strongly algebraic and the pull-back section $\pi^*s$ is of class $C^\infty$ and transverse to the zero section of $\pi^*\xi$ and $(\pi^*s)^{-1}(0) = Z_1$. By Lemma 2, there exists an algebraic section $\nu$ of $\pi^*\xi$ arbitrarily close to $s$ in the $C^\infty$ topology and vanishing on $Z_1$. Thus we can assume that $\nu$ is transverse to the zero section of $\pi^*\xi$ and $\nu^{-1}(0) = Z_2$. 


Let $\eta$ be the restriction of $\xi$ to $H^n$ and let $\rho = (\pi|_{\pi^{-1}(H^n)})^{-1}$.
Then $\eta = \rho^*(\pi^*\xi|_{\pi^{-1}(H^n)})$ and $\nu = \rho^*\nu$ is an algebraic section of $\eta$
which is transverse to the zero section of $\eta$ and satisfies $X = \nu^{-1}(0)$.

Let $Q$ be the $A$-module of all algebraic sections of $\eta$. It
follows from the definition of a strongly algebraic vector bundle that $Q$
is a finitely generated projective module of rank $k$ (see also [3,
Proposition 12.1.11]) and hence so is the module $P = \text{Hom}(Q, A)$.
Since $u$ is transverse to the zero section of $\eta$ and $\nu^{-1}(0) = X$
the element $\alpha(u)$ belongs to $I(X)$ and all
elements of this form generate $I(X)$. To conclude the proof, we define
$h: P \to I(X)$ by $h(\alpha) = \alpha(u)$ for $\alpha$ in $P$.

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