Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1132–1136

EXTENSION CLOSED AND CLUSTER CLOSED SUBSPACES

DOUGLAS HARRIS

Introduction. One of the most useful properties of a compact Hausdorff space is that such a space is closed whenever embedded into a Hausdorff space. This property does not extend to compact spaces with respect to embeddings into arbitrary spaces. Thus, an interesting topological problem is to characterize the types of absolute "closure" properties that are possessed by compact spaces. This is the problem that is solved in the present paper.

The following notation and terminology will be used below. We shall consider a fixed space X and subspace A, representing arbitrary nonempty open subsets of X (respectively A) by W (respectively V). Collections of nonempty open subsets of X (respectively A) are denoted by χ (respectively α) and an *extension of* α *to* X is a collection χ whose trace $\chi|A = \{W \cap A : W \in \chi\}$ (with empty intersections deleted) is equal to α; such extensions always exist. A *cover* of X (respectively A) is a collection χ (respectively α) whose union is X (respectively A); a cover is *infinite* if it has no finite subcover.

For each V define V^X to be the union of all W such that $W \cap A \subset V$, and for each α define $\alpha^X = \{V^X : V \in \alpha\}$.

A filterbase λ on A *converges* (*clusters*) in X if there is $x \in X$ such that for each W with $x \in W$ there is $F \in \lambda$ with $F \subset W$ (for each W with $x \in W$, if $F \in \lambda$ then $W \cap F \neq \emptyset$).

The proofs of the results given in this paper are straightforward, and are omitted (except for that of Proposition A).

1. Extension closed subspaces.

Among Hausdorff spaces, the closed subspaces are characterized by the property that such a subspace is *extension closed*; that is, every cover of the subspace extends to a cover of the entire space.

A number of equivalent characterizations of extension closed subspaces can be given. A few preliminary properties must be set forth.

1.1. $V^X = X - \mathrm{cl}_X(A - V)$.

1.2. $X - \mathrm{cl}_X A \subset V^X$.

1.3. $V^X \cap A = V$.

1.4. Any extension χ of α refines α^X.

Received October 20, 1971 and in revised form, August 9, 1972.

https://doi.org/10.4153/CJM-1972-119-8 Published online by Cambridge University Press

1.5. A collection α extends to a cover χ of X if and only if α^X is a cover of X.

1.6. If α, α' are covers of A, α refines α', and α extends to a cover of X, then α' extends to a cover of X.

PROPOSITION A. *The following are equivalent for the subspace A of the space X.*

(i) *The subspace A is extension closed.*

(ii) *If λ is a family of closed subsets of A with empty intersection, then $\mathrm{cl}_X\lambda = \{\mathrm{cl}_X F : F \in \lambda\}$ has empty intersection.*

(iii) *If λ is a filterbase on A that converges in X then λ also converges in A.*

Proof. The equivalence of (i) and (ii) is immediate from 1.5.

To see that (i) implies (iii), suppose that λ is a filterbase on A that converges to $x \in X$ but does not converge in A. Since λ does not converge in A there is a cover α such that $F \not\subset V$ for each $F \in \lambda$ and each $V \in \alpha$. It follows that $x \notin V^X$ for each $V \in \alpha$, and therefore α^X is not a cover of X, so by 1.5 the collection α does not extend to a cover of X.

To see that (iii) implies (i) suppose that (iii) is satisfied and α is a cover of A. It follows from 1.2 that the trace on $B = A \cup (X - \mathrm{cl}_X A)$ of α^X covers B. Now $X - B \subset \mathrm{cl}_X A$, and if $x \in X - B$ it follows that

$$\lambda = \{W \cap A : x \in W\}$$

is a filterbase on A that converges in X; thus, there is $a \in A$ such that λ converges to a. Now there is $V \in \alpha$ with $a \in V$, and it follows that $x \in V^X$. Thus the trace on $X - B$ of α^X covers $X - B$, and it now follows that α^X covers X.

1.7. *Remark.* It is perhaps worth noting that the axiom of choice appears necessary in showing that (i) implies (iii).

The next two results relate extension closed subspaces and closed subspaces.

1.8. A closed subspace is extension closed.

1.9. An extension closed subspace of a Hausdorff space is closed.

The following result is frequently useful.

1.10. A retract of a space is an extension closed subspace.

A particular application of 1.10 is to product spaces.

1.11. Any factor of a product space is embeddable as an extension closed subspace of the product.

The next application of 1.10 should be compared with the result that the diagonal in the product space $X \times X$ is closed if and only if the space is Hausdorff.

1.12. For any space X, the diagonal in $X \times X$ is extension closed.

https://doi.org/10.4153/CJM-1972-119-8 Published online by Cambridge University Press

We now turn to the hereditary properties of extension closed subspaces.

1.13. An extension closed subspace of a compact space is compact.

1.14. *Remark.* It is clear that a property of covers that is preserved under taking of traces leads to an extension closed hereditary property. For example, paracompactness, countable compactness, countable paracompactness, and the Lindelöf property are extension closed hereditary properties.

The extension closed subspaces are extremely important in connection with general compact spaces, especially in view of the results in [**2**], that a space is compact if and only if it is homeomorphic to an extension closed subspace of a product of finite spaces, and the result of similar character for T_1 spaces in [**3**]; however, the requirements are just a little too strong in the absolute closure context as the following example shows.

1.15. *Example.* Let $X = \{1, -1, 2, -2\}$ with a subbase for open sets $\{\emptyset, \{1\}, \{-1\}, \{1, -1, 2\}, \{1, -1, -2\}\}$. Then the compact subspace $\{1, -1\}$ is not extension closed. Note also that it is the union of the extension closed subspaces $\{1\}$ and $\{-1\}$ and the intersection of the extension closed subspaces $\{1, -1, 2\}$ and $\{1, -1, -2\}$.

2. Cluster closed subspaces. The subspace A of the space X is *cluster closed* if every infinite cover of A extends to a cover of X. The next two results show that this concept is characteristic of compact subspaces.

2.1. Suppose $A \subset X$ is compact. Then A is cluster closed.

2.2. Suppose $A \subset X$ is cluster closed and X is compact. Then A is compact.

The following characterizations of cluster closed subspaces are sometimes useful; the characterization (iii) is the source of the term cluster closed.

PROPOSITION B. *The following are equivalent for the subspace A of the space X.*

(i) *The subspace A is cluster closed.*

(ii) *If λ is a family of closed subsets of A with finite intersection property and empty intersection then $\mathrm{cl}_X\lambda = \{\mathrm{cl}_X F : F \in \lambda\}$ has empty intersection.*

(iii) *If λ is a filterbase on A that clusters in X then λ also clusters in A.*

The following results relate closed, extension closed, and cluster closed subspaces.

2.3. An extension closed (thus also a closed) subspace is cluster closed.

2.4. A cluster closed subspace of a Hausdorff space is closed.

2.5. *Remark.* It is clear that a property of covers that is possessed by all finite covers and that is preserved under taking of traces leads to a cluster closed hereditary property. The properties mentioned in 1.14 are thus cluster closed hereditary properties.

The intersection of two cluster closed subspaces need not be cluster closed. The union is better behaved, as the following immediate consequence of Proposition B (iii) shows.

2.6. The union of two cluster closed subspaces is cluster closed.

2.7. *Remark.* It is clear from 2.6 that the cluster closed subspaces of X may be taken as closed base for a topology X^* on the set X, which by 2.3 will be larger than the given topology; the space X^* will, in fact, be a T_1 space, since points are cluster closed subspaces. By iteration of this process one eventually obtains a space X^c in which every cluster closed subspace is closed; such a space need not be T_2. Examples can readily be found to show that this construction is not functorial; for example, if X is the finite complement topology on an infinite set and Y is a non-discrete T_2 topology on the same underlying set then $X^* = X^c$ is the discrete space on the set X and $Y^* = Y^c$, so id : $Y \to$ is not continuous from Y^c to X^c.

3. Absolute closure properties. If $\mathscr{P}$ is a class of spaces, then a space X is *$\mathscr{P}$-absolutely closed* (*extension closed*, *cluster closed*) if it belongs to $\mathscr{P}$ and is closed (extension closed, cluster closed) whenever embedded into a space of the class $\mathscr{P}$.

It is clear that if $\mathscr{P}$ and $\mathscr{R}$ are classes of spaces with $\mathscr{P} \subset \mathscr{R}$, then any $\mathscr{R}$-absolutely closed space that belongs to $\mathscr{P}$ is $\mathscr{P}$-absolutely closed; similarly for the other two absolute properties. Characterizations are given below for the three absolute closure properties for the classes $\mathscr{T}, \mathscr{K}, \mathscr{F}$ and $\mathscr{C}$ of all spaces, all T_0 spaces, all T_1 spaces, and all completely regular Hausdorff spaces. The conditions are expressed so that the progressive weakening as we pass from $\mathscr{K}$ to $\mathscr{F}$ to $\mathscr{C}$ is apparent.

THEOREM I. (a) *A space is $\mathscr{T}$-absolutely closed if and only if it is empty.*

(b) *A space is $\mathscr{K}$-absolutely closed if and only if it is empty.*

(c) *A space is $\mathscr{F}$-absolutely closed if and only if it is finite and belongs to $\mathscr{F}$.*

(d) *A space is $\mathscr{C}$-absolutely closed if and only if it is compact and belongs to $\mathscr{C}$.*

THEOREM II. (a) *A space is $\mathscr{T}$-absolutely extension closed if and only if the intersection of all non-empty closed subsets is non-empty.*

(b) *A space is $\mathscr{K}$-absolutely extension closed if and only if the intersection of all non-empty closed subsets is non-empty and the space belongs to $\mathscr{K}$.*

(c) *A space is $\mathscr{F}$-absolutely extension closed if and only if the intersection of all infinite closed subsets is non-empty and the space belongs to $\mathscr{F}$.*

(d) *A space is $\mathscr{C}$-absolutely extension closed if and only if the intersection of any family of infinite closed sets with finite intersection property is non-empty and the space belongs to $\mathscr{C}$.*

THEOREM III. *A space is $\mathscr{P}$-absolutely cluster closed if and only if it is compact, and belongs to $\mathscr{P}$, for $\mathscr{P} = \mathscr{T}, \mathscr{K}, \mathscr{F}$, or $\mathscr{C}$.*

https://doi.org/10.4153/CJM-1972-119-8 Published online by Cambridge University Press

References

1. Douglas Harris, *Structures in topology*, Mem. Amer. Math. Soc., No. 115.

2. ——— *Compact spaces and products of finite spaces*, Proc. Amer. Math. Soc. *35* (1972), 275–280.

3. ———*Universal compact T_1 spaces* (to appear in General Topology and Appl.).

Marquette University,
Milwaukee, Wisconsin