ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A FOURIER SERIES (II) ¹

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Let \( \sum_{n=0}^{\infty} a_n \) be a given series with its partial sums \( \{s_n\} \) and \( \{p_n\} \) a sequence of real or complex parameters. Write

\[
P_n = p_0 + p_1 + p_2 + \cdots + p_n; \quad p_{-k} = P_{-k} = 0 \quad (k \geq 1).
\]

The transformation given by

\[
t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r} s_r
\]
defines the Nörlund means of \( \{s_n\} \) generated by \( \{p_n\} \). The series \( \sum a_n \) is said to be absolutely summable \( (N, p_n) \) or summable \( |N, p_n|, \) if \( \{t_n\} \) is of bounded variation, i.e., \( \sum |t_n - t_{n-1}| \) converges.

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Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable in the sense of Lebesgue over \( (-\pi, \pi) \). Let

\[
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \approx \sum_{n=1}^{\infty} A_n(t).
\]

We write

\[
\varphi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\}.
\]

In this note, we prove the following theorem concerning the \( |N, p_n| \) summability of the Fourier series of \( f(t) \) at \( t = x \).

**Theorem.** Let \( \{p_n\} \) be a sequence of non-negative and non-increasing real parameters such that \( \{Ap_n\} \) is monotonic. If (i) \( \varphi(t) \) is of bounded variation in \( (0, \pi) \) and (ii) \( \{P_n \sum_{r=1}^{\infty} (vp_r)^{-1}\} \) is bounded, then the Fourier series of \( f(t) \) is summable \( |N, p_n| \) at \( t = x \).

¹ The first paper appears under the same title in this journal, vol. 7 (1967), 252-256.
The following lemmas are required.

**Lemma 1 (McFadden) [1].** For \(0 \leq a < b < \infty, 0 \leq t \leq \pi\),

\[
\left| \sum_{j=a}^{b} p_j e^{(n-v)t} \right| \leq A P_r
\]

where \(r = \lceil t^{-1} \rceil\).

**Lemma 2.** If \(\{p_v\}\) is monotonic increasing and \(\{\Delta p_v\}\) monotonic, then, for a fixed \(n\), \(\{(P_n - P_v)(n-v)^{-1}\}\) is non-increasing and \(\{(p_v - p_n)(n-v)^{-1}\}\) monotonic in the same direction as \(\{\Delta p_v\}\).

**Proof.** If \(\{p_v\}\) is monotonic, then the sequence

\[
\sigma_k = \frac{\dot{p}_1 + \dot{p}_2 + \cdots + \dot{p}_k}{k}
\]

is also monotonic in the same direction as \(\{p_v\}\). Thus, we see that, for a fixed \(n\), if \(p_v \geq 0, \dot{p}_v \geq \dot{p}_{v+1}\), then

\[
\frac{P_n - P_v}{n-v} = \frac{\dot{p}_{v+1} + \dot{p}_{v+2} + \cdots + \dot{p}_n}{n-v}
\]

is non-increasing for \(v < n\) and since \(\{\Delta p_v\}\) is monotonic,

\[
\frac{\dot{p}_v - \dot{p}_n}{n-v} = \frac{(\dot{p}_v - \dot{p}_{v+1}) + (\dot{p}_{v+1} - \dot{p}_{v+2}) + \cdots + (\dot{p}_{n-1} - \dot{p}_n)}{n-v}
\]

is also monotonic in the same direction as \(\{p_v\}\). This proves the lemma.

**Lemma 3 (McFadden) [1].**

\[
I = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{\sin (n-v)t}{n-v} \right|
\]

\[
\leq \sum_{n=1}^{\tau} \frac{1}{P_n P_{n-1}} \left| \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{\sin (n-v)t}{n-v} \right|
\]

\[
+ \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{v=0}^{k-1} (P_n p_v - P_v p_n) \frac{\sin (n-v)t}{n-v} \right|
\]

\[
+ \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{v=k}^{n-1} (p_v - p_n) \frac{\sin (n-v)t}{n-v} \right|
\]

\[
+ \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{v=k}^{n-1} \left( \frac{P_n - P_v}{n-v} \right) \sin (n-v)t \right|
\]

\[
= I_1 + I_2 + I_3 + I_4,
\]

say, where \(\tau = \lceil t^{-1} \rceil\) and \(k = \lceil n/2 \rceil\).
We have
\[ t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{\nu}a_{n-\nu}, \]
and
\[ t_{n-1} = \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} P_{\nu}a_{n-\nu-1} \]
Thus,
\[ |t_n - t_{n-1}| = \left| \sum_{\nu=0}^{n-1} \left( \frac{P_{\nu}}{P_n} - \frac{P_{\nu-1}}{P_{n-1}} \right) a_{n-\nu} \right| \]
\[ = \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} (P_{\nu}P_n - P_{\nu}P_{n-1}) a_{n-\nu} \right|. \]
Also, for the Fourier series of \( f(t) \) at \( t = x \),
\[ A_n(x) = \frac{1}{\pi} \int_{0}^{\pi} \varphi(t) \cos nt \, dt. \]

In order to establish the theorem, it is enough to prove that, under the conditions of the theorem,
\[ \sum_{n=1}^{\infty} \left| \int_{0}^{\pi} \varphi(t) \Omega(n, t) \, dt \right| \leq A, \]
where
\[ \Omega(n, t) = \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} (P_{\nu}P_n - P_{\nu}P_{n-1}) \cos (n-\nu) t, \]
and here and elsewhere \( A \) is an absolute constant not necessarily the same at each occurrence. Noticing that
\[ \int_{0}^{\pi} \varphi(t) \Omega(n, t) \, dt = - \int_{0}^{\pi} \left( \int_{0}^{t} \Omega(n, u) \, du \right) d\varphi(t), \]
and that
\[
\sum_{n=1}^{\infty} \left| \int_0^t \left( \int_0^n \Omega(n, u) \, du \right) \, d\varphi(t) \right| \leq \int_0^t \left| d\varphi(t) \right| \left( \sum_{n=1}^{\infty} \left| \int_0^n \Omega(n, u) \, du \right| \right),
\]
by (i), since \( \varphi(t) \) is of bounded variation in \((0, \pi)\).
\[
\int_0^\pi \left| d\varphi \right| < \infty,
\]
we establish the theorem if we can show that
\[
\sum_{n=1}^{\infty} \left| \int_0^t \Omega(n, u) \, du \right| \leq A,
\]
uniformly for \(0 < t < \pi\), or that is the same thing,
\[
I = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} (P_n \varphi_\nu - P_\nu \varphi_n) \frac{\sin (n-\nu)t}{n-\nu} \right| \leq A.
\]

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We denote \( \tau = \lfloor t^{-1} \rfloor \) and \( k = \lfloor n/2 \rfloor \) and separate \( I \) in McFaddens' way as in Lemma 3. Since \( P_n \varphi_n \leq P_\nu \varphi_\nu \) for \( \nu \leq n \) and \( |\sin (n-\nu)t/(n-\nu)| \leq At \),
\[
I_1 = \sum_{n=1}^{\tau} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} (P_n \varphi_\nu - P_\nu \varphi_n) \frac{\sin (n-\nu)t}{n-\nu} \right| \leq At \sum_{n=1}^{\tau} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} P_n \varphi_\nu = At \sum_{n=1}^{\tau} \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} \varphi_\nu \leq A.
\]

By Abel's transformation and Lemma 1,
\[
I_2 = \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{k-1} \frac{P_n - P_\nu \varphi_n}{n-\nu} \varphi_\nu \sin (n-\nu)t \right| \leq AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left( \frac{P_n - P_{k-1} \varphi_n}{n-k+1} \right)^\nu \left( \varphi_{k-1} \right)^\nu \sin (n-\nu)t \left| \frac{\sin (n-\nu)t}{n-\nu} \right| \leq AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{k-2} \varphi_\nu \left( \frac{P_n - P_\nu \varphi_n}{n-\nu} \right)^\nu \left( \frac{\varphi_{k-1}}{n-k+1} \right)^\nu \sin (n-\nu)t \left| \frac{\sin (n-\nu)t}{n-\nu} \right| \leq AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \cdot \frac{P_n}{n-k+1} \leq A.
\]
\[\begin{align*}
&+ A P_r \sum_{n=r+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{v=0}^{k-2} \frac{P_n}{(n-v)(n-v-1)} \\
&+ A P_r \sum_{n=r+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{v=0}^{k-2} \frac{P_n}{(n-v)\left(\frac{P_{v+1}}{P_v} - \frac{P_v}{P_{v+1}}\right)} \\
&\leq A P_r \sum_{n=r+1}^{\infty} \frac{1}{nP_{n-1}} + A P_r \sum_{n=r+1}^{\infty} \frac{1}{n} \sum_{v=0}^{k-2} \frac{1}{(n-v)(n-v-1)} \\
&+ A P_r \sum_{n=r+1}^{\infty} \frac{\varphi_n}{P_n P_{n-1}} \cdot \frac{1}{n-k+2} \left(\frac{P_{k-1}}{\varphi_{k-1}} - \frac{P_0}{\varphi_0}\right) \\
&\leq A + A P_r \sum_{n=r+1}^{\infty} \frac{1}{nP_n} \\
&< A,
\end{align*}\]

by (ii). By Lemma 2, since \(\{(\varphi_v - \varphi_n)(n-v)^{-1}\}\) is monotonic, Abel’s transformation gives

\[\begin{align*}
\left|\sum_{n=k}^{n-1} \varphi_v - \varphi_n \sin (n-v)t\right| &\leq \frac{A}{t} \frac{\varphi_k - \varphi_n}{n-k} + \frac{A}{t} (\varphi_{n-1} - \varphi_n) + \frac{A}{t} \sum_{v=k}^{n-2} \left|D_v \left(\frac{\varphi_v - \varphi_n}{n-v}\right)\right| \\
&\leq \frac{A}{t} \cdot \frac{\varphi_k}{k} + \frac{A}{t} (\varphi_{n-1} - \varphi_n) + \frac{A}{t} \left|\frac{\varphi_k - \varphi_n}{n-k} - (\varphi_{n-1} - \varphi_n)\right| \\
&\leq \frac{A}{t} \cdot \frac{\varphi_k}{k} + \frac{A}{t} (\varphi_{n-1} - \varphi_n).
\end{align*}\]

Hence,

\[\begin{align*}
I_3 &\leq \frac{A}{t} \sum_{n=r+1}^{\infty} \frac{\varphi_k}{kP_{n-1}} + \frac{A}{t} \sum_{n=r+1}^{\infty} \frac{\varphi_{n-1} - \varphi_n}{P_{n-1}} \\
&\leq \frac{A}{t} \sum_{n=r+1}^{\infty} \frac{1}{k(k-1)} + \frac{A}{t} \frac{\varphi_r}{P_r} \\
&\leq A + A r \frac{\varphi_r}{P_r} \\
&< A.
\end{align*}\]

Moreover, by Lemma 2, since \(\{(P_n - P_r)(n-v)^{-1}\}\) is non-increasing,
\[
\left| \sum_{v=k}^{n-1} \frac{P_n - P_v}{n-v} \sin (n-v)t \right|
\]
\[\leq \frac{A}{t} \sum_{v=k}^{n-1} \left| \Delta_v \left( \frac{P_n - P_v}{n-v} \right) \right| + \frac{A}{t} \cdot \frac{P_n - P_k}{n-k} + \frac{A}{t} \cdot \hat{p}_n
\]
\[\leq \frac{A}{t} \cdot \frac{P_n - P_k}{n-k} + \frac{A}{t} \cdot \hat{p}_n + \frac{A}{t} \cdot \frac{P_n - P_k}{n-k} + \frac{A}{t} \cdot \hat{p}_n
\]
\[\leq \frac{AP_n}{nt}.
\]

Thus, we obtain
\[
I_4 \leq \frac{A}{t} \sum_{n=r+1}^{\infty} \frac{1}{nP_n}
\]
\[\leq \frac{A}{t} \sum_{n=r+1}^{\infty} \frac{1}{n(n-1)}
\]
\[< A.
\]

This completes the proof of the theorem.

Reference


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