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A note on quasi-uniform continuity

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It is shown that:

(i) a continuous function from a compact quasi-uniform space into a quasi-uniform space is not necessarily quasi-uniformly continuous;

(ii) if the range of the function is a uniform space, the function will be necessarily quasi-uniformly continuous.

(This contradicts an example in the literature and generalizes a classical result.) Finally, a generalization of (ii) is given by means of a suitable boundedness notion.

A *quasi-uniform space* is defined to be a pair $(X, \mathcal{U})$, where X is a non-empty set and $\mathcal{U}$ is a filter on $X \times X$ which satisfies:

(i) each $U \in \mathcal{U}$ defines a reflexive relation on X,

(ii) to each $U \in \mathcal{U}$ there corresponds $W \in \mathcal{U}$ such that $W \circ W \subseteq U$.

The pair $(X, \mathcal{U})$ becomes a *uniform space* when:

(iii) $U^{-1} \in \mathcal{U}$ whenever $U \in \mathcal{U}$.

Examples of quasi-uniform spaces which are not uniform are easily found using the fact that every topological space is quasi-uniformizable [5, Theorem 1.19]. On the other hand, each quasi-uniform space $(X, \mathcal{U})$ gives rise to a topology on X, the neighbourhood system for each $x \in X$ being given by the family

$$U(x) \equiv \{y \in X : (x, y) \in U\}, \quad U \in \mathcal{U}.$$

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It is well-known [3, p. 198] that each continuous function from a compact uniform space into a uniform space is uniformly continuous and that there exist compact quasi-uniform spaces the quasi-uniformity of which is not a uniformity [2, p. 399, Example]. Here we give

THEOREM 1. *Every continuous function from a compact quasi-uniform space into a uniform space is quasi-uniformly continuous.*

REMARK 1. This result contradicts an example, in [5, p. 55], given without proof, of a continuous function from a compact quasi-uniform space into a uniform space which is not quasi-uniformly continuous.

Proof. See Remark 3.

COUNTEREXAMPLE. The above result is best possible in the sense that a continuous function from a compact quasi-uniform space (even a compact uniform space) into a quasi-uniform space is not necessarily quasi-uniformly continuous. For suppose that X is a uniform (with uniformity $\mathcal{U}$) non-discrete compact T_0 space and that $\mathcal{P}$ is the Pervin quasi-uniformity for X [5, p. 15]. The identity map from $(X, \mathcal{U})$ into $(X, \mathcal{P})$ is continuous but it is not quasi-uniformly continuous since, according to [2, p. 398, Theorem 5], $\mathcal{P} \not\subseteq \mathcal{U}$.

In order to state and prove Theorem 2 below, we have need of the following, which are proved in [4].

DEFINITION. A subset A of a space (X, r) is said to be r-bounded if from each open cover of X one can select a finite subcover of A ; equivalently: if every filter in A has at least one adherent point in the space (X, r) .

REMARK 2. From the above and Exercise 1b in [1, p. 109] it follows that there exist spaces such that an r-bounded subset is not necessarily compact or relatively compact (that is, subset of a compact set). However each (relatively) compact subset is r-bounded.

PROPOSITION. *Although a space* (X, r) *is compact if and only if* X *is* r*-bounded, a proper subset* A *may be* r*-bounded without the relative space* (A, r_A) *being compact, that is, without* A *being* r_A*-bounded.*

(See Remark 2.)

THEOREM 2. *Let* $f : (X, R) \to (Y, \mathcal{U})$ *be a continuous function from*

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the quasi-uniform space $(X, \mathcal{R})$ *into the uniform space* $(Y, \mathcal{U})$ *and let* A *be an* r*-bounded subset of* X. *Then the restriction* $f/A : (A, \mathcal{R}_A) \to (Y, \mathcal{U})$ *of the function* f *to* A *is quasi-uniformly continuous (with respect to the relative quasi-uniformity* $\mathcal{R}_A$, *induced by* $\mathcal{R}$ *on* A).

Proof. To show that f/A is quasi-uniformly continuous we must show that for every $U \in \mathcal{U}$, there exists $V \in \mathcal{R}$ satisfying

$$\{(f(x), f(y)) : (x, y) \in V \cap (A\times A)\} \subseteq U . \tag{1}$$

There exists symmetric $W \in \mathcal{U}$ such that $W \circ W \subseteq U$. Let $x \in X$. Since f is continuous, corresponding to the neighbourhood $W(f(x))$ of $f(x)$, there exists $V_x \in \mathcal{R}$ such that

$$f(V_x(x)) \subseteq W(f(x)) . \tag{2}$$

Choose $T_x \in \mathcal{R}$ such that $T_x \circ T_x \subseteq V_x$. The union of the interiors of the $T_x(x)$ covers X and so the r-boundedness of A implies that there exists a finite set of points $\{x_1, \ldots, x_n\}$ in X satisfying $\bigcup_{k=1}^{n} T_{x_k}(x_k) \supseteq A$. Define $V = \bigcap_{k=1}^{n} T_{x_k} \in \mathcal{R}$. We will show that V satisfies (1).

Suppose that $(a, b) \in V \cap (A\times A)$. Then there exists $k \in \{1, \ldots, n\}$, $k = k(a)$, such that $a \in T_{x_k}(x_k)$. Thus $(x_k, a) \in T_{x_k}$. But also $(a, b) \in T_{x_k}$, hence $(x_k, b) \in T_{x_k} \circ T_{x_k} \subseteq V_{x_k}$ and so $b \in V_{x_k}(x_k)$. Accordingly $a, b \in V_{x_k}(x_k)$. Using (2) we have

$$f(a), f(b) \in f\left(V_{x_k}(x_k)\right) \subseteq W(f(x_k)) ,$$

hence $(f(a), f(x_k)) \in W^{-1} = W$ and $(f(x_k), f(b)) \in W$. Therefore $(f(a), f(b)) \in W \circ W \subseteq U$ as required.

REMARK 3. If X is r-bounded, by the Proposition we get Theorem 1 as a special case of Theorem 2 above.

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