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On Small Complete Sets of Functions

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Abstract. Using Local Residues and the Duality Principle a multidimensional variation of the completeness theorems by T. Carleman and A. F. Leontiev is proven for the space of holomorphic functions defined on a suitable open strip $T_\alpha \subset \mathbf{C}^2$. The completeness theorem is a direct consequence of the Cauchy Residue Theorem in a torus. With suitable modifications the same result holds in $\mathbf{C}^n$.

1 Introduction

Let $\alpha > 0$, then an open strip T_α of $\mathbf{C}^n$ is defined as follows

$$T_\alpha = \{(z_1, \dots, z_n) \in \mathbf{C}^n : |\Im z_i| < \alpha, i = 1, \dots, n\}.$$

By $A(T_\alpha)$ we denote the space of functions holomorphic in T_α and continuous in $\bar{T}_\alpha$ equipped with the topology of uniform convergence. By $\mathcal{H}(T_\alpha)$ we denote the space of functions holomorphic in the open strip T_α and equipped with the topology of locally uniform convergence.

For $n = 1$, a well-known result of T. Carleman [5] states that given a sequence $\{\lambda_n\}$, $\lambda_n > 0$, $\lambda_n \uparrow \infty$ such that $\tau = \limsup_{R\to\infty} \frac{\sum_{\lambda_i < R} \frac{1}{\lambda_i}}{\ln R} < \infty$, the set of exponential functions $\{e^{\lambda_n z}\}_n$ is complete in the space $\mathcal{H}(T_{\pi\tau})$, where $T_{\pi\tau}$ is a strip of the complex plane.

By the completeness of the set of exponential functions $\{e^{\lambda_n z}\}$ in the space $\mathcal{H}(T_{\pi\tau})$ we mean that every function $f \in \mathcal{H}(T_{\pi\tau})$ can be approximated arbitrarily closely over each compact subset of $T_{\pi\tau}$ by linear combinations of functions of this set. Variations of this theorem on a certain type of nonconvex strips in the complex plane can be found in [8], [7].

In the present paper we will be interested in a multidimensional version of the following completeness theorem by A. F. Leontiev [6].

Theorem 1.1 *Let $\{\lambda_n\}$ be an increasing sequence of positive real numbers converging to infinity. Assume that $\{\lambda_n\} = \{\lambda_n'\} \cup \{\lambda_n''\}$, where the subsequence $\{\lambda_n'\}$ is the set of all roots in the right-half plane of some entire function $L(z)$ of exponential type so that $|L(iy)| \geq Be^{\alpha|y|}$, $B \neq 0$. Assume that the terms of the subsequence $\{\lambda_n''\}$ satisfy the following condition: $\sum \frac{1}{\lambda_n''} = \infty$. Then the set of exponential functions $\{e^{\lambda_n z}\}$ is complete in the space $A(T_\alpha)$.*

There are two essential differences between the results of T. Carleman and A. F. Leontiev. One is related to the space of holomorphic functions they consider. The other one is more subtle and it is related to the frequencies of the exponentials. Carleman's result essentially

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requires positive density, that is, if $\lim_{n\to\infty} \frac{n}{\lambda_n} = \sigma > 0$ then $\sigma = \tau$. In Leontiev's formulation this is not the case. It seems that his completeness theorem allows for the density of the frequencies $\{\lambda'_n\}$ to be zero, as soon as the other conditions are satisfied. That is, in Leontiev's theorem it is the growth on the imaginary axis which is of importance. However, we were not able to construct an example of entire function of exponential type whose zeroes on the right half-plane have density zero and this function satisfies the growth condition of Leontiev's theorem. Furthermore, given a sequence $\{\lambda_n\}$ it is not an easy matter to check if it satisfies the conditions of the Theorem 1.1. These considerations led us to modify slightly the approach in [6] and consider the case $\lim_{n\to\infty} \frac{n}{\lambda'_n} = 0$.

Let $m \in \mathbf{N}$. Then it is easy to find an increasing sequence of positive numbers $\{\alpha_i\}$ tending to infinity and having exponent of convergence equal to 1. Such a sequence can be chosen to satisfy $\lim_{r\to\infty} \frac{n_{\{\alpha_i\}}(r)}{r} = \frac{2m}{\pi}$ also. Then, from the Theorem 8.2.1 in [4] for the infinite product (entire function of infinite type)

$$f(z) = \prod_{i=1}^{\infty}\Bigl(1 + \frac{z}{\alpha_i}\Bigr)e^{-\frac{z}{\alpha_i}} = \lim_{k\to\infty}\prod_{i=1}^{k}\Bigl(1 + \frac{z}{\alpha_i}\Bigr)e^{-\frac{z}{\alpha_i}} = \lim_{k\to\infty} f_k(z), \tag{1.1}$$

we have that on the imaginary axis the following inequality holds

$$|f(iy)| \geq Be^{m|y|}, \quad B \neq 0,\ iy \in i\mathbf{R}. \tag{1.2}$$

For f from (1.1) we state without proof the following

Proposition 1.1 *Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n > 0$ for every $n \in \mathbf{N}$ and $\lambda_n \uparrow \infty$. Assume that $\{\lambda_n\} = \{\lambda'_n\} \cup \{\lambda''_n\}$, where the subsequence $\{\lambda'_n\}$ satisfies the conditions $\lim_{n\to\infty} \frac{n}{\lambda'_n} = 0$ and the subsequence $\{\lambda''_n\}$ satisfies the condition $\sum_n \frac{1}{\lambda''_n} = \infty$. Let $m \in \mathbf{N}$. Define the sequence of functions*

$$\psi_n(z) = \sum_{j=1}^{n} b_j e^{\lambda''_n z} + \lim_{k\to\infty}\sum_{i=1}^{\infty} a_{k,i}e^{\lambda'_i z}, \quad z \in \bar{T}_m,\ n \in \mathbf{N},$$

where the coefficients $b_j, a_{k,i}$ are expressed in terms of $\frac{1}{f(\lambda_n)}, \frac{1}{f_k(\lambda_n)}, \frac{1}{L'(\lambda_n)}, \frac{1}{q_n(\lambda_n)}, \frac{1}{q'_n(\lambda_n)}$, $L(z) = \prod_{i=1}^{\infty}(1 - \frac{z^2}{(\lambda'_i)^2})$, $q_n(z) = \prod_{i=1}^{n}(1 - \frac{z}{\lambda''_i})$. Then $\psi_n \in A(T_m)$ for every $n \in \mathbf{N}$ and every function $g \in A(\bar{T}_m)$ can be approximated arbitrarily closely over the compact subsets of $\bar{T}_m$ by finite linear combinations of the functions ψ_n.

Remark 1.1 This form of the theorem can be proven using the method in [6].

In order to obtain a multidimensional variation of the above results, we will use Leontiev's approach combined with an application of the Local Residues in the space $\mathcal{H}(T_\alpha)$, $T_\alpha \subset \mathbf{C}^2$. The main tool will be a Cauchy Residue Theorem in a torus.

Let $V = \{(\lambda_{1n}, \ldots, \lambda_{mn})\} \subset \mathbf{R}^m_+ = \{(x_1, \ldots, x_m) : x_j > 0, j = 1, \ldots, m\}$, be a sequence of points converging to infinity. For such sequence a direct variation of Leontiev's theorem would be the following

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Proposition 1.2 *Let* $V = \{(\lambda_{1n}, \dots, \lambda_{mn})\} \subset \mathbf{R}^m_+ = \{(x_1, \dots, x_m) : x_j > 0, j = 1, \dots, m\}$*, be a sequence of points converging to infinity. Assume that for every sequence* $\{\lambda_{jn}\}$*,* $j = 1, \dots, m$ *both conditions of the Theorem 1.1 hold. Then the system* $\{e^{\lambda_{1n_{i_1}} z + \cdots + \lambda_{mn_{i_m}} z}\}$*,* $(\lambda_{1n_{i_1}}, \dots, \lambda_{mn_{i_m}}) \in \{\lambda_{1n}\} \times \cdots \times \{\lambda_{mn}\}$ *is complete in* $A(T_\alpha)$*.*

Therefore, a nontrivial extension of the above theorems to the case of several complex variables is to find complete sets of exponential functions or complete sets of limits of Dirichlet series similar to the functions ψ_n from the Proposition 1.1 whose frequencies do not form a lattice, that is, when the sequence of exponents $\{(\lambda_{1n}, \dots, \lambda_{mn})\}$ belonging to $\mathbf{R}^m_+ = \{(x_1, \dots, x_m) : x_j > 0, j = 1, \dots, m\}$ is not a set-theoretic product of sequences from $\mathbf{R}_+$. For reasons of simplicity we will state and prove our results in $\mathbf{C}^2$.

We also point out that there exists a variation of the theory of Leontiev in Several Complex Variables, due to A. B. Sekerin [13]. Sekerin's approach is based on the use of the Radon Transform. He shows that for some type of domains $D \subset \mathbf{C}^n$, every $f \in \mathcal{H}(D)$ with certain growth conditions, can be written as a Dirichlet series, whose frequencies depend on the geometry of D only. However, his results do not give completeness results covered by Theorem 1.2.

Definition 1.1 *Let* $V = \{(\lambda_n, \mu_n)\} \subset \mathbf{R}^2_+ = \{(x_1, x_2) : x_j > 0, j = 1, 2\}$*, be a sequence of points converging to infinity. We say that the sequence V satisfies the cone condition if*

1) there exists $\delta > 0$ *such that* $0 < \frac{1}{\delta} < \frac{\mu_n}{\lambda_n} < \delta$ *for every* $n \in \mathbf{N}$*. That is, the terms of the sequence lie inside a cone* $C \subseteq \mathbf{R}^2_+$*.*

2) If $n_{\{\lambda_n\}}(R)$ *and* $n_{\{\mu_n\}}(R)$ *denote the number of the terms of the sequences* $\{\lambda_n\}, \{\mu_n\}$*, respectively, in the interval* $(0, R)$ *then we have the following density condition:*

$$n_{\{\lambda_n\}}(R) = o(R^\rho) \text{ and } n_{\{\mu_n\}}(R) = o(R^\rho), \quad 0 < \rho \le \frac{1}{4}.$$

Now we are ready to state the main result of the paper.

Theorem 1.2 *Let* $V = \{(\lambda_n, \mu_n)\}$ *be a sequence of distinct points in* $\mathbf{R}^2_+$ *tending to infinity. Assume that*

$$V = \{(\lambda'_n, \mu'_n)\} \cup \{(\lambda''_n, \mu''_n)\}, \quad \{(\lambda'_n, \mu'_n)\} \cap \{(\lambda''_n, \mu''_n)\} = \varnothing$$

where the subsequence $\{(\lambda'_n, \mu'_n)\}$ *satisfies the cone condition and* $\sum_n \frac{1}{\sqrt[4]{\mu'_n}} < \infty$*. The subsequence* $\{(\lambda''_n, \mu''_n)\}$ *satisfies the conditions* $\sum_n \frac{1}{(\lambda''_n)^2} = \infty$, $\sum_n \frac{1}{\sqrt[8]{\mu''_n}} < \infty$*. Let* $\beta'_n = \sqrt{(\lambda'_n)^2 + (\mu'_n)^2}$*,* $\beta''_n = \sqrt{(\lambda''_n)^2 + \mu''_n}$ *for every* $n \in \mathbf{N}$ *and assume that* $n_{\{\lambda''_n\}}(\beta'_n) = O(\sqrt{\beta'_n})$*,* $n_{\{\lambda''_n\}}(\beta''_n) = O(\sqrt{\beta''_n})$*. Assume that for the sequences* $\{\beta'_n\}$ *and* $\{\beta''_n\}$ *the separation conditions*

$$|\beta'_n - \lambda'_j| = |\sqrt{(\lambda'_n)^2 + (\mu'_n)^2} - \lambda'_j| \ge d_1 > 0, \quad (n, j) \in \mathbf{N} \times \mathbf{N} \tag{1.3}$$

$$|\beta''_n - \lambda'_j| = |\sqrt{(\lambda''_n)^2 + \mu''_n} - \lambda'_j| \ge d_2 > 0, \quad (n, j) \in \mathbf{N} \times \mathbf{N}, \tag{1.4}$$

$$|\beta'_n - \lambda''_j| = |\sqrt{(\lambda'_n)^2 + (\mu'_n)^2} - \lambda''_j| \ge (\beta'_n)^{\frac{1}{4}} > 0, \quad (n, j) \in \mathbf{N} \times \mathbf{N} \tag{1.5}$$

$$|\beta''_n - \lambda''_j| = |\sqrt{(\lambda''_n)^2 + \mu''_n} - \lambda''_j| \ge (\beta''_n)^{\frac{1}{4}} > 0, \quad (n, j) \in \mathbf{N} \times \mathbf{N} \tag{1.6}$$

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hold, where d_1, d_2 are positive constants and that their counting functions satisfy $n_{\{\beta'_n\}}(r) = O(\ln r)$, $n_{\{\beta''_n\}}(r) = O(\ln r)$.

Then, there exists $\alpha_0 \in \mathbf{N}$, depending on V, such that

1) For every $\alpha \geq \alpha_0$, $\alpha \in \mathbf{N}$, the sequence of functions $\{\psi_m(z_1, z_2)\}$, $(z_1, z_2) \in T_\alpha$ given by

$$\psi_m(z_1, z_2) = \sum_{j=1}^{m} b_j e^{\lambda''_j z_1 + \sqrt{\mu''_j} z_2} + \lim_{l\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} a_{l,i,m} e^{\lambda'_i z_1 + \mu'_i z_2}, \quad m \in \mathbf{N}, \tag{1.7}$$

where the coefficients $b_j, a_{l,i,m}$ are Local Residues of meromorphic $(2, 0)$ forms at the points of V and depend on the strip T_α, satisfies $\{\psi_m\} \subset \mathcal{H}(T_\alpha)$.

2) The sequence of functions $\{\psi_m\}$ is complete in the space $\mathcal{H}(T_\alpha)$.

Furthermore, every $f \in \mathcal{H}(T_\alpha)$ can be approximated pointwise by finite linear combinations of the exponential functions taken from $\{e^{\lambda'_n z_1 + \mu'_n z_2}\} \cup \{e^{\lambda''_n z_1 + \sqrt{\mu''_n} z_2}\}$.

Example 1.1 Consider the sequences $\{\sqrt{2}3^n\}, \{5^n\}$, $n \in \mathbf{N}$. If $\{b_n\} = \{\sqrt[4]{n}\}$, $n \in \mathbf{N}$, then $\sum_n \frac{1}{b_n^2} = \infty$. Hence, we can take $b_n^{(i)} \in \{b_n\}$, $i = 1, \ldots, k_1$ such that $\sum_{i=1}^{k_1} (\frac{1}{b_n^{(i)}})^2 \geq 1$. Choose $\beta'_1 = \sqrt{2}3^{n_1}$, where $n_1 \in \mathbf{N}$ is the smallest integer such that $k_1 < \sqrt{\beta'_1}$, $\beta'_1 > b_n^{(i)}$, $i = 1, \ldots, k_1$. Take now $\beta'_i = \sqrt{2}3^{n_i}$, $i = 2, \ldots, k_1$ satisfying $\beta'_1 < \beta'_2 < \cdots < \beta'_{k_1}$ and such that

$$b_n^{(i)} \notin \bigcup_{j=1}^{k_1} [\sqrt{2}3^{n_j} - \sqrt[8]{2}3^{\frac{n_j}{4}}, \sqrt{2}3^{n_j} + \sqrt[8]{2}3^{\frac{n_j}{4}}], \quad i = 1, \ldots, k_1.$$

Similarly, we can choose $\beta''_1 = 5^{m_1}$, where $m_1 \in \mathbf{N}$ is the smallest integer such that $k_1 < \sqrt{\beta''_1}$, $\beta''_1 > b_n^{(i)}$, $i = 1, \ldots, k_1$, $5^{m_1} \neq 3^n, n \in \mathbf{N}$. We can take $\beta''_i = 5^{m_i}$, $i = 2, \ldots, k_1$ satisfying $\beta''_1 < \beta''_2 < \cdots < \beta''_{k_1}$ and such that

$$b_n^{(i)} \notin \bigcup_{j=1}^{k_1} [5^{m_j} - 5^{\frac{m_j}{4}}, 5^{m_j} + 5^{\frac{m_j}{4}}], \quad i = 1, \ldots, k_1.$$

Since

$$\sum_{\substack{b_n \geq \beta'_{k_1} + \sqrt[4]{\beta'_{k_1}} \\ b_n \geq \beta''_{k_1} + \sqrt[4]{\beta''_{k_1}}}} \frac{1}{b_n^2} = \infty,$$

there exist $b_n^{(i)} \in \{b_n\}$, $i = k_1 + 1, \ldots, k_2$ such that $\sum_{i=k_1+1}^{k_2} (\frac{1}{b_n^{(i)}})^2 \geq 1$. We can again take as $\beta'_{k_1+1} = \sqrt{2}3^{n_{k_1+1}} > \beta'_{k_1}$, $\beta''_{k_1+1} = 5^{m_{k_1+1}} > \beta''_{k_1}$, where $n_{k_1+1}, m_{k_1+1} \in \mathbf{N}$ are the smallest integers such that $k_2 < \sqrt{\beta'_{k_1+1}}, \sqrt{\beta''_{k_1+1}}$ and $\beta'_{k_1+1}, \beta''_{k_1+1} > b_n^{(i)}$, $i = 1, \ldots, k_2$.

Furthermore, we choose $\beta_i'' = 5^{m_i}$, $i = k_1 + 2, \dots, k_2$, $\beta_i' = \sqrt{2}3^{n_i}$, $i = k_1 + 2, \dots, k_2$ satisfying

$$\beta_1' < \beta_2' < \cdots < \beta_{k_1}' < \beta_{k_1+1}' < \cdots < \beta_{k_2}'$$
$$\beta_1'' < \beta_2'' < \cdots < \beta_{k_1}'' < \beta_{k_1+1}'' < \cdots < \beta_{k_2}''$$
$$b_n^{(i)} \notin \bigcup_{j=1}^{k_2} \big([\beta_j' - \sqrt[4]{\beta_j'}, \beta_j' + \sqrt[4]{\beta_j'}] \cup [\beta_j'' - \sqrt[4]{\beta_j''}, \beta_j'' + \sqrt[4]{\beta_j''}] \big),$$
$$i = 1, \dots, k_2$$

and subject to the restriction $5^{m_i} \neq 3^n$, $n \in \mathbf{N}$. We can repeat this process countably many times and obtain the sequences $\lambda_i'' = b_n^{(i)}$, $i \in \mathbf{N}$, $\mu_i'' = 5^{2m_i} - (\lambda_i'')^2$, $m_i \in \mathbf{N}$. Furthermore, we put $\{(\lambda_i', \mu_i')\} = \{(3^{n_i}, 3^{n_i})\}$. It is clear that the conditions of the Theorem 1.2 are satisfied when the constants $d_1 = d_2 = \frac{1}{4}$. The number α_0 is determined as in Lemma 3.6.

Therefore the set of exponential functions $\{e^{3^{n_k} z_1 + 3^{n_k} z_2}\}_{n_k} \cup \{e^{\lambda_i'' z_1 + \sqrt{\mu_i''} z_2}\}$ generates the sequence $\{\psi_n\}$ which is complete in the space $\mathcal{H}(T_\alpha)$ for every $\alpha \geq \alpha_0$, $\alpha \in \mathbf{N}$.

Remark 1.2 In view of Theorem 1.1 it is quite possible that the density of exponential is not a good way to count the magnitude of the complete sets. However, it is interesting to observe that sets of exponential functions as in the above example are really small, since the sequence $\{\frac{\mu_n}{\lambda_n}\}$ has a finite number of accumulation points. In the case of an exponential complete set, with frequencies from a lattice, there are infinitely many accumulation points for the sequences $\{\frac{\mu_m}{\lambda_n}\}$, $(n, m) \in \mathbf{N} \times \mathbf{N}$.

The Lebesgue measure on the unit circle, contained in the real plane, of the limit points of the normalized vectors $\{(\frac{\lambda_n}{\sqrt{\lambda_n^2 + \mu_n^2}}, \frac{\mu_n}{\sqrt{\lambda_n^2 + \mu_n^2}})\}$ can be considered as a measure of the complete sets of functions. Therefore, the resulting sequences $\{\psi_n\}$ in the above example are small complete sets of functions in contrast to the complete sets of functions from Section 2.1.

The idea of the proof is very simple. We realize the sequence $\mathcal{V} = \{(\lambda_n', \mu_n')\} \cup \{(\lambda_n'', \sqrt{\mu_n''})\}$ as a subset of the union of discrete intersection of the zero varieties of entire functions $F_{1,n}, F_2$ (complete intersection of codimension 0). Then $\mathcal{V} \subset \bigcup_n (Z_{F_{1,n}} \cap Z_{F_2})$ is contained in the polar set of $\frac{1}{F_{1,n}F_2}$. Since the lattice $\bigcup_n (Z_{F_{1,n}} \cap Z_{F_2})$ is much larger than the set V, we modify suitably the function $\frac{1}{F_{1,n}F_2}$ by introducing a numerator of the appropriate growth, so that the only points of the lattice $\bigcup_n (Z_{F_{1,n}} \cap Z_{F_2})$ which are in the polar set are the points of $\mathcal{V}$. We then prove a multidimensional variation of the Cauchy Residue Theorem. At this point we introduce Local Residues as a tool to compute integrals over noncompact, two real-dimensional cycles. To do that we compute the limit of finite sums of Residue Integrals over separating cycles [2].

Next, we use the result of L. Aizenberg-A. Martineau-G. Tillman [1], [9], [12] on the integral representation of the analytic functionals to complete the proof the theorem.

The paper is organized as follows. In Section 2 all the basic tools and necessary theorems are presented. Section 3 contains a basic construction of a global meromorphic form. In Section 4 one of the key lemmas, the lemma on "the change of the contour of integration" is proven. Section 5 contains a Residue Theorem on a two real-dimensional, noncompact cycle and the final step of the proof of the Theorem 1.2.

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2 Preliminaries

2.1 An Example of a Very Large Multidimensional Complete System in $\mathcal{H}(T_\alpha)$

Let T_α be an open tube in $\mathbf{C}^2$. We claim that the set of exponential functions $\{e^{cz_1+dz_2}, (c,d) \in [\frac{1}{8},\frac{1}{4}] \times [\frac{1}{3},\frac{1}{2}]\}$ is complete in $\mathcal{H}(T_\alpha)$. Remark here that this set of functions is by far "larger" than the one in the statement of the Theorem 1.2. We also point out that the particular choice of the rectangle $[\frac{1}{8},\frac{1}{4}] \times [\frac{1}{3},\frac{1}{2}]$ is not essential. It would be enough to assume that endpoints of the intervals are positive and smaller than 1. If

$$CT_\alpha = \{z \in \mathbf{C}^2 : |\Im z_i| > \alpha, i = 1,2\} \tag{2.1}$$

then a result from [12], implies the isomorphism of topological vector spaces

$$\mathcal{H}'(T_\alpha) \cong \mathcal{H}_0(\bar{\mathbf{C}}^2 \setminus CT_\alpha) = \{\phi \in \mathcal{H}(\overline{\mathbf{C}^2} \setminus CT_\alpha) : \phi(\infty) = 0\}.$$

It follows from this that every analytic functional $L \in \mathcal{H}'(T_\alpha)$ can be written as

$$L(f) = \int_{\Gamma_1\times\Gamma_2} f(\zeta)\phi(\zeta)\, d\zeta_1 \wedge d\zeta_2, \quad f \in \mathcal{H}(T_\alpha), \tag{2.2}$$

where $\phi \in \mathcal{H}_0(\overline{\mathbf{C}^2 \setminus CT_\alpha})$ and $\Gamma_1 \times \Gamma_2$ is a compact cycle contained in T_α. Therefore the Borel-Laplace Transform $\hat{L}$ of $L \in \mathcal{H}'(T_\alpha)$ is an entire function of the following form

$$\hat{L}(t_1,t_2) = \int_{\Gamma_1\times\Gamma_2} \phi(\zeta_1,\zeta_2) e^{\zeta_1 t_1+\zeta_2 t_2}\, d\zeta_1 \wedge d\zeta_2, \quad (t_1,t_2) \in \mathbf{C}^2 \tag{2.3}$$

Now we are ready to state the following

Lemma 2.1 *Let $\{(c,d) \in [\frac{1}{8},\frac{1}{4}] \times [\frac{1}{3},\frac{1}{2}]\} \subset \mathbf{R}^2_+$. Then the set of exponential functions $\{e^{cz_1+dz_2}, [\frac{1}{8},\frac{1}{4}] \times [\frac{1}{3},\frac{1}{2}]\}$ is complete in $\mathcal{H}(T_\alpha)$.*

Proof Montel's theorem implies that the set $\{(c,d) \in [\frac{1}{8},\frac{1}{4}] \times [\frac{1}{3},\frac{1}{2}]\}$ is a set of uniqueness for the Borel-Laplace transform of every $L \in \mathcal{H}'(T_\alpha)$. This means that if an entire function of the form

$$\hat{L}(t_1,t_2) = \int_{\Gamma_1\times\Gamma_2} \phi(\zeta_1,\zeta_2) e^{\zeta_1 t_1+\zeta_2 t_2}\, d\zeta_1 \wedge d\zeta_2, \quad (t_1,t_2) \in \mathbf{C}^2,$$

where $\Gamma_1 \times \Gamma_2$ is a compact cycle, vanishes on the set $\{(c, d) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]\}$ then it is identically equal to 0. Assume that the above set of exponential functions is not complete. That is, $\mathcal{H}(T_\alpha) \neq \overline{\{e^{cz_1+dz_2}, (c, d) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]\}}$. Then Hahn-Banach theorem implies that there exists an element $L \in \mathcal{H}'(T_\alpha)$ so that

$$L(f) = 0, \quad f \in \overline{\left\{e^{cz_1+dz_2}, (c, d) \in \left[\frac{1}{8}, \frac{1}{4}\right] \times \left[\frac{1}{3}, \frac{1}{2}\right]\right\}}$$
$$L(g) \neq 0, \quad g \notin \overline{\left\{e^{cz_1+dz_2}, (c, d) \in \left[\frac{1}{8}, \frac{1}{4}\right] \times \left[\frac{1}{3}, \frac{1}{2}\right]\right\}},$$

where $\overline{\{e^{cz_1+dz_2}, (c, d) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]\}}$ denotes the closure of the linear span of the exponential set of functions under the topology of the uniform convergence over the compact sets. Tillman's result [12] implies then that L is of the form (2.2). Therefore for every $(c, d) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]$ we have

$$L(e^{cz_1+dz_2}) = \int_{\Gamma_1\times\Gamma_2} e^{cz_1+dz_2} \phi(\zeta_1, \zeta_2)\, d\zeta_1 \wedge d\zeta_2 = \hat{L}(c, d).$$

For every $c_0 > 0$ fixed, we have that the entire function $\hat{L}(c_0, t_2)$ of one complex variable vanishes on $[\frac{1}{3}, \frac{1}{2}]$ and therefore is identically equal to 0. Thus the real dimension of the zero variety $Z_{\hat{L}} = \{(t_1, t_2) \in \mathbf{C}^2 \mid \hat{L}(t_1, t_2) = 0\}$ is equal to 3. Hence $\hat{L} \equiv 0$. This implies that $L \equiv 0$ because the Borel-Laplace Transform is an isomorphism between topological vector spaces. Contradiction. ■

The same result can be obtained using a different duality [1], [9], since the domain T_α is convex.

2.2 Grothiendieck Residue For Separating Cycles

The material of this section is contained in [2]. The use of separating cycles allows the replacement of the integral over a compact, two dimensional (real) cycle with the sum of local (Grothendieck) residues. The setting is the following:

Definition 2.1 *Let $(f_1, f_2)\colon \bar{\mathcal{U}}_\alpha \to \mathbf{C}^2$ be a holomorphic mapping defined in a bounded neighborhood of a point $\alpha \in \mathbf{C}^2$. Assume that α is an isolated common zero of the functions f_1, f_2, the local residue of a meromorphic form $\omega = \frac{h}{f_1 f_2}\, dz_1 \wedge dz_2$, $h \in \mathcal{H}(\mathcal{U}_\alpha)$ at the point α is the integral*

$$\operatorname{res}_\alpha \omega = (2\pi i)^{-n} \int_{\Gamma_\alpha} \frac{h}{f_1 f_2}\, dz_1 \wedge dz_2,$$

where

$$\Gamma_\alpha = \Gamma_\alpha^\epsilon(f) = \{(z_1, z_2) \in \mathcal{U}_\alpha \subset \mathbf{C}^2 : |f_1(z)| = \epsilon_1, |f_2(z)| = \epsilon_2\},$$

is a local cycle.

Consider now the holomorphic functions $F_1, F_2\colon G \to \mathbf{C}$, where $G = G_1 \times G_2$ is a convex domain (Stein domain) in $\mathbf{C}^2$. For the hypersurfaces $Z_{F_i} = \{z \in \mathbf{C}^2 \mid F_i(z) = 0\}, i = 1, 2$ assume that the set $V = Z_{F_1} \cap Z_{F_2}$ is a discrete subset of G. Furthermore for the Stein domain $G = G_1 \times G_2$ the domains $G \setminus Z_{F_i}$ are Stein also. Let $\Gamma = \partial G_1 \times \partial G_2$ be a two dimensional real cycle so that $\Gamma \not\sim 0$ in $G \setminus Z_{F_1} \cup Z_{F_2}$, that is the cycle $\Gamma = \partial G_1 \times \partial G_2$ is nontrivial. If $\Gamma \sim 0$ in $G \setminus Z_{F_i}$ for $i = 1, 2$, then the cycle Γ is called *separating* for the hypersurfaces $\{F_i = 0\}, i = 1, 2$.

We now recall a theorem taken from [2], [15] stating that the separating cycles are homologous to a sum of a local ones.

Theorem 2.1 *Let X be a 2-dimensional complex manifold. Let also $\mathcal{D}_j$, $j = 1, 2$ be divisors with discrete intersection, i.e $Z = \mathcal{D}_1 \cap \mathcal{D}_2$ is a discrete subset of X. If the homology group $H_3(X)$ is trivial and $X \setminus \mathcal{D}_j$, $j = 1, 2$ are Stein manifolds, then a nontrivial, two-real dimensional cycle Γ is separating if and only if*

$$\Gamma \sim \sum_{\alpha \in Z} n_\alpha \Gamma^{\epsilon}_{\alpha}(F), \tag{2.4}$$

where $n_\alpha \in \mathbf{Z}$ and

$$\Gamma^{\epsilon}_{\alpha}(F) = \{(z_1, z_2) \in \mathcal{U}_\alpha \subset G \subset \mathbf{C}^2 : |F_1(z)| = \epsilon_1, |F_2(z)| = \epsilon_2\},$$

for ϵ_i sufficiently small positive real numbers and $\mathcal{U}_\alpha$ some open bounded neighborhood of $\alpha \in Z$.

Therefore for $F_1, F_2, h \in \mathcal{O}(\bar{G})$ we have the following equality that reduces the calculation of an integral over a compact cycle to the calculation of the sum of local residues.

$$\int_\Gamma \frac{h}{F_1 F_2}\, dz_1 \wedge dz_2 = \sum_{\alpha \in Z} n_\alpha \int_{\Gamma^{\epsilon}_{\alpha}(F)} \frac{h}{F_1 F_2}\, dz_1 \wedge dz_2. \tag{2.5}$$

3 The Sequence $\mathcal{V} = \{(\lambda'_n, \mu'_n)\} \cup \{(\lambda''_n, (\mu''_n)^{\frac{1}{2}})\}$ as a subset of a complete intersection

The purpose of this section is to construct a global meromorphic form whose polar set in the first quadrant $\mathbf{R}^2_+$ contains the points of the sequence $\mathcal{V}$ and has the appropriate growth condition on the imaginary plane. Keeping the notation of the previous sections we have

Lemma 3.1 *Let $V' = \{(\lambda'_n, \mu'_n)\} \subset \mathbf{R}^2_+$ be a sequence of points satisfying the cone condition and such that $\mu'_n \uparrow \infty$, $\lambda'_n \uparrow \infty$. Assume also that $\sum_n \frac{1}{\sqrt[4]{\mu'_n}} < \infty$. Then, for any $\alpha \in \mathbf{N}$ there exist entire functions $L_1 \in \mathcal{H}(\mathbf{C})$ and $L_2 \in \mathcal{H}(\mathbf{C}^2)$, so that*

$$\{(\lambda'_n, \mu'_n)\} \subset Z^+_{L_1} \cap Z^+_{L_2},$$

$$Z^+_{L_1} = \{(z_1, z_2) \in \mathbf{C}^2 \mid L_1(z_1) = 0, \Re z_1 > 0, \Re z_2 > 0\}$$

$$Z^+_{L_2} = \{(z_1, z_2) \in \mathbf{C}^2 \mid L_2(z_1, z_2) = 0, \Re z_1 > 0, \Re z_2 > 0\}. \tag{3.6}$$

and satisfying the following estimate for $(z_1, z_2) \in iR^2$

$$|L_1(z_1)L_2(z_1, z_2)| \geq Be^{2\alpha(|z_1|+|z_2|)}, \quad \text{for some } B > 0 \text{ constant.} \tag{3.7}$$

Proof Consider any sequence $\{a_m\}$ of distinct, positive numbers, increasing to $+\infty$ in the z_1- coordinate plane with its density equal to $\lim_{r\to\infty} \frac{n_{\{a_m\}}(r)}{r} = \frac{4\alpha}{\pi}$, $\alpha \in \mathbf{N}$ and whose exponent of convergence is equal to 1. Also, we assume that $\{\alpha_m\} \cap \{\lambda_i\} = \varnothing$. Then we take the canonical product $f_1(z_1) = \prod_m (1 + \frac{z_1}{a_m})e^{-\frac{z_1}{a_m}}$. Theorem 8.2.1 in [4] implies that

$$|f_1(z_1)| \geq be^{2\alpha|z_1|}, \quad z_1 = ir_1 \in i\mathbf{R},\ b > 0.$$

Similarly, for any sequence $\{\gamma_k\}$ of distinct, positive numbers, increasing to $+\infty$ in the z_2- coordinate plane with its density equal to $\lim_{r\to\infty} \frac{n_{\{\gamma_k\}}(r)}{r} = \frac{4\alpha}{\pi}$, $\alpha \in \mathbf{N}$ and whose exponent of convergence is equal to 1, we define the canonical product $f_2(z_2) = \prod_k (1 + \frac{z_2}{\gamma_k})e^{-\frac{z_2}{\gamma_k}}$. Naturally, we have assumed here that $\{\gamma_k\} \cap \{\mu_i\} = \varnothing$. For the same reason as above we have

$$|f_2(z_2)| \geq ce^{2\alpha|z_2|}, \quad z_2 = ir_2 \in i\mathbf{R},\ c > 0.$$

We define the desired entire functions L_1 of one complex variable and the function L_2 of two complex variables as follows

$$L_1(z_1) = f_1(z_1) \prod_{n=1}^{\infty} \left(1 - \frac{z_1^2}{(\lambda_n')^2}\right) \tag{3.8}$$

$$L_2(z_1, z_2) = f_2(z_2) \prod_{n=1}^{\infty} \left(1 - \frac{z_1^2 + z_2^2}{(\lambda_n')^2 + (\mu_n')^2}\right). \tag{3.9}$$

These functions are well defined because of the cone condition and the fact that $\sum_n \frac{1}{\mu_n'} < \infty$. The first claim (3.6) is obvious. The relation (3.7) follows directly from the fact that

$$\prod_{n=1}^{\infty} \left(1 - \frac{(ir_1)^2}{(\lambda_n')^2}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{r_1^2}{(\lambda_n')^2}\right) \geq 1$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{(ir_1)^2 + (ir_2)^2}{(\lambda_n')^2 + (\mu_n')^2}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{r_1^2 + r_2^2}{(\lambda_n')^2 + (\mu_n')^2}\right) \geq 1.$$

This completes the proof of the lemma. ■

The next lemma is of similar nature

Lemma 3.2 *Let* $V'' = \{(\lambda_n'', \mu_n'')\}$ *be a sequence of points satisfying conditions of Theorem 1.2. For any* $n \in \mathbf{N}$ *there exist entire functions* $Q_{1,n} \in \mathcal{H}(\mathbf{C})$, $Q_2 \in \mathcal{H}(\mathbf{C}^2)$, *so that* $(\lambda_n'', \sqrt{\mu_n''}) \in Z^+_{Q_{1,n}} \cap Z^+_{Q_2}$, *where* Q_2 *is independent of* n *and*

$$\{(\lambda_n'', \sqrt{\mu_n''})\}_n \subset \bigcup_n (Z^+_{Q_{1,n}} \cap Z^+_{Q_2}),$$

$$Z^+_{Q_{1,n}} = \{(z_1, z_2) \in \mathbf{C}^2 \mid Q_{1,n}(z_1) = 0, \Re z_1 > 0, \Re z_2 > 0\}$$

$$Z^+_{Q_2} = \{(z_1, z_2) \in \mathbf{C}^2 \mid Q_2(z_1, z_2) = 0, \Re z_1 > 0, \Re z_2 > 0\}. \tag{3.10}$$

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Furthermore, for $(z_1, z_2) \in iR^2$ *the following estimate*

$$|Q_{1,n}(z_1)Q_2(z_1,z_2)| \geq 1$$

holds.

Proof Consider the functions given by

$$Q_{1,n}(z_1) = \prod_{i=1}^{n}\left(1 - \frac{z_1^2}{(\lambda_i'')^2}\right)$$

$$Q_2(z_1,z_2) = \prod_{i=1}^{\infty}\left(1 - \frac{z_1^2 + z_2^2}{(\lambda_i'')^2 + \mu_i''}\right).$$

We remark here that the function $Q_{1,n}$ is of one complex variable. The proof of the lemma follows. ■

We define now two holomorphic functions from $\mathbf{C}^2$ into $\mathbf{C}$ as follows:
For every $n \in \mathbf{N}$ and for every $(\sigma_1, \sigma_2) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]$ we have

$$\begin{aligned} F_{1,n}(z_1,z_2) &= (z_2 - \sigma_2)L_1(z_1)Q_{1,n}(z_1) \\ &= f_1(z_1)(z_2 - \sigma_2)\prod_{i=1}^{\infty}\left(1 - \frac{z_1^2}{(\lambda_i')^2}\right)\prod_{i=1}^{n}\left(1 - \frac{z_1^2}{(\lambda_i'')^2}\right) \end{aligned}$$

$$\begin{aligned} F_2(z_1,z_2) &= (z_1 - \sigma_1)L_2(z_1,z_2)Q_2(z_1,z_2) \\ &= f_2(z_2)(z_1 - \sigma_1)\prod_{i=1}^{\infty}\left(1 - \frac{z_1^2 + z_2^2}{(\lambda_i')^2 + (\mu_i')^2}\right)\prod_{i=1}^{\infty}\left(1 - \frac{z_1^2 + z_2^2}{(\lambda_i'')^2 + \mu_i''}\right). \end{aligned}$$

The intersection of the zero varieties $Z_{F_{1,n}} \cap Z_{F_2}$ is discrete and defines a sequence $\{P\}$ by

$$\begin{aligned} \{P\} &= \{(\sigma_1,\sigma_2)\} \cup \{(\lambda_n', \mu_n')\} \cup \{(\lambda_n'', \sqrt{\mu_n''})\} \\ &\subset \Big(\bigcup_n (Z_{F_{1,n}} \cap Z_{F_2})\Big) \cap \{(x_1,x_2) \in \mathbf{R}^2 \mid x_1 > 0, x_2 > 0\}. \end{aligned}$$

It is clear that the set $\big(\bigcup_n (Z_{F_{1,n}} \cap Z_{F_2})\big) \cap \{(x_1,x_2) \in \mathbf{R}^2 \mid x_1 > 0, x_2 > 0\}$ contains more points than the sequence $\{P\}$. However, none of them belongs to Z_{f_1}, Z_{f_2}. Therefore, while the functions f_1, f_2 are of importance in Lemma 3.6 they do not contribute to the sum of Local Residues in (5.16).

Our purpose is to introduce a numerator, (suitably chosen) into the meromorphic (2,0) form $\frac{dz_1 \wedge dz_2}{F_{1,n}F_2}$ in order that the integral over certain compact, two real dimensional cycles, be equal to a sum of a local residues at the points of the sequence $\{P\}$. This numerator is independent of the degree n of $Q_{1,n}(z_1,z_2)$. The next lemmas are of importance in Section 5. We assume that the sequences $\{(\lambda_n'', \mu_n'')\}$, $\{(\lambda_n', \mu_n')\}$ appearing in the statements of the Lemmas 3.3–3.5 satisfy the conditions of the Theorem 1.2.

Lemma 3.3 *For any* $(\sigma_1, \sigma_2) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]$ *consider the sequences*

$$\begin{aligned} z_1 &= \sigma_1, \\ \nu_n^2 &= (\beta_n')^2 - \sigma_2^2 = (\lambda_n')^2 + (\mu_n')^2 - \sigma_2^2, \quad n \in \mathbf{N}, \\ \omega_m^2 &= (\beta_m'')^2 - \sigma_2^2 = (\lambda_m'')^2 + \mu_m'' - \sigma_2^2, \quad m \in \mathbf{N}. \end{aligned}$$

Then the entire function

$$y(z_1, z_2) = e^{z_1} \prod_n \Bigl(1 - \frac{z_1^2}{\nu_n^2}\Bigr) \prod_m \Bigl(1 - \frac{z_1^2}{\omega_m^2}\Bigr)$$

is of exponential type zero. Furthermore, the inclusion

$$(Z_{(z_2-\sigma_2)} \cap Z_{F_2} \cap \mathbf{R}_+^2) \setminus \{P\} \subset Z_y \subset \mathbf{C}^2 \tag{3.11}$$

holds.

Proof The proof is elementary, since we can assume without loss of generality that $\lambda_n' > 1, \lambda_n'' > 1, \mu_n' > 1, \mu_n'' > 1$. ■

Lemma 3.4 *1) Let* $V' = \{(\lambda_m', \mu_m')\}$ *be a sequence satisfying the cone condition. For every* $m \in \mathbf{N}$ *consider the finite sequence of positive real numbers*

$$\tau_{m,k}^2 = (\beta_m')^2 - (\lambda_m')^2 = (\lambda_m')^2 + (\mu_m')^2 - (\lambda_k')^2, \quad m \neq k,\ 1 \leq k \leq m_0(\beta_m'),$$

where $m_0(\beta_m')$ *is the largest index such that* $(\lambda_m')^2 + (\mu_m')^2 - (\lambda_k')^2 \geq 0$. *Then the infinite product*

$$g(z_1, z_2) = e^{z_2} \prod_{m=1}^{\infty} \prod_{k=1}^{m_0(\beta_m')} \Bigl(1 - \frac{z_2^2}{\tau_{m,k}^2}\Bigr)$$

is a well defined entire function of exponential type η_g.

2) Let $V'' = \{(\lambda_m'', \mu_m'')\}$ *be sequence satisfying conditions of the Theorem 1.2. For every* $m \in \mathbf{N}$ *consider the finite sequence of positive real numbers*

$$\zeta_{m,j}^2 = (\beta_m'')^2 - (\lambda_j')^2 = (\lambda_m'')^2 + \mu_m'' - (\lambda_j')^2, \quad m \neq j,\ 1 \leq j \leq m_0'(\beta_m''),$$

where $m_0'(\beta_m'')$ *is the largest index such that* $(\lambda_m'')^2 + \mu_m'' - (\lambda_j')^2 \geq 0$. *Then the infinite product*

$$h(z_1, z_2) = e^{z_2} \prod_{m=1}^{\infty} \prod_{j=1}^{m_0(\beta_m'')} \Bigl(1 - \frac{z_2^2}{\zeta_{m,j}^2}\Bigr)$$

is a well defined entire function of exponential type η_h.

3) In addition the following holds for $l_1(z_1) = \prod_{i=1}^{\infty}(1 - \frac{z_1^2}{(\lambda_i')^2})$

$$(Z_{l_1} \cap Z_{F_2} \cap \mathbf{R}_+^2) \setminus \{P\} \subset Z_h \cup Z_g \subset \mathbf{C}^2. \tag{3.12}$$

Proof 1) First we will show that infinite product converges. We have that, for $m \neq k$, $1 \leq k \leq m_0(\beta'_m)$,

$$\begin{aligned}\tau^2_{m,k} &= (\lambda'_m)^2 + (\mu'_m)^2 - (\lambda'_k)^2 \\ &= \big(\sqrt{(\lambda'_m)^2 + (\mu'_m)^2} - \lambda'_k\big)\big(\sqrt{(\lambda'_m)^2 + (\mu'_m)^2} + \lambda'_k\big)\end{aligned}$$

Combining the last identity with the separation condition (1.3)

$$|\sqrt{(\lambda'_m)^2 + (\mu'_m)^2} - \lambda'_k| > d_1,$$

we deduce that

$$|\tau_{m,k}| \geq \sqrt{d_1}\sqrt{\mu'_m}.$$

therefore, since $m_0(\beta'_m) = o(2(\mu'_m)^\rho)$, we have that

$$\sum_j^{m_0(\beta'_m)} \frac{1}{\tau^2_{m,k}} \leq \frac{(2\epsilon\mu'_m)^\rho}{d_1(\sqrt{\mu'_m})^2}, \quad m >> 0.$$

The condition $\sum_m \frac{1}{\sqrt{\mu'_m}} < \infty$ implies

$$\sum_m \frac{2\epsilon(\mu'_m)^\rho}{d_1\mu'_m} < \infty.$$

The convergence of the infinite product to an entire function of order 1 follows. The function is of finite type η_g by Borel's theorem [4], since the exponent of convergence of the sequence $\{\tau_{m,k}\}$ is smaller than or equal to 1.

2) The proof is similar to part 2) and uses the conditions $|\sqrt{(\lambda''_m)^2 + \mu''_m} - \lambda'_k| > d_2$, $\sum_m \frac{1}{\sqrt[4]{\mu''_m}} < \infty$.

The last claim follows. ■

Lemma 3.5 *1) For every $m \in \mathbf{N}$ consider the finite sequence*

$$\chi^2_{m,j} = (\beta'_m)^2 - (\lambda''_j)^2 = (\lambda'_m)^2 + (\mu'_m)^2 - (\lambda''_j)^2, \quad 1 \leq j \leq m_0(\beta'_m),$$

where $m_0(\beta'_m)$ is the largest index so that $\chi^2_{m,j} \geq 0$. Then the infinite product

$$u(z_2) = e^{z_2} \prod_{m=1}^{\infty} \prod_{j=1}^{m_0(\beta'_m)} \Big(1 - \frac{z_2^2}{\chi^2_{m,j}}\Big)$$

defines an entire function of exponential type η_u .

2) For every $m \in \mathbf{N}$ consider the finite sequence

$$\gamma^2_{m,j} = (\beta''_m)^2 - (\lambda''_j)^2 == (\lambda''_m)^2 + \mu''_m - (\lambda''_j)^2, \quad 1 \leq j \leq m_0(\beta''_m),\ j \neq m,$$

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where $m_0(\beta_m'')$ is the largest index so that $\gamma_{m,j}^2 \geq 0$. Then the infinite product

$$v(z_2) = e^{z_2} \prod_{m=1}^{\infty} \prod_{j=1}^{m_0(\beta_m'')} \Bigl(1 - \frac{z_2^2}{\gamma_{m,j}^2}\Bigr)$$

defines an entire function of exponential type η_v.

3) Furthermore, if $\mathcal{A}_n = Z_{Q_{1,n}} \cap Z_{F_2} \cap \mathbf{R}_+^2$. Then

$$\bigcup_n \mathcal{A}_n \setminus \{P\} \subset Z_u \cup Z_v. \tag{3.13}$$

Proof 1) We observe that

$$\begin{aligned}\chi_{m,j}^2 &= (\lambda_m')^2 + (\mu_m')^2 - (\lambda_j'')^2\\ &= \Bigl(\sqrt{(\lambda_m')^2 + (\mu_m')^2} - \lambda_j''\Bigr)\Bigl(\sqrt{(\lambda_m')^2 + (\mu_m')^2} + \lambda_j''\Bigr).\end{aligned}$$

By (1.5) we have that $|\sqrt{(\lambda_m')^2 + (\mu_m')^2} - \lambda_j''| \geq (\beta_m')^{\frac{1}{4}} > 0$. Therefore

$$|\chi_{m,j}| \geq \sqrt{(\beta_m')^{\frac{1}{4}}} \sqrt{\sqrt{(\lambda_m')^2 + (\mu_m')^2} + \lambda_j''} \geq \sqrt{\beta_m'}^{\frac{1}{4}} (\beta_m')^{\frac{1}{2}}.$$

The order of magnitude for $m_0(\beta_m')$ is equal to $O(\sqrt{\beta_m'})$. Therefore

$$\sum_{i=1}^{m_0(\beta_m')} \frac{1}{\chi_{m,j}^2} \leq \frac{O(\sqrt{\beta_m'})}{(\beta_m')^{\frac{1}{4}}(\beta_m')} \leq \frac{K'}{(\beta_m')^{\frac{1}{4}}(\beta_m')^{\frac{1}{2}}}, \quad m >> 0.$$

Thus

$$\sum_{m>>0} \frac{O(\sqrt{\beta_m'})}{(\beta_m')^{\frac{5}{4}}} \leq K' \sum_{m>>0} \frac{1}{\sqrt[4]{\beta_m'}} < \infty,$$

because $\sum_m \frac{1}{\sqrt[4]{\mu_m'}} < \infty$. We conclude now that the infinite product $u(z_2)$ converges to an entire function of order one. It remains to show that it is of finite type. To this end it is enough to apply Borel's theorem [4], since the exponent of convergence of the sequence $\{\chi_{m,j}\}$ is smaller or equal than 1.

2) The proof of the second part goes along the same lines. We observe that

$$\begin{aligned}\gamma_{m,j}^2 &= (\lambda_m'')^2 + \mu_m'' - (\lambda_j'')^2\\ &= \Bigl(\sqrt{(\lambda_m'')^2 + \mu_m''} - \lambda_j''\Bigr)\Bigl(\sqrt{(\lambda_m'')^2 + \mu_m''} + \lambda_j''\Bigr).\end{aligned}$$

By (1.6) we have that $|\sqrt{(\lambda_m'')^2 + \mu_m''} - \lambda_j''| \geq (\beta_m'')^{\frac{1}{4}} > 0$. Therefore

$$|\gamma_{m,j}| \geq \sqrt{(\beta_m'')^{\frac{1}{4}}} \sqrt{\sqrt{(\lambda_m'')^2 + \mu_m''} + \lambda_j''} \geq (\beta_m'')^{\frac{1}{8}} (\beta_m'')^{\frac{1}{2}}.$$

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The order of magnitude for $m_0(\beta''_m)$ is equal to $O(\sqrt{\beta''_m})$. Therefore

$$\sum_{i=1}^{m_0(\beta''_m)} \frac{1}{\gamma_{m,j}^2} \leq \frac{O(\sqrt{\beta''_m})}{(\beta''_m)^{\frac{1}{4}}(\beta''_m)} \leq \frac{K''}{(\beta''_m)^{\frac{1}{4}}(\beta''_m)^{\frac{1}{2}}}, \quad m >> 0$$

Thus

$$\sum_{m>>0} \frac{O(\sqrt{\beta''_m})}{(\beta''_m)^{\frac{3}{4}}} \leq K'' \sum_{m>>0} \frac{1}{\sqrt[4]{\beta''_m}} < \infty,$$

because $\sum_m \frac{1}{\sqrt[4]{\mu''_m}} < \infty$. We conclude now that the infinite product $u(z_2)$ converges to an entire function of order one. It remains to show that it is of finite type. Again, Borel's theorem [4] leads to the desired conclusion, since the exponent of convergence of the sequence $\{\gamma_{m,j}\}$ is smaller than or equal to 1.

The last claim follows. ■

For any $(\sigma_1, \sigma_2) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]$ and for every $n \in \mathbf{N}$ consider the (2,0) form

$$\Phi_n(z_1, z_2) = \qquad A_n(\sigma) \frac{(vugh)(z_2)y(z_1)\, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(L_1L_2)(z_1, z_2)(Q_{1,n}Q_{2,n})(z_1, z_2)},$$

where

$$A_n(\sigma) = \frac{(L_1L_2)(\sigma)(Q_{1,n}Q_{2,n})(\sigma)}{(uvgh)(\sigma_2)y(\sigma_1)},$$

where $Q_{2,n}(z_1, z_2) = \prod_{i=1}^n (1 - \frac{z_1^2+z_2^2}{(\lambda''_i)^2+\mu''_i})$, $Q_{1,n}(z_1) = \prod_{i=1}^n (1 - \frac{z_1^2}{(\lambda''_i)^2})$. It is clear that the numerator of the form $\Phi_n(z_1, z_2)$ vanishes at all the points of $\bigcup_n (Z_{F_{1,n}} \cap Z_{F_2}) \setminus \{P\}$.

Let

$$b(z_1, z_2) = g(z_2)v(z_2)u(z_2)h(z_2)y(z_1), \tag{3.14}$$

then for every $n \in \mathbf{N}$ we can define the functions

$$\mathcal{L}_n(s_1, s_2) = A_n(\sigma) \int_{i\mathbf{R}^2} \frac{e^{z_1s_1+z_2s_2} b(z_1, z_2)\, dz_1\, dz_2}{(\sigma - z)(L_1L_2)(z_1, z_2)Q_{2,n}(z_1, z_2)Q_{1,n}(z_1)}, \tag{3.15}$$

$$= A_n(\sigma) \int_{i\mathbf{R}^2} \frac{e^{z_1s_1+z_2s_2} b(z_1, z_2)\, dz_1\, dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(L_1L_2)(z_1, z_2)Q_n(z_1, z_2)}, \tag{3.16}$$

where

$$Q_n(z_1, z_2) = (Q_{1,n}Q_{2,n})(z_1, z_2) = \prod_{i=1}^n \left(1 - \frac{z_1^2}{(\lambda''_i)^2}\right) \prod_{i=1}^n \left(1 - \frac{z_1^2 + z_2^2}{(\lambda''_i)^2 + \mu''_i}\right) \tag{3.17}$$

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Lemma 3.6 *Put*

$$\alpha_0 = min\{n \in \mathbf{N} : n > \theta = \eta_g + \eta_h + \eta_u + \eta_v\}.$$

For every $n \in \mathbf{N}$, the function $\mathcal{L}_n(s_1, s_2)$ defined in (3.15)–(3.16) is well defined and holomorphic in the open strip

$$T_\alpha = \{(s_1, s_2) \in \mathbf{C}^2 : |\Im s_i| < \alpha, i = 1, 2\}, \quad \textit{for } \alpha \geq \alpha_0,\ \alpha \in \mathbf{N}.$$

Furthermore, $\mathcal{L}_n(s_1, s_2) \to 0$ uniformly in the open strip T_α whenever $n \to \infty$.

Proof The proof follows from the fact that the modulus of the integrand over $i\mathbf{R}^2$ is bounded by

$$\left|\frac{b(z_1, z_2)e^{z_1 s_1 + z_2 s_2}}{(\sigma - z)(L_1 L_2)(z_1, z_2)Q_n(z_1, z_2)}\right| \leq B_\epsilon e^{-r_1 \Im s_1 - r_2 \Im s_2 + \theta|r_2| + \epsilon|r_1| - 2\alpha(|r_1| + |r_2|)},$$

where B_ϵ is a positive constant depending only on ϵ. The integral (3.15) converges absolutely in $T_{\alpha-\epsilon}$, provided that $|\Im s_i| < \alpha - \epsilon$, $i = 1, 2$. It is also clear that $\mathcal{L}_n$ is holomorphic in the tube $T_{\alpha-\epsilon}$. Since this holds for any $\epsilon > 0$ the first part of the lemma follows. The uniform convergence follows from the fact that

$$|\mathcal{L}_n(s_1, s_2)| \leq C\left|\frac{(L_1 L_2)(\sigma)}{b(\sigma)}\right| |Q_n(\sigma)| \tag{3.18}$$

and that $|Q_n(\sigma)| \to 0$ as $n \to \infty$. The conclusion of the lemma follows. ■

4 Changing the Contour of Integration

The main result of this section is the Lemma 4.1 which is the basic step for the introduction of the local residues and separating cycles in Section 5. At this stage we introduce the angles

$$\Gamma_i = \left\{z_i \in \mathbf{C} : |\arg z_i| < \frac{\pi}{4}\right\}, \quad i = 1, 2.$$

Consider the infinite (noncompact) cycle $\Gamma = \partial\Gamma_1 \times \partial\Gamma_2$. The following lemma is of the type "change of the contour of integration" and is a direct analogue of the corresponding lemma in one complex variable.

For every $l \in \mathbf{N}$ we define the entire functions

$$G_{1,l}(z_1, z_2) = \prod_{i=1}^{l}\left(1 + \frac{z_1}{a_i}\right)e^{-\frac{z_1}{a_i}} \prod_{i=1}^{\infty}\left(1 - \frac{z_1^2}{(\lambda_i')^2}\right)$$

$$G_{2,l}(z_1, z_2) = \prod_{i=1}^{l}\left(1 + \frac{z_2}{\gamma_i}\right)e^{-\frac{z_2}{\gamma_i}} \prod_{i=1}^{\infty}\left(1 - \frac{z_1^2 + z_2^2}{(\lambda_i')^2 + (\mu_i')^2}\right),$$

where the sequences $\{a_i\}$, $\{\gamma_i\}$ appear in the Lemma 3.1.

For the rest of the paper $\alpha_0 \in \mathbf{N}$ denotes the number defined in Lemma 3.6.

Lemma 4.1 *Let $\alpha \in \mathbf{N}$ and $\alpha \geq \alpha_0$. Then, for $(s_1, s_2) \in T_\alpha$ we have that*

$$\begin{aligned} \mathcal{L}_n(s_1,s_2) &= A_n(\sigma) \int_{i\mathbf{R}^2} \frac{b(z_1,z_2)e^{z_1s_1+z_2s_2}\, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(L_1L_2)Q_n(z_1,z_2)} \\ &= 3A_n(\sigma) \lim_{l\to\infty} \int_\Gamma \frac{b(z_1,z_2)e^{z_1s_1+z_2s_2}\, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(G_{1,l}G_{2,l}Q_n)(z_1,z_2)}, \end{aligned} \tag{4.1}$$

where $Q_n(z_1,z_2) = Q_{1,n}(z_1)Q_2(z_1,z_2)$ and $b(z_1,z_2)$ are defined by (3.14)–(3.17).

Proof Using the simple fact that the square of a modulus of a complex number is equal to the sum of the squares of its real and imaginary parts, we deduce that when $z_1 = r_1e^{\pm i\theta_1}$, $|\theta_1| \in [\frac{\pi}{4}, \frac{\pi}{2}]$ the estimate

$$\left|1 - \frac{z_1^2}{(\lambda_n')^2}\right|^2 \geq 1, \forall n \in \mathbf{N}$$

holds. Hence, (3.8) and the fact that $\{a_m\}$, $\{\gamma_n\}$ are sequences of positive numbers imply that

$$\left|G_{1,l}(r_1e^{i\theta_1})\right| \geq e^{-K_l|z_1|}, \frac{\pi}{4} \leq |\theta_1| \leq \frac{\pi}{2}, \quad \text{where } K_l = \max\{\sum_{i=1}^{l} \frac{1}{a_i}, \sum_{i=1}^{l} \frac{1}{\gamma_i}\}. \tag{4.2}$$

Similarly, for $\frac{\pi}{4} \leq |\theta_1|, |\theta_2| \leq \frac{\pi}{2}$ and $(z_1,z_2) = (r_1e^{i\theta_1}, r_2e^{i\theta_2})$ elementary computations show that

$$\left|1 - \frac{z_1^2 + z_2^2}{(\lambda_n')^2 + (\mu_n')^2}\right|^2 \geq 1, \quad \forall n \in \mathbf{N}. \tag{4.3}$$

Therefore, for $(\theta_1,\theta_2) \in \{\frac{\pi}{4} \leq |\theta_1| \leq \frac{\pi}{2}\} \times \{\frac{\pi}{4} \leq |\theta_2| \leq \frac{\pi}{2}\}$ we have from (3.8) and (3.9)

$$|G_{1,l}(r_1e^{i\theta_1})G_{2,l}(r_1e^{i\theta_1}, r_2e^{i\theta_2})| \geq e^{-K_l(|z_1|+|z_2|)}. \tag{4.4}$$

Furthermore, arguments of the same nature show that for every $(\theta_1,\theta_2) \in \{\frac{\pi}{4} \leq |\theta_1| \leq \frac{\pi}{2}\} \times \{\frac{\pi}{4} \leq |\theta_2| \leq \frac{\pi}{2}\}$, we have that

$$|Q_n(z_1,z_2)| = |Q_{1,n}(z_1,z_2)Q_{2,n}(z_1,z_2)| \geq 1. \tag{4.5}$$

Thus (4.4) and (4.5) imply that for $(z_1,z_2) = (r_1e^{i\theta_1}, r_2e^{i\theta_2})$ such that $(\theta_1,\theta_2) \in \{\frac{\pi}{4} \leq |\theta_1| \leq \frac{\pi}{2}\} \times \{\frac{\pi}{4} \leq |\theta_2| \leq \frac{\pi}{2}\}$ we have

$$|Q_n(z_1,z_2)G_{1,l}(z_1,z_2)G_{2,l}(z_1,z_2)| > e^{-K_l(|z_1|+|z_2|)}. \tag{4.6}$$

Choose the sequences of positive numbers $\{R_{1,m}\}_m$, $\{R_{2,m}\}_m$ strictly increasing to infinity, such that for every $m \in \mathbf{N}$ the following are satisfied: $R_{1,m} \neq \lambda_n'$, $R_{2,m} = 2R_{1,m}$, $R_{2,m}^2 \neq (\lambda_n')^2 + (\mu_n')^2$ and $R_{1,m}^2 + R_{2,m}^2 \neq (\lambda_n')^2 + (\mu_n')^2$, $i = 1, 2 \ldots$. In addition $R_{1,m} \neq \lambda_i''$

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$R_{2,m}^2 \neq (\lambda_n'')^2 + \mu_n''$ and $R_{1,m}^2 + R_{2,m}^2 \neq (\lambda_i'')^2 + \mu_i''$ For for $i = 1, 2$ consider the union of two domains $D_{i,m} = D_{i,m}^{+} \cup D_{i,m}^{(-)} \subset \mathbf{C}$, and $m \in \mathbf{N}$ surrounded by the contours

$$\partial D_{i,m}^{(+)} = [0, iR_{i,m}] \cup \left\{R_{i,m}e^{i\theta_i}, \frac{\pi}{4} \leq \theta_i \leq \frac{\pi}{2}\right\} \cup \{re^{i\frac{\pi}{4}}, 0 \leq r \leq R_{i,m}\} \tag{4.7}$$

$$\partial D_{i,m}^{(-)} = [-iR_{i,m}, 0] \cup \left\{re^{-i\frac{\pi}{4}}, 0 \leq r \leq R_{i,m}\right\} \cup \{R_{i,m}e^{-i\theta_i}, \frac{\pi}{4} \leq \theta_i \leq \frac{\pi}{2}\} \tag{4.8}$$

If $\partial D_{i,m} = \partial D_{i,m}^{(+)} \cup \partial D_{i,m}^{(-)}$, then $\partial D_{1,m} \times \partial D_{2,m}$ is a two-real dimensional compact cycle which is the boundary of both $D_{1,m} \times \partial D_{2,m}$ and $\partial D_{1,m} \times D_{2,m}$. The orientation of $\partial D_{1,m} \times \partial D_{2,m}$ is the induced one. The simplexes $D_{1,m} \times \partial D_{2,m}$, $\partial D_{1,m} \times D_{2,m}$ are homologous to 0 in $\bar{D}_{1,m} \times \bar{D}_{2,m}$ because if

$$\mathcal{A} = \{z \in \mathbf{C}^2 \mid (\sigma_1 - z_1)(\sigma_2 - z_2)Q_n(z)G_{1,l}(z)G_{2,l}(z) = 0\},$$

then $\bar{D}_{1,m} \times \bar{D}_{2,m} \backslash \mathcal{A} = \bar{D}_{1,m} \times \bar{D}_{2,m}$. This follows from direct verification. Multidimensional Cauchy Theorem implies that

$$A_n(\sigma) \int_{\partial D_{1,m} \times \partial D_{2,m}} \frac{b(z_1, z_2)e^{s_1z_1+s_2z_2}}{(\sigma_1 - z_1)(\sigma_2 - z_2)(Q_nG_{1,l}G_{2,l})(z_1, z_2)} \, dz_1 \wedge dz_2 = 0. \tag{4.9}$$

It is straightforward to see now that

$$\begin{aligned} A_n(\sigma) \int_{-iR_{1,m}}^{iR_{1,m}} \int_{-iR_{2,m}}^{iR_{2,m}} & \frac{b(z_1, z_2)e^{s_1z_1+s_2z_2} \, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(Q_nG_{1,l}G_{2,l})(z_1, z_2)} \\ &= A_n(\sigma) \int_{\Gamma \cap B(R_{1,m}, R_{2,m})} \frac{b(z_1, z_2)e^{s_1z_1+s_2z_2} \, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(Q_nG_{1,l}G_{2,l})(z_1, z_2)} \\ &\quad + A_n(\sigma) \int_{\varrho_m} \frac{b(z_1, z_2)e^{s_1z_1+s_2z_2} \, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(Q_nG_{1,l}G_{2,l})(z_1, z_2)} \\ &\quad + A_n(\sigma) \int_{\varphi_m} \frac{b(z_1, z_2)e^{s_1z_1+s_2z_2} \, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(Q_nG_{1,l}G_{2,l})(z_1, z_2)} \\ &\quad + A_n(\sigma) \int_{\vartheta_m} \frac{b(z_1, z_2)e^{s_1z_1+s_2z_2} \, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(Q_nG_{1,l}G_{2,l})(z_1, z_2)}, \end{aligned} \tag{4.10}$$

where $B(R_{1,m}, R_{2,m})$ denotes the bidisc $B(0, R_{1,m}) \times B(0, R_{2,m})$ and

$$\varrho_m = \left\{R_{1,m}e^{i\theta_1}, \frac{\pi}{4} \leq |\theta_1| \leq \frac{\pi}{2}\right\} \times \{r_2e^{\pm i\frac{\pi}{4}}, 0 \leq r_2 \leq R_{2,m}\} \tag{4.11}$$

$$\cup \left\{R_{2,m}e^{i\theta_2}, \frac{\pi}{4} \leq |\theta_2| \leq \frac{\pi}{2}\right\} \times \{r_1e^{\pm i\frac{\pi}{4}}, 0 \leq r_1 \leq R_{1,m}\} \tag{4.12}$$

$$\cup \left\{R_{1,m}e^{i\theta_1}, \frac{\pi}{4} \leq |\theta_1| \leq \frac{\pi}{2}\right\} \times \left\{R_{2,m}e^{i\theta_2}, \frac{\pi}{4} \leq |\theta_2| \leq \frac{\pi}{2}\right\}. \tag{4.13}$$

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Furthermore,

$$\varphi_m = [-R_{1,m}, R_{1,m}] \times \{r_2 e^{\pm \frac{\pi}{4} i}, \quad 0 \le r_2 \le R_{2,m}\} \cup [-R_{1,m}, R_{1,m}] \times \left\{R_{2,m} e^{\pm i\theta_2}, \frac{\pi}{4} \le |\theta_2| \le \frac{\pi}{2}\right\} \tag{4.14}$$

$$\vartheta_m = \{r_1 e^{\pm \frac{\pi}{4} i}, 0 \le r_1 \le R_{1,m}\} \times [-R_{2,m}, R_{2,m}] \cup \left\{R_{1,m} e^{\pm i\theta_1}, \frac{\pi}{4} \le |\theta_1| \le \frac{\pi}{2}\right\} \times [-R_{2,m}, R_{2,m}]. \tag{4.15}$$

Consequently, (4.6) implies for $(z_1, z_2) \in \partial D_{1,m} \times \partial D_{2,m}$ that for every (s_1, s_2) in the half-tube

$$\{(s_1, s_2) \in \mathbf{C}^2 : |\Im s_i| < \alpha, \Re s_i < \sigma_0 = -\sqrt{2} K_l - \alpha - 1 - \epsilon, = 1, 2\} \tag{4.16}$$

we have the following estimates

$$|b(z_1, z_2) e^{s_1 z_1 + s_2 z_2 + K_l(|z_1| + |z_2|)}| \le e^{-|z_1| - |z_2|}, \quad \forall (z_1, z_2) \in \partial D_{1,m} \times \partial D_{2,m}. \tag{4.17}$$

Taking the limit in (4.10) when $m \mapsto \infty$ we get the desired result. Actually the limit on the parts of ϱ_m, described by (4.11), (4.12) and (4.13) is zero, because of (4.16). Similarly, the integrals over the subsets of φ_m, ϑ_m

$$[-R_{1,m}, R_{1,m}] \times \left\{R_{2,m} e^{i\theta}, \frac{\pi}{4} \le |\theta| \le \frac{\pi}{2}\right\}$$

$$\left\{R_{1,m} e^{i\theta}, \frac{\pi}{4} \le |\theta| \le \frac{\pi}{2}\right\} \times [-R_{2,m}, R_{2,m}]$$

tend to zero for the same reasons whenever $m \to \infty$. On the other hand we claim that the integral over the set $[-R_{1,m}, R_{1,m}] \times \{r_2 e^{\pm \frac{\pi}{4} i}, 0 \le r_2 \le R_{2,m}\}$, while $m \mapsto \infty$ is equal to the integral over Γ. This follows from multidimensional Cauchy theorem over the union of two cylinders $\partial D_{1,m} \times \{r_2 e^{\pm \frac{\pi}{4} i}, 0 \le r_2 \le R_{2,m}\}$ and then passing to the limit. Similarly the other case. This is why the coefficient 3 appears in (4.1). Now, if in the definition of the half-tube (4.16) we replace (s_1, s_2) by $(s_1 - h, s_2 - h)$, $h \in \mathbf{R}$, we get that $\sigma_0 = h - (\sqrt{2} K_l + \alpha + 1 + \epsilon)$. This means that σ_0 becomes arbitrary. Therefore, for $(s_1, s_2) \in T_\alpha$ we have that

$$\int_{i\mathbf{R}^2} \frac{b(z_1, z_2) e^{z_1 s_1 + z_2 s_2}}{(\sigma_1 - z_1)(\sigma_2 - z_2)(G_{1,l} G_{2,l} Q_n)(z_1, z_2)} \, dz_1 \wedge dz_2 = 3 \int_\Gamma \frac{b(z_1, z_2) e^{z_1 s_1 + z_2 s_2}}{(\sigma_1 - z_1)(\sigma_2 - z_2)(G_{1,l} G_{2,l} Q_n)(z_1, z_2)} \, dz_1 \wedge dz_2.$$

Apply now the Lebesgue Dominated Convergence Theorem and Fatou's lemma with respect to l. The conclusion of the lemma follows. ∎

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5 A Residue Theorem for the Integral (4.1)

Some of the ideas of this section were already present in [16] where a weaker form of Residue Theorem was proven. Throughout this section the index m of the polynomials $Q_m(z_1,z_2) = \prod_{i=1}^m (1 - \frac{z_1^2}{(\lambda_i'')^2}) \prod_{i=1}^m (1 - \frac{z_1^2+z_2^2}{(\lambda_i'')^2+\mu_i''})$ remains fixed and $\alpha \geq \alpha_0$. For every $n \in \mathbf{N}$ consider the functions

$$L_{1,n}(z_1) = f_1(z_1) \prod_{i=1}^n \Big(1 - \frac{z_1^2}{(\lambda_i')^2}\Big)$$

$$L_{2,n}(z_1,z_2) = f_2(z_2) \prod_{i=1}^n \Big(1 - \frac{z_1^2+z_2^2}{(\lambda_i')^2 + (\mu_i')^2}\Big).$$

We observe that $L_{1,n}L_{2,n} \to L_1L_2$ uniformly over compact subsets. Hence the partial products $\frac{1}{L_{1,n}L_{2,n}}$ converge pointwise to $\frac{1}{L_1L_2}$ on $\partial\Gamma_1 \times \partial\Gamma_2$. Therefore we deduce the following

Lemma 5.1 *For every $(s_1,s_2) \in T_\alpha$ and fixed $m \in \mathbf{N}_1$ fixed the equality*

$$\mathcal{L}_m(s_1,s_2) = \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{e^{s_1z_1+s_2z_2} A_m(\sigma) b(z_1,z_2)}{(z_1-\sigma_1)(z_2-\sigma_2)(Q_mL_1L_2)(z_1,z_2)}\, dz_1 \wedge dz_2 \tag{5.1}$$

$$= \lim_n \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{e^{s_1z_1+s_2z_2} A_m(\sigma) b(z_1,z_2)\, dz_1 \wedge dz_2}{(z_1-\sigma_1)(z_2-\sigma_2)Q_mL_{1,n}L_{2,n})(z_1,z_2)}$$

$$= \lim_n \mathcal{L}_{n,m}(s_1,s_2), \tag{5.2}$$

holds and the convergence is uniform over the compact subsets of T_α.

Proof Since for every $(s_1,s_2) \in T_\alpha$ we have that

$$\frac{e^{s_1z_1+s_2z_2} b(z_1,z_2)}{(z_1-\sigma_1)(z_2-\sigma_2)(L_1L_2Q_m)(z_1,z_2)} \in L^1(i\mathbf{R}^2)$$

the proof follows directly from The Lebesgue Dominated Convergence Theorem for complex functions [11].

Next we show the uniform convergence.

If $\chi(z_1,z_2)$ denotes the integrand in (5.1) and $\chi_n(z_1,z_2)$ the integrand in (5.2) we have for every $r_0 > 0$ and $(s_1,s_2) \in T_\alpha$ the following

$$\int_{-i\mathbf{R}^2} \chi(z_1,z_2)\, dz_1\, dz_2 = \int_{-i\infty}^{-r_0}\int_{-i\infty}^{i\infty} \chi(z_1,z_2)\, dz_1\, dz_2 + \int_{-r_0}^{r_0}\int_{-i\infty}^{i\infty} \chi(z_1,z_2)\, dz_1\, dz_2$$
$$+ \int_{r_0}^{i\infty}\int_{-i\infty}^{i\infty} \chi(z_1,z_2)\, dz_1\, dz_2$$

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$$\int_{-i\mathbf{R}^2} \chi_n(z_1,z_2)\,dz_1\,dz_2 = \int_{-i\infty}^{-r_0}\int_{-i\infty}^{i\infty} \chi_n(z_1,z_2)\,dz_1\,dz_2 + \int_{-r_0}^{r_0}\int_{-i\infty}^{i\infty} \chi_n(z_1,z_2)\,dz_1 dz_2$$
$$+ \int_{r_0}^{i\infty}\int_{-i\infty}^{i\infty} \chi_n(z_1,z_2)\,dz_1\,dz_2. \tag{5.3}$$

The absolute convergence of every integral above is very easy to show. On the sets $i(-\infty,-r_0]\times i\mathbf{R}$, $i[r_0,\infty)\times i\mathbf{R}$ we have that integrals of $|\chi_n-\chi|$ are bounded by $4e^{-r_0}$. On the other hand on the set $i[-r_0,r_0]\times i\mathbf{R}$ the uniform convergence of $L_{1,n}$ implies the uniform convergence of the integrals. Summarizing, given $\epsilon>0$, there exists $r_0(\epsilon)>0$ so that for every $r\geq r_0(\epsilon)$ the estimate $\frac{4}{e^{r_0}}<\epsilon$ holds. Furthermore, on $i[-r_0,r_0]\times i\mathbf{R}$ there exists $n_0(\epsilon)$ so that

$$\Big|\frac{1}{L_{1,n}L_{2,n}}-\frac{1}{L_1L_2}\Big| = \frac{|L_1L_2-L_{1,n}L_{2,n}|}{|L_1L_2L_{1,n}L_{2,n}|} < \frac{\epsilon}{D},\quad n\geq n_0(\epsilon)$$

where the constant $D=\int_{i[-r_0,r_0]\times i\mathbf{R}}|\frac{e^{s_1z_1+s_2z_2}b(z_1,z_2)}{(\beta-z)(Q_m)(z_1,z_2)}|dz_1dz_2$ depends on $(s_1,s_2)\in T_\alpha$. ■

Direct adaptation of the proof of the Lemma 4.1 gives for $(s_1,s_2)\in T_\alpha$

$$\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{e^{s_1z_1+s_2z_2}A_m(\sigma)b(z_1,z_2)\,dz_1\wedge dz_2}{(\sigma_1-z_1)(\sigma_2-z_2)(Q_mL_{1,n}L_{2,n})(z_1,z_2)}$$
$$=3\int_{\partial\Gamma_1\times\partial\Gamma_2}\frac{e^{s_1z_1+s_2z_2}A_m(\sigma)b(z_1,z_2)\,dz_1\wedge dz_2}{(\sigma_1-z_1)(\sigma_2-z_2)(Q_mL_{1,n}L_{2,n})(z_1,z_2)}.$$

Now we are going to compute for every $n\in\mathbf{N}$ and for any $(s_1,s_2)\in T_\alpha$ the value of the last integral. We begin by proving a certain decomposition result.

Lemma 5.2 *Let $L_1Q_{1,m}\in\mathcal{H}(\mathbf{C})$, $L_2Q_{2,m}\in\mathcal{H}(\mathbf{C}^2)$, be the entire functions from Lemmas 3.1, 3.2. Then, there are bounded domains $G_n=G_{1,n}\times G_{2,n}\subset\mathbf{C}^2$, $n\in\mathbf{N}$ so that*

$$\Gamma_1\times\Gamma_2=\bigcup_n G_{1,n}\times G_{2,n},\ \text{and } G_n\subseteq G_{n+1},\quad \forall n\in\mathbf{N}.$$

Furthermore, if

$$L_{1,\nu,l}(z_1)Q_{1,m}(z_1)=\prod_{j=1}^{l}\Big(1+\frac{z_1}{a_j}\Big)e^{-\frac{z_1}{a_j}}\prod_{i=1}^{\nu}\Big(1-\frac{z_1^2}{(\lambda_i')^2}\Big)\prod_{i=1}^{m}\Big(1-\frac{z_1^2}{(\lambda_i'')^2}\Big),$$
$$\lambda_i',\lambda_i''\in G_{1,n},\forall i=1,2,\dots,\nu,\quad j=1,\dots,m$$

$$L_{2,\nu,l}Q_{2,m}(z_1,z_2)=\prod_{j=1}^{l}\Big(1+\frac{z_2}{\gamma_j}\Big)e^{-\frac{z_2}{\gamma_j}}\prod_{i=1}^{\nu}\Big(1-\frac{z_1^2+z_2^2}{(\lambda_i')^2+(\mu_i')^2}\Big)\prod_{j=1}^{m}\Big(1-\frac{z_1^2+z_2^2}{(\lambda_j'')^2+\mu_j''}\Big),$$
$$(\lambda_i',\mu_i'),(\lambda_j'',\mu_j'')\in G_{1,n}\times G_{2,n},i=1,\dots,\nu,\quad j=1,\dots,m.$$

are the partial products converging to $L_1 Q_{1,m}$ and $L_2 Q_{2,m}$ respectively, then on the compact cycles $\partial G_{1,n} \times \partial G_{2,n}$ we have

$$|(z_2 - \sigma_2) L_{1,\nu,l}(z_1) Q_{1,m}(z_1, z_2)| \geq \frac{d_m}{2^\nu} e^{-K_l|z_1|}, \tag{5.4}$$

$$|(z_1 - \sigma_1) L_{2,\nu,l} Q_m(z)| \geq \frac{d_m}{2^\nu} e^{-K_l|z_2|}, \quad z \in \partial G_{1,n} \times \partial G_{2,n}, \tag{5.5}$$

where the constants $d_m = \frac{\sigma_1 \sigma_2}{2^{m+1}}$ depend only on m and

$$K_l = \max\Big\{\sum_{i=1}^{l} \frac{1}{a_i}, \sum_{i=1}^{l} \frac{1}{\gamma_i}\Big\}.$$

Proof For any $n \in \mathbf{N}$ it is easy to choose $R_{1,n} > 0$ so that

$$|L_{1,\nu,l}(z)| \geq \frac{e^{-K_l|z_1|}}{2^\nu}, \quad |z| = r_1 \geq R_{1,n}, \quad |Q_{1,m}(z)| \geq \frac{1}{2^m}, \quad |z| = r_1 \geq R_{1,n}$$

Furthermore, for $z_1 = r_1 e^{\pm i\frac{\pi}{4}}$, $0 \leq r_1 < \infty$, we have

$$\Big|1 - \frac{z_1^2}{(\lambda_i')^2}\Big|^2 = 1 + \frac{r_1^4}{(\lambda_i')^4} \geq \frac{1}{2}, \quad \forall i \in \mathbf{N},$$

$$\Big|1 - \frac{z_1^2}{(\lambda_j'')^2}\Big|^2 = 1 + \frac{r_1^4}{(\lambda_i'')^4} \geq \frac{1}{2}, \quad \forall i = 1, \ldots, m.$$

Pick $R_{1,n}$ large enough to satisfy $4\big((\lambda_\nu')^2 + (\mu_\nu')^2\big) < R_{1,n}$. Then $R_{1,n} > \lambda_i'$, $i = 1, \ldots, n$. For the function $\prod_{i=1}^{n}(1 - \frac{z_1^2 + z_2^2}{(\lambda_i')^2 + (\mu_i')^2})$ we choose $R_{2,n} > 0$ to satisfy $\big((\lambda_\nu')^2 + (\mu_\nu')^2\big) < R_{2,n}$ and $4R_{2,n} = R_{1,n}$. We need one more restriction on the magnitude of the sequence $\{R_{2,n}\}$. If $M = \max\{(\lambda_j'')^2 + \mu_j''\}_{j=1}^m$ then $R_{2,n} > M$.

For every $j = 1, \ldots, m$ elementary calculations show that

$$\begin{aligned}\Big|1 - \frac{z_1^2 + z_2^2}{(\lambda_j'')^2 + \mu_j''}\Big|^2 = 1 &+ \frac{r_1^4 + r_2^4}{\big((\lambda_j'')^2 + \mu_j''\big)^2} - \frac{2r_2^2 \cos 2\theta_2}{(\lambda_j'')^2 + \mu_j''} - \frac{2r_1^2 \cos 2\theta_1}{(\lambda_j'')^2 + \mu_j''} \\ &+ \frac{2r_1^2 r_2^2 \cos 2\theta_1 \cos 2\theta_2}{\big((\lambda_j'')^2 + \mu_j''\big)^2} + \frac{2r_1^2 r_2^2 \sin 2\theta_1 \sin 2\theta_2}{\big((\lambda_j'')^2 + \mu_j''\big)^2}.\end{aligned} \tag{5.6}$$

From (5.6) for $z_1 = r_1 e^{\pm i\frac{\pi}{4}}$, $z_2 = r_2 e^{\pm i\frac{\pi}{4}}$, and $0 \leq r_1 \leq R_{1,n}$, $0 \leq r_2 \leq R_{2,n}$ we have for $j = 1, \ldots, m$ that

$$\Big|1 - \frac{r_1^2 e^{\pm 2i\frac{\pi}{4}} + r_2^2 e^{\pm 2i\frac{\pi}{4}}}{(\lambda_j'')^2 + \mu_j''}\Big|^2 \geq 1.$$

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Now we look at the case $z_2 = r_2 e^{\pm i\frac{\pi}{4}}$, $0 \le r_2 \le R_{2,n}$ and $z_1 = R_{1,n}e^{i\theta_1}$, $\theta_1 \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. For every $j = 1, \dots, m$ the relation (5.6) implies that

$$\begin{aligned}\left|1 - \frac{R_{1,n}e^{\pm i\frac{\pi}{4}} + r_2 e^{\pm i\frac{\pi}{4}}}{(\lambda_j'')^2 + \mu_j''}\right|^2 &= \frac{\big((\lambda_j'')^2 + \mu_j''\big)^2 + R_{1,m}^4 + r_2^4}{\big((\lambda_j'')^2 + \mu_j''\big)^2} - \frac{2R_{1,n}^2 r^2 \cos 2\theta_1}{(\lambda_j'')^2 + \mu_j''} \\ &\quad \pm \frac{2R_{1,n}^2 r^2 \sin 2\theta_1 - 2\big((\lambda_j'')^2 + \mu_j''\big)R_{1,n}^2 \cos 2\theta_1}{\big((\lambda_j'')^2 + \mu_j''\big)^2} \\ &\ge \frac{R_{1,n}^4 - 2R_{1,n}^2 R_{2,n}^2 - 2R_{2,n}R_{1,n}^2}{\big((\lambda_j'')^2 + \mu_j''\big)^2} \ge 1,\end{aligned} \tag{5.7}$$

after the substitution $R_{1,n} = 4R_{2,n}$ and since $R_{2,n} \ge M^2$.

Similarly the case $z_1 = r_1 e^{\pm i\frac{\pi}{4}}$, $0 \le r_1 \le R_{1,n}$, $z_2 = R_{2,n}e^{\theta_2}$, $\theta_2 \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Actually, for every $j = 1, \dots, m$ we have from (5.6) that

$$\begin{aligned}\left|1 - \frac{r_1^2 e^{\pm 2i\theta_1} + R_{2,n}^2 e^{\pm 2i\theta_2}}{(\lambda_j'')^2 + \mu_j''}\right|^2 &\ge \left(\Re\Big(1 - \frac{r_1^2 e^{\pm 2i\theta_1} + R_{2,n}^2 e^{\pm 2i\theta_2}}{(\lambda_j'')^2 + \mu_j''}\Big)\right)^2 \\ &= \frac{\big((\lambda_j'')^2 + \mu_j''\big)^2 + R_{2,n}^4 \cos^2 2\theta_2 - 2R_{2,n}^2\big((\lambda_j'')^2 + \mu_j''\big)\cos 2\theta_2}{\big((\lambda_j'')^2 + \mu_j''\big)^2}.\end{aligned}$$

If $\frac{2R_{2,n}^2 \cos 2\theta_2}{(\lambda_j'')^2 + \mu_j''} < \frac{1}{2}$ then $\cos 2\theta_2 < \frac{1}{4}\frac{(\lambda_j'')^2 + \mu_j''}{R_{2,n}^2} < \frac{1}{4}$ for every $j = 1, \dots, m$. Therefore, for suitably chosen ϑ_0, if $\vartheta_0 \le |\theta_2| \le \frac{\pi}{4}$ then $|1 - \frac{z_1^2 + z_2^2}{(\lambda_j'')^2 + \mu_j''}|^2 \ge \frac{1}{2}$. Note that ϑ_0 depends on m. For $0 \le |\theta_2| \le \vartheta_0$ we have

$$\left|1 - \frac{z_1^2 + z_2^2}{(\lambda_j'')^2 + \mu_j''}\right|^2 \ge 1 + \frac{R_{2,n}^2}{(\lambda_j'')^2 + \mu_j''}\left(\frac{R_{2,n}^2 \cos^2 2\vartheta_0}{(\lambda_j'')^2 + \mu_j''} - 2\right) \ge \frac{1}{2},$$

for $R_{2,n}$ sufficiently large.

Now, if $(z_1, z_2) = (R_{1,n}e^{i\theta_1}, R_{2,n}e^{i\theta_2})$, $(\theta_1, \theta_2) \in [-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$ then for every $j = 1, \dots, m$ the relation (5.6) implies

$$\begin{aligned}\left|1 - \frac{z_1^2 + z_2^2}{(\lambda_j'')^2 + \mu_j''}\right|^2 &\ge \frac{R_{1,n}^4 + R_{2,n}^4 - 2R_{1,n}^2 R_{2,n}^2}{\big((\lambda_j'')^2 + \mu_j''\big)^2} - \\ &\quad - \frac{2(\lambda_j'')^2 + \mu_j'')R_{1,n}^2 + 2\big((\lambda_j'')^2 + \mu_j''\big)R_{2,n}^2}{\big((\lambda_j'')^2 + \mu_j''\big)^2} \\ &\ge \frac{257R_{2,n}^4 - 32R_{2,n}^4 - 32\big((\lambda_j'')^2 + \mu_j''\big)R_{2,n}^2 - 2\big((\lambda_j'')^2 + \mu_j''\big)R_{2,n}^2}{\big((\lambda_j'')^2 + \mu_j''\big)^2} \\ &\ge 1,\end{aligned} \tag{5.8}$$

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after the substitution $4R_{2,n} = R_{1,n}$.

The modulus of $|L_{2,\nu,l}|$ is estimated on the compact cycles $\partial G_{1,n} \times \partial G_{2,n}$ in the same way. We conclude now that

$$|L_{i,\nu,l}(z_1, z_2)| \geq \frac{e^{-K_l|z_i|}}{2^\nu}, \ (z_1, z_2) \in \partial G_{1,n} \times \partial G_{2,n}, \quad i = 1, 2. \tag{5.9}$$

It is easy to see that $|Q_{2,m}(z_1, z_2)| > \frac{1}{2^m}$. Since $|\sigma_1 - z_1||\sigma_2 - z_2| \geq \frac{\sigma_1\sigma_2}{2}$, we put $d_m = \frac{\sigma_1\sigma_2}{2}\frac{1}{2^m}$.

Furthermore we can take the sequences $\{R_{1,n}\}_n$, $\{R_{2,n}\}_n$ strictly increasing. Define now the domains $G_n = G_{1,n} \times G_{2,n}$, $n \in \mathbf{N}$ where

$$G_{1,n} = \{z_1 \in \Gamma_1 \mid 0 < \Re z_1 < R_{1,n}\}$$
$$G_{2,n} = \{z_2 \in \Gamma_1 \mid 0 < \Re z_2 < R_{2,n}\},$$

Thus (5.4) holds on $\partial G_{1,n} \times \partial G_{2,n}$. Also it is easy to see that $G_n \subseteq G_{n+1}$ for every $n \in \mathbf{N}$ and that the equality $\Gamma_1 \times \Gamma_2 = \bigcup_n G_n$ holds. The proof of the lemma is now complete. ■

The next lemma is about decomposition of the compact cycles $\partial G_{1,n} \times \partial G_{2,n}$ into local ones.

Lemma 5.3 *Keeping the notation of the Lemma 5.2, for n fixed, let $\partial G_n = \partial G_{1,n} \times \partial G_{2,n}$ be the two-dimensional compact, real cycle. Then $\partial G_n = \partial G_{1,n} \times \partial G_{2,n}$ separate the divisors*

$$Z_{(\sigma_2 - z_2)L_{1,\nu,l}Q_{1,m}} = \{z \in \mathbf{C}^2 \mid (z_2 - \sigma_2)(L_{1,\nu,l}Q_{1,m})(z) = 0\},$$
$$Z_{(\sigma_1 - z_1)L_{2,\nu,l}Q_{2,m}} = \{z \in \mathbf{C}^2 \mid (z_1 - \sigma_1)(L_{2,\nu,l}Q_{2,m})(z) = 0\}$$

Furthermore, if

$$U_{\nu,m,l} = Z_{(\sigma_2 - z_2)L_{1,\nu,l}Q_{1,m}} \cap Z_{(\sigma_1 - z_1)L_{2,\nu,l}Q_{2,m}} \tag{5.10}$$

is a subset of the lattice $Z_{(\sigma_2 - z_2)L_1Q_{1,m}} \cap Z_{(\sigma_1 - z_1)L_2Q_{2,m}}$, then the following decomposition of $\partial G_n = \partial G_{1,n} \times \partial G_{2,n}$ into the local cycles holds

$$\partial G_{1,n} \times \partial G_{2,n} \sim \sum_{\nu_i \in U_{n,m,l}} \Gamma^\epsilon_{\nu_i}\big((\sigma - z)L_{\nu,l}Q_m\big), \tag{5.11}$$

where $(z - \sigma)L_{\nu,l}Q_m$ denotes the mapping

$$\big((z_2 - \sigma_2)L_{1,\nu,l}Q_{1,m}, (z_1 - \sigma_1)L_{2,\nu,l}Q_{2,m}\big) \colon \mathbf{C}^2 \mapsto \mathbf{C}^2.$$

Proof Consider the domain $X_\epsilon = \{(z_1, z_2) \in \mathbf{C}^2 \mid \Re z_i > -\epsilon, i = 1, 2\}$ for sufficiently small $\epsilon > 0$. It is easy to see that for every n, the cycle $\partial G_{1,n} \times \partial G_{2,n} \not\sim 0$ in $X_\epsilon \setminus Z_{(\sigma_2 - z_2)L_{1,\nu,l}Q_{1,m}} \cup Z_{(\sigma_1 - z_1)L_{2,\nu,l}Q_{2,m}}$, and $\partial G_{1,n} \times \partial G_{2,n} \sim 0$ in $X_\epsilon \setminus Z_{(\sigma_2 - z_2)L_{1,\nu,l}Q_{1,m}}$ and $\partial G_{1,n} \times \partial G_{2,n} \sim 0$ in $X_\epsilon \setminus Z_{(\sigma_1 - z_1)L_{2,\nu,l}Q_{2,m}}$. Actually, the cycle $\partial G_{1,n} \times \partial G_{2,n}$ is the boundary of the three dimensional chain $G_{1,n} \times \partial G_{2,n} \setminus Z_{(\sigma_2 - z_2)L_{1,\nu,l}Q_{1,m}}$ in $X_\epsilon \setminus Z_{(\sigma_2 - z_2)L_{1,\nu,l}Q_{1,m}}$ because of (5.4). That is, the 2-dimensional "torus" is the boundary of 3-dimensional

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"torus" $\setminus\{n+m+1+l$ complex "circles"$\}$ in the Stein manifold $X_\epsilon \setminus \{n+m+1+l$ complex "circles"$\}$. Similarly, the same cycle is the boundary of the three dimensional chain $\partial G_{1,n} \times G_{2,n} \setminus Z_{(\sigma_1-z_1)L_{2,\nu,l}Q_{2,m}}$ in $X_\epsilon \setminus Z_{(\sigma_1-z_1)L_{2,\nu,l}Q_{2,m}}$. Therefore the cycle $\partial G_{1,n} \times \partial G_{2,n}$ separate the hypersurfaces $Z_{(\sigma_2-z_2)L_{1,\nu,l}Q_{1,m}}$ and $Z_{(\sigma_1-z_1)L_{2,\nu,l}Q_{2,m}}$. By Section 2.2 we have that

$$\partial G_{1,n} \times \partial G_{2,n} \sim \sum_{\nu_i \in U_{\nu,m,l}} n_{\nu_i} \Gamma^\epsilon_{\nu_i}\big((\sigma - z)L_{\nu,l}Q_m\big), \quad n_{\nu_i} \in \mathbf{Z}$$

By the result from [14] the coefficient n_{ν_i} is equal to the intersection index of ∂G_n and of the line l_i joining the point ν_i with a point $\nu^0 \in X \setminus G_n$. By the geometry of the domain G_n it is easy to see that the intersection index $n_{\nu_i} = \operatorname{ind}(\partial G_n, l_i) = 1$. This completes the proof of the lemma. ■

Remark 5.1 In general, when the number of the hypersurfaces is bigger than 2 it is not possible to show that the coefficients n_{ν_i} are equal to the intersection index. However in our case, since G_n is always a direct product of bounded cones, this can be done following the proof of Proposition 1 in [14]. Thus the proof of Theorem 1.2 in $\mathbf{C}^n$, $n \geq 3$ is essentially the same.

Lemma 5.4 *Let n be fixed, then for every $\partial G_{1,n} \times \partial G_{2,n}$ there exists a sequence of the relatively compact domains $\{G^k_{1,n} \times G^k_{2,n}\}_k$ containing $G_{1,n} \times G_{2,n}$ such that*

$$G^k_{1,n} \times G^k_{2,n} \subset G^{k+1}_{1,n} \times G^{k+1}_{2,n}, \quad \bigcup_k G^k_{1,n} \times G^j_{2,n} = \Gamma_1 \times \Gamma_2.$$

Furthermore, the compact cycles $\{\mathcal{Z}_{n,k}\}_{k\in\mathbf{N}}$, $\mathcal{Z}_{n,k} = \partial G^k_{1,n} \times \partial G^k_{2,n}$ separate the divisors $Z_{(z_1-\sigma_1)L_{1,\nu,l}Q_{1,m}}, Z_{(z_2-\sigma_2)L_{2,\nu,l}Q_{2,m}}$ for every $k \in \mathbf{N}$ and satisfy

$$\partial G_{1,n} \times \partial G_{2,n} \sim \mathcal{Z}_{n,k} \quad \textit{and}\ \forall (z_1, z_2) \in \mathcal{Z}_{n,k}$$

$$|(z_2 - \sigma_2)L_{1,\nu,l}Q_{1,m}(z)| \geq \frac{d_m}{2^\nu} e^{-K_l|z_1|}, \tag{5.12}$$

$$|(z_1 - \sigma_1)L_{2,\nu,l}(z)Q_{2,m}(z)| \geq \frac{d_m}{2^\nu} e^{-K_l|z_2|}. \tag{5.13}$$

Proof Since n is fixed, taking a strictly increasing sequences of positive numbers $\{R^{(k)}_{i,n}\}_{k\in\mathbf{N}}$ such that $R^{(k)}_{i,n} \mapsto \infty$, $i = 1, 2$ and $R^{(1)}_{i,n} \geq R_{i,n}$ where $R_{i,n}$ are taken from the Lemma 4.2. Every $R^{(k)}_{i,n}$ is chosen as in Lemma 4.2. Observe that the lower bounds in the statement of the lemma depend only on n and not on k. For every $j \in \mathbf{N}$ define the domains

$$G^k_{1,n} \times G^k_{2,n} = \Gamma_1 \times \Gamma_1 \cap \big(B(0, R^{(k)}_{1,n}) \times B(0, R^{(k)}_{2,n})\big).$$

Then $\partial G_{1,n} \times \partial G_{2,n} \sim \partial G^k_{1,n} \times \partial G^k_{2,n}$ for every $k \in \mathbf{N}$. The estimates on the cycles $\mathcal{Z}_{n,k}$ follow from the construction of $\mathcal{Z}_{n,k} = \partial G^k_{1,n} \times \partial G^k_{2,n}$ and therefore imply the separation of the divisors $Z_{(z_2-\sigma_2)L_{1,\nu,l}Q_{1,m}}$ and $Z_{(z_1-\sigma_1)L_{2,\nu,l}Q_{2,m}}$ for every $k \in \mathbf{N}$. ■

Lemma 5.5 *Let $(s_1, s_2) \in T_\alpha$. Then for every $n \in \mathbf{N}$ we have*

$$
\begin{aligned}
\mathcal{L}_{\nu,m,l}(s_1, s_2) &= 3 \int_{\partial\Gamma_1 \times \partial\Gamma_2} \frac{e^{s_1 z_1 + s_2 z_2} b(z_1, z_2) A_m(\sigma)\, dz_1 \wedge dz_2}{(z_1 - \sigma_1)(z_2 - \sigma_2)(L_{1,\nu,l} L_{2,\nu,l} Q_m)(z_1, z_2)} \\
&= 3 \lim_{k \to \infty} \int_{\mathcal{Z}_{n,k}} \frac{e^{s_1 z_1 + s_2 z_2} A_m(\sigma) b(z_1, z_2)\, dz_1 \wedge dz_2}{(z_1 - \sigma_1)(z_2 - \sigma_2)(L_{1,\nu,l} L_{2,\nu,l} Q_m)(z_1, z_2)},
\end{aligned} \tag{5.14}
$$

where $A_m(z) = \frac{(L_1 L_2 Q_m)(\sigma)}{b(\sigma)}$.

Proof The limit in the second equality exists since by lemma 4.4 the integrals over the compact homologous cycles $\mathcal{Z}_{n,k}$, $k \in \mathbf{N}$ are equal. On the other hand we have the estimates for $(z_1, z_2) \in \mathcal{Z}_{n,k}$ $\forall k \in \mathbf{N}$

$$
\left| \frac{b(z_1, z_2) e^{s_1 z_1 + s_2 z_2}}{(z_1 - \sigma_1)(z_2 - \sigma_2)(L_{1,\nu,l} L_{2,\nu,l} Q_m)(z_1, z_2)} \right| \le \frac{4^\nu}{d_m^2} e^{-|z_1| - |z_2|}, \tag{5.15}
$$

provided that (s_1, s_2) belong to the half-tube

$$
\{(s_1, s_2) \in \mathbf{C}^2 : |\Im s_1| < \alpha, \Re s_i < \tau_0 = -K_l - \alpha - 5\epsilon - 1, i = 1, 2\}.
$$

If $l_{in}^k = \partial B(0, R_{i,n}^k) \cap \Gamma_i$, $i = 1, 2$, then the estimates (4.16) imply that the integrals over the parts of the cycles described by $l_{1,n}^k \times l_{2,n}^k$, $l_{1,n}^k \times (\partial G_{2,n}^j \setminus l_{2,n}^k)$, $(\partial G_{1,n} \setminus l_{1,n}^k) \times l_{2,n}^k$ tend to 0 while $k \mapsto \infty$. At these stage we replace (s_1, s_2) by $(s_1 - t, s_2 - t)$, $t > 0$. Then τ_0 in the definition of the half-tube becomes arbitrary. The conclusion of the lemma follows. ∎

Now we are ready to state and prove the Residue Theorem.

Theorem 5.1 *For $(s_1, s_2) \in T_\alpha$ we have*

$$
\begin{aligned}
\mathcal{L}_{\nu,m,l}(s_1, s_2) &= 3 \lim_{k \to \infty} \int_{\mathcal{Z}_{n,k}} \frac{A_m(\sigma) b(z_1, z_2) e^{s_1 z_1 + s_2 z_2}\, dz_1 \wedge dz_2}{(\sigma_1 - z_1)(\sigma_2 - z_2)(L_{1,\nu,l} L_{2,\nu,l} Q_m)(z_1, z_2)} \\
&= 3 \frac{A_m(\sigma)}{A_{\nu,m,l}(\sigma)} e^{s_1 \sigma_1 + s_2 \sigma_2} - 3 \sum_{(\lambda_i', \mu_i') \in U_{\nu,m,l}} \frac{A_m(\sigma) b(\lambda_i', \mu_i') e^{\lambda_i' s_1 + \mu_i' s_2}}{\mathcal{J}_{(\sigma - z) L_{\nu,l} Q_m}\big((\lambda_i', \mu_i')\big)} \\
&\quad - 3 \sum_{(\lambda_j'', \mu_j'') \in U_{\nu,m,l}} \frac{A_m(\sigma) b(\lambda_j'', \sqrt{\mu_j''}) e^{\lambda_j'' s_1 + \sqrt{\mu_j''} s_2}}{\mathcal{J}_{(\sigma - z) L_{\nu,l} Q_m}\big((\lambda_j'', \sqrt{\mu_j''})\big)},
\end{aligned} \tag{5.16}
$$

where $\mathcal{J}_{(z-\sigma) L_{\nu,l} Q_m}$ is the Jacobian of the mapping

$$
\big((\sigma_2 - z_2)(L_{1,\nu,l} Q_{1,m})(z_1, z_2), (\sigma_1 - z_1)(L_{2,\nu,l} Q_{2,m})\big) : \mathbf{C}^2 \mapsto \mathbf{C}^2,
$$

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$A_{\nu,m,l} = \frac{L_{1,\nu,l}(\sigma)L_{2,\nu,l}(\sigma)Q_{1,m}(\sigma)Q_{2,m}(\sigma)}{b(\sigma)}$ *and* $U_{\nu,l,m}$ *is defined in (5.10). Furthermore,*

$$\mathcal{L}_m(s_1,s_2) = \lim_{l\to\infty}\lim_{\nu\to\infty}\mathcal{L}_{\nu,m,l}(s_1,s_2)$$

$$(5.17)\qquad = 3e^{s_1\sigma_1+s_2\sigma_2} - 3\lim_{l\to\infty}\lim_{\nu\to\infty}\Big[\sum_{(\lambda_i',\mu_i')\in U_{\nu,m,l}} \frac{A_m(\sigma)e^{\lambda_i' s_1+\mu_i' s_2}b(\lambda_i',\mu_i')}{\mathcal{J}_{(z-\sigma)L_{\nu,l}Q_m}(\lambda_i',\mu_i')})$$

$$+ \sum_{(\lambda_j'',\mu_j'')\in U_{\nu,m,l}} \frac{A_m(\sigma)e^{\lambda_j'' s_1+\sqrt{\mu_j''}s_2}b(\lambda_j'',\sqrt{\mu_j''})}{\mathcal{J}_{(z-\sigma)L_{\nu,l}Q_m}(\lambda_j'',\sqrt{\mu_j''})}\Big],$$

where the limit with respect to ν *is uniform over compact subsets of* T_α.

Proof Recall that by construction the two dimensional real cycles $\partial G_{1,n}^k \times \partial G_{2,n}^k$ separate the hypersurfaces $Z_{(z_2-\sigma_2)L_{1,\nu,l}Q_{1,m}}$, $Z_{(z_1-\sigma_1)L_{2,\nu,l}Q_{2,m}}$ for any $k = 1,\dots,$. If $U_{\nu,m,l} = Z_{(z_2-\sigma_2)L_{1,\nu,l}Q_{1,m}} \cap Z_{(z_1-\sigma_1)L_{2,\nu,l}Q_{2,m}}$ then

$$\partial G_{1,n}^k \times \partial G_{2,n}^k \sim \sum_{u_i\in U_{\nu,m,l}} \Gamma_{u_i}^{\epsilon}\big((z-\sigma)L_{\nu,l}Q_m\big).$$

Denoting by Σ the last sum of local cycles we have

$$\int_{\partial G_{1,n}\times\partial G_{2,n}} \frac{b(z_1,z_2)e^{s_1z_1+s_2z_2}A_m(\sigma)}{(z-\sigma)(L_{1,\nu}L_{2,\nu,l}Q_m)(z)}\,dz_1\wedge dz_2$$

$$(5.18)\qquad = \int_{\mathcal{Z}_{n,k}} \frac{b(z_1,z_2)e^{s_1z_1+s_2z_2}A_m(\sigma)}{(z-\sigma)(L_{1,\nu,l}L_{2,\nu,l}Q_m)(z)}\,dz_1\wedge dz_2$$

$$= \int_{\Sigma} \frac{b(z_1,z_2)e^{s_1z_1+s_2z_2}A_m(\sigma)}{(z-\sigma)(L_{1,\nu,l}L_{2,\nu,l}Q_m)(z)}\,dz_1\wedge dz_2.$$

On the other hand by "Leray"'s rule (Section 14.2,[15]) we have on every local cycle τ from the sum Σ

$$(5.19)\qquad \int_{\tau} \frac{b(z_1,z_2)e^{s_1z_1+s_2z_2}A_m(\sigma)}{(z-\sigma)(L_{1,\nu,l}L_{2,\nu,l}Q_m)(z)}\,dz_1\wedge dz_2 = \frac{e^{u_i\cdot s}A_m(\sigma)b(u_i)}{\mathcal{J}_{(z-\sigma)(L_{1,\nu,l}L_{2,\nu,l}Q_{1,m}Q_{2,m})}(u_i)}.$$

The right hand side in (5.19) is possibly different than zero only when $u_i \in \{(\sigma_1,\sigma_2)\} \cup \{(\lambda_j',\mu_j')\} \cup \{(\lambda_j'',\sqrt{\mu_j''})\}$. This happens because at the other points of the intersection of the zero hypersurfaces the function $b(z_1,z_2)$ vanishes.

The conclusion of the theorem follows from the Lemma 5.1. ■

6 Proof of the Theorem 1.2

Keeping the notation of the previous sections, the Residue Theorem 5.1 leads to the following

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Theorem 6.1 *For every $(\sigma_1, \sigma_2) \in [\frac{1}{8}, \frac{1}{4}] \times [\frac{1}{3}, \frac{1}{2}]$, the function $e^{\sigma_1 s_1 + \sigma_1 s_2}$, $(s_1, s_2) \in T_\alpha$ is*

1) the uniform limit of the sequence $\{\psi_m\}$,

$$\psi_m(s_1, s_2) = \sum_{j=1}^{m} \frac{A_m(\sigma) e^{\lambda_j'' s_1 + \sqrt{\mu_j''} s_2} b(\lambda_j'', \sqrt{\mu_j''})}{\mathcal{J}_{(z-\sigma)LQ_m}(\lambda_j'', \sqrt{\mu_j''})} + \lim_{l\to\infty} \lim_{\nu\to\infty} \sum_{(\lambda_i', \mu_i') \in U_{\nu,m,l}} \frac{A_m(\sigma) e^{\lambda_i' s_1 + \mu_i' s_2} b(\lambda_i', \mu_i')}{\mathcal{J}_{(z-\sigma)L_{\nu,l}Q_m}(\lambda_i', \mu_i')}), \quad m \in \mathbf{N} \tag{6.1}$$

over compact subsets of T_α.

2) the pointwise limit of the finite linear combinations of

$$\{e^{\lambda_i' s_1 + \mu_i' s_2}\} \cup \{e^{\lambda_j'' s_1 + \sqrt{\mu_j''} s_2}\}.$$

Proof The proof follows from the Theorem 5.1 and the estimate (3.18). ■

Proof of the Theorem 1.2 Lemma 2.1, Lemma 3.6 and the Theorem 6.1 imply the desired result. ■

Remark 6.1 The Completeness Theorem 1.2 can be proven for the closed strips $\bar{T}_\alpha$, $\alpha \geq \alpha_0$.

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