MINIMALITY AND STABILITY OF MINIMAL HYPERSURFACES IN $\mathbb{R}^n$

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In this paper we show that the hypercone over $S^2 \times S^4$ is strictly area-minimizing in $\mathbb{R}^8$. We also show the existence of smooth embedded stable hypersurfaces in $\mathbb{R}^8$ which are not area-minimizing.

1. Introduction

Given a regular minimal hypercone $C$ in $\mathbb{R}^{n+2}$ (that is $C = \partial \times \Sigma$)

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for some smoothly embedded minimal hypersurface $E$ of $S^{n+1}$, we say that $C$ is strictly area-minimizing if there exists a constant $\theta > 0$ such that

\[(*) M(C^t) \leq M(T) - \theta \epsilon^{n+1}\]

for $T \in I_{n+1}^2(\mathbb{R}^{n+2})$, where $C^t = C \setminus B_1(0)$, whenever $\epsilon \in (0,1)$,

\[\exists T = \delta C^t \quad \text{and} \quad \text{spt}(T) \cap B_\epsilon(0) = \emptyset.\]

Let $E_+, E_-$ be the two connected components of $\mathbb{R}^{n+2} - C$. Then we say that $C$ is one-sided strictly area minimizing in $E_+$ (respectively, in $E_-$) if $(\ast)$ holds for all such $T$ above satisfying, in addition, the condition $\text{spt}(T) \subseteq E_+$ (spt $(T) \subseteq E_-$, respectively).

The aim of this note is to prove the following:

**THEOREM.** Let $E = S^n\left(\sqrt{m\over n}\right) \times S^{n-m}\left(\sqrt{n-m\over n}\right)$ where $n \geq 2m$ and either $n \geq 6$, $m \geq 2$ or $n \geq 7$, $m \geq 1$. Then $C = O \times E$ is strictly area minimizing in $\mathbb{R}^{n+2}$. If $E = S^1\left(\sqrt{1\over \delta}\right) \times S^5\left(\sqrt{5\over \delta}\right)$, then $C = O \times E$ is one-sided strictly area minimizing in $E = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 : |y| \leq \delta^{1\over 2} |x|\}$.

The strictly area minimality of $C(1,\delta) = O \times \Sigma$, $\Sigma = S^1\left(\sqrt{1\over \delta}\right) \times S^5\left(\sqrt{5\over \delta}\right)$, in $E$ implies that $C(1,\delta)$ is stable (see [5]). In fact, it is strictly stable by [7] and [6]. Moreover, we have the following:

**COROLLARY.** $E = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 : |y| < \delta^{1\over 2} |x|\}$ is foliated by smoothly embedded minimal hypersurfaces. Each of these hypersurfaces is one-sided area minimizing (hence stable) but not globally area minimizing.

The above corollary solves the open problem [1.6] of [1].

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2. Proofs

First we recall some results and notation from the recent work of Hardt and Simon [3]. They show that, if $C$ is area-minimizing, then there exist minimal hypersurfaces $S_\pm \subset E_\pm$ which coincide near infinity with

$$\{x \pm V_\pm(x) \nu_\sigma(x) : x \in C\}$$

where $V_\pm$ are functions on $C$ and $\nu_\sigma$ is an orienting unit normal vector field for $C$. Let $\gamma_\pm$ denote the characteristic exponents of the O.D.E. obtained by separating variables in the Jacobi field equation for $C$. By [3], we have the following alternative characterizations of strict minimality:

(i) $V_\pm$ both have the slower decay at infinity. That is

$$\liminf_{|x| \to \infty} |x|^{\gamma_-} V_+(x) > 0$$

in the case that $\gamma_+ > \gamma_-$

$$\liminf_{|x| \to \infty} (\log |x|^{-1}) |x|^{(n-1)/2} V_+(x) > 0$$

in the case that $\gamma_+ = \gamma_- = (n-1)/2$.

(ii) There are a closed, homothetically invariant $K \subset \mathbb{R}^{n+2}$ with $H^{n+1}$-measure zero and a $C^1$-vector field $X$ on $\mathbb{R}^{n+2} - K$ such that $X = \nu_C$ on $C \setminus K$ and $|X| \leq 1$, $\pm \div X \geq 0$ on $E_\pm$, and at least one of these inequalities is strict in at least one point $x_+ \in E_+ \setminus K$ and at least one point $x_- \in E_- \setminus K$.

By (ii) and the construction of Lawson [4], we see that all known examples of minimizing hypercones, except the case

$$\Sigma = S^2\left(\sqrt{\frac{2}{3}}\right) \times S^2\left(\sqrt{\frac{3}{5}}\right),$$

are strictly area minimizing.

Our theorem is, actually, a directly consequence of the characterization (i) and the O.D.E. results due to Simoes [7].
Proof of Theorem. For $\Sigma = S^m\left(\sqrt{n\over n-1}\right) \times S^{n-m}\left(\sqrt{n-m\over n}\right)$, the square of the length of the second fundamental form of $\Sigma = n$, see [6]. Since $\gamma_+ \geq \gamma_-$ are the roots of the characteristic equation:

$$\gamma^2 - (n-1)\gamma + n = 0,$$

we have that

$$\gamma_\pm = {1\over 2}(n-1 \pm \sqrt{(n-1)^2 - 4n}) = {1\over 2}(n-1 \pm \sqrt{n^2 - 6n + 1})$$

Now for $n \geq 6$, $n \geq 2m$ and $m \geq 1$, we have, by [7, Theorem 2.9.3],

on $\Sigma_+$ the following:

(a) $\lim [\arctan (dv/du) - \pi/4]/[\arctan (V/U) - \pi/4] = -\gamma_-$

where $v = |y|$, $u = s^{1/2} |x|$ and $U > V$; and $\Sigma_+$ denotes the leaf of the global foliation (see [3], [7]) in $U > V$, which passes through the point $U = 1$ and $V = 0$.

Then (a) is equivalent to

$$(a') \lim (dY/du)/(Y/u) = -\gamma_-$$

where $Y = u - v > 0$.

The latter implies that

$$u - v = u - \gamma_- + o(u - \gamma_-) \text{ as } u \to +\infty.$$
Minimal hypersurfaces

because \( V_\lambda(X) = |X|^{-\lambda} \) follows from the fact that \( |u - v| = |X|^{-\lambda} \), where \( X = (x, y) \). Thus we conclude, by (ii) that for \( n \geq 2m \) and either \( n \geq 6, m \geq 2 \) or \( n \geq 7, m \geq 1 \) the corresponding minimal hypercones \( C \) are strictly minimizing. We also obtain that, when \( n = 6, m = 1 \), \( C \) is one-sided strictly area minimizing in \( \overline{E} \).

Proof of Corollary. Using the same technique as [2], one concludes that \( \overline{E} = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^6 : \|y\| \leq \sqrt{5} |X|\} \) is foliated by \( S_\lambda = \mu_\lambda \times S^1, \ 0 < \lambda < \infty \). Each \( S_\lambda \) will be a smoothly embedded one-sided area minimizing hypersurface, for \( 0 < \lambda < \infty \), hence stable (see [3]). But \( S_\lambda \) cannot be area minimizing in \( \mathbb{R}^6 \), since \( C(1,5) \) is not area minimizing in \( \mathbb{R}^6 \), and \( C(1,5) \) is the tangent cone of \( S_\lambda \) at infinity.

3. An open problem

The following problem, which was raised by Simon, remains open.

\[ (P) \text{ Is there an example (other than } \mathbb{R}^2 \text{ in } \mathbb{R}^6 \text{) of a minimal hypercone } C \text{ in } \mathbb{R}^N \text{ which is minimizing but not strictly minimizing?} \]

The candidate \( S^2 \times S^4 \) is now ruled out by our result.

References


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