AXIOMATISATIONS OF THE AVERAGE AND A FURTHER GENERALISATION OF MONOTONIC SEQUENCES

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1. Introduction. A bounded monotonic sequence is convergent. This paper shows that a bounded sequence which is g-monotonic (to be defined) also converges. The proof generalises one attributed to Professor R. A. Rankin by Copson **[1]**. The theorem requires two definitions: the first axiomatises the notion of " average " and the second generalises the concept of monotonicity.

DEFINITION 1. A function f: $\mathbb{R}^r \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, is said to be an averaging function if it is continuous, strictly increasing in each argument and satisfies

$$x = f(x, x, \ldots, x) \qquad \text{for all } x \in \mathbb{R}. \tag{1}$$

DEFINITION 2. A sequence $\{a_n\}$ is said to be g-decreasing if there exists an averaging function f such that

$$a_n \leqq f(a_{n-1}, a_{n-2}, \ldots, a_{n-r}) \qquad \text{for all } n > r. \tag{2}$$

If the inequality in (2) is reversed, we say the sequence is g-increasing. A sequence is g-monotonic if it is either g-decreasing or g-increasing.

2. The theorem and its proof. We can now concisely state the

THEOREM. *If a real sequence is bounded and g-monotonic then it is convergent.*

Proof. We first prove the theorem for g-decreasing sequences. Let

$$A_n = \max(a_{n-1}, a_{n-2}, \ldots, a_{n-r}).$$

Clearly, for all n,

$$A_{n+1} \leqq \max(a_n, A_n) \tag{3}$$

and

$$A_n = a_{n-t(n)} \quad \text{for some } t(n) \text{ between 1 and } r. \tag{4}$$

By the properties of f,

$$f(a_{n-1}, a_{n-2}, \ldots, a_{n-r}) \leqq f(A_n, A_n, \ldots, A_n) = A_n. \tag{5}$$

Therefore, by (2),

$$a_n \leqq A_n$$

and so, by (3),

$$A_{n+1} \leqq A_n.$$

Therefore either A_n tends to a finite limit A or it diverges to $-\infty$. But, if the latter were true, a_n would also diverge, contrary to hypothesis. Thus $A_n \to A$. Therefore, by (5), $\overline{\lim\limits_{n\to\infty}}\, a_n \leqq A$.

We now prove that $\varliminf_{n\to\infty} a_n \geqq A$. Putting $n = m+r+1$ and $t(n) = r+1-s$ in (4) we obtain

$$a_{m+s} = A_{m+r+1} \geqq A. \tag{6}$$

Now, by (2) and the monotonicity of f,

$$\begin{aligned} a_{m+s} &\leqq f(a_{m+s-1}, a_{m+s-2}, \ldots, a_m, \ldots, a_{m+s-r}) \\ &\leqq f(A_{m+s}, A_{m+s}, \ldots, a_m, \ldots, A_{m+s}), \end{aligned} \tag{7}$$

where A_{m+s} is in every place except the sth, where there is a_m. Here s is a function of m and its values can be $1, 2, \ldots, r$.

Now if

$$\varliminf_{n\to\infty} a_m = A - 2\delta < A,$$

then there exists a strictly increasing subsequence $\{m_k\}$ of the positive integers such that

$$a_{m_k} < A - \delta \quad \text{for} \quad k = 1, 2, 3, \ldots.$$

Moreover, we may choose the subsequence so that each m_k corresponds to the same value of s in (7). Hence, from (6) and (7),

$$A \leqq f(A_{m_k+s}, \ldots, A-\delta, \ldots, A_{m_k+s}),$$

where, for all $k \geqq 1$, $A-\delta$ occurs in the same sth place. Letting $k \to \infty$, we deduce from the continuity of f that

$$A \leqq f(A, \ldots, A-\delta, \ldots, A),$$

which, with (1), contradicts the definition that f is strictly increasing in its sth argument. Hence $A \leqq \varliminf_{n\to\infty} a_n$ and this, together with the result $\varlimsup_{n\to\infty} a_n \leqq A$, shows that $\lim_{n\to\infty} a_n = A$ for g-decreasing sequences.

To complete the proof we observe that, if $\{a_n\}$ is g-increasing with respect to the averaging function f, then $\{-a_n\}$ is g-decreasing with respect to the averaging function $\bar{f}$, where

$$\bar{f}(b_1, b_2, \ldots, b_r) = -f(-b_1, -b_2, \ldots, -b_r).$$

Hence if $\{a_n\}$ is g-increasing, $\{-a_n\}$ converges and so therefore does $\{a_n\}$.

3. Some further remarks. Among functions satisfying the properties required of f are weighted and unweighted arithmetic, geometric, and harmonic means. Perversities such as the median, mode and mid-range are either discontinuous or else not strictly increasing, and so do not satisfy our definition of an averaging function.

However, the conditions required of f are not necessary, as has been shown for the linear case on page 163 of **[1]**. This raises the question of whether one can derive necessary and sufficient conditions for convergence using an approach such as this. The obvious conjecture that if a real sequence converges it is g-monotonic after some point with respect to *some* averaging function is shown to be false by the sequence $1, p_1, 1, q_1, 1, 1, p_2, 1, 1, q_2, 1, 1, 1, p_3,$

1, 1, 1, $q_3, \ldots$, where $\{p_n\}$ and $\{q_n\}$ are monotonically increasing and decreasing respectively, each with limit 1.

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REFERENCE

1. E. T. Copson, On a generalisation of monotonic sequences, *Proc. Edinburgh Math. Soc.* **17** (1970), 159–164.

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