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A note on a result of K.D. Magill, Jr

Eric Warren

The result of Magill concerning differentiable maps on the reals; namely, that all automorphisms of this semigroup are inner; is shown to be untrue in the case of analytic maps on the complexes. Also a characterization of the inner automorphisms of the analytic maps is given.

Let R denote the reals and let $D(R)$ denote the set of all (finitely) differentiable maps from R to R. In [2] Magill proves that every automorphism of $D(R)$ is inner ([2, Theorem 2.1]). He also shows that the set of inner automorphisms of $D(R)$ is isomorphic to the set of all strictly monotonic homeomorphisms of R onto itself that have a (finite) derivative everywhere ([2, Corollary 2.2]). After reading these results the following question is implicit: is the same result true if R is replaced by C, the complex numbers, and $D(R)$ is replaced by $D(C)$, the set of all entire (that is, analytic everywhere) maps from C to C? It turns out that it is not true. We note that an automorphism ψ of $D(C)$ is inner if there exists a bijection $h \in D(C)$, with $h^{-1} \in D(C)$, such that $\psi(f) = h \circ f \circ h^{-1}$, $\forall f \in D(C)$.

Before we begin we recall a result proven in [2]; that is, if $\mathcal{D}$ is a semigroup of maps from a space X to itself (composition is the semigroup operation) and if $\mathcal{D}$ contains all the constant maps from X to X, then for every automorphism ψ of $\mathcal{D}$ there exists a unique bijection h from X to X such that $\psi(f) = h \circ f \circ h^{-1}$, $\forall f \in \mathcal{D}$.

EXAMPLE. There exists an automorphism of $D(C)$ which is not inner.

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Proof. Let $h : C \to C$ be the complex conjugate function; that is if $z = x + iy$ then $h(z) = x - iy = \bar{z}$. Then it is easily checked that $\psi : D(C) \to D(C)$ defined by $\psi(f) = h \circ f \circ h^{-1}$, $\forall f \in D(C)$, is an automorphism. Also, since $D(C)$ certainly contains all constant maps, no other h represents ψ in this way. But $h \notin D(C)$, so ψ is not inner.

Analogous to Corollary 2.2 in [2] we have:

COROLLARY. *The set of all inner automorphisms of* $D(C)$ *is isomorphic to the set of all maps in* $D(C)$ *of the form* $f(z) = az + b$, *where* $a, b \in C$ *and* $a \neq 0$, $\forall z \in C$. *Call the set of all such maps* $A(C)$.

Proof. First we note that the set of all entire bijections is precisely $A(C)$ (see, for example, Theorems 7.5 and 7.6 of [1], pp. 178–179). Now consider the map α mapping $A(C)$ onto the set of all inner automorphisms defined by, $\forall h \in A(C)$, $\alpha(h) = \psi$, where $\psi(f) = h \circ f \circ h^{-1}$, $\forall f \in D(C)$. That α is a homomorphism onto is easily checked. That α is one-to-one follows from the fact that such an h for a given ψ is unique. So α is an isomorphism.

Another method of extension has been attempted; namely, to the semigroup of Fréchet-differentiable maps on a real Banach space. Whether or not all automorphisms are inner or not is not yet known, but for the best result so far in this regard one should see Yamamuro [3].

References

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Department of Mathematics,
University of British Columbia,
Vancouver, B.C., Canada.

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