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ORDER PRESERVING MAPS ON HERMITIAN MATRICES

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Abstract

We prove that a continuous map ϕ defined on the set of all $n \times n$ Hermitian matrices preserving order in both directions is up to a translation a congruence transformation or a congruence transformation composed with the transposition.

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1. Introduction and statement of the main result

Let H_n denote the space of all $n \times n$ Hermitian matrices. This set is a poset with the usual partial order defined by $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for every $x \in \mathbb{C}^n$. In other words, $A \le B$ if and only if $B - A$ is a positive semidefinite matrix.

One can find in the literature hundreds of papers dealing with linear maps preserving order, most of them treating the infinite-dimensional case. Recently, a first result on order preserving maps in the absence of the linearity assumption has been obtained. We say that a map $\phi : H_n \to H_n$ preserves order in both directions if for every pair $A, B \in H_n$ we have

$$A \le B \iff \phi(A) \le \phi(B). \tag{1.1}$$

When studying such maps there is no loss of generality in assuming that $\phi(0) = 0$. Indeed, if ϕ preservers order in both directions, then the same is true for the map $A \mapsto \phi(A) - \phi(0)$, $A \in H_n$. Quite surprisingly, every bijective map $\phi : H_n \to H_n$ preserving order in both directions and satisfying $\phi(0) = 0$ must be a congruence transformation, possibly composed with the transposition. More precisely, the following result was proved by Molnár [3].

Theorem 1.1. *Let $\phi : H_n \to H_n$, $n \ge 2$, be a bijective map satisfying (1.1) and $\phi(0) = 0$. Then there exists an invertible $n \times n$ complex matrix T such that either*

$$\phi(A) = TAT^*$$

for every $A \in H_n$, *or*

$$\phi(A) = TA^tT^*$$

for every $A \in H_n$.

It is a remarkable fact that after a harmless normalization $\phi(0) = 0$ the real-linear character of ϕ is not an assumption but a conclusion. This result was motivated by some problems in mathematical physics. It was proved in [3] in a more general infinite-dimensional setting. An interested reader can find more information on the background of this problem in [4]. The original proof by Molnár [3] depends heavily on some deep results from functional analysis. An elementary self-contained proof can be found in [5].

It is the aim of this note to prove that in the presence of the continuity assumption we can get the above result without the bijectivity assumption.

Theorem 1.2. *Let* $\phi : H_n \to H_n$, $n \geq 2$, *be a continuous map satisfying* (1.1) *and* $\phi(0) = 0$. *Then there exists an invertible* $n \times n$ *complex matrix* T *such that either*

$$\phi(A) = TAT^*$$

for every $A \in H_n$, *or*

$$\phi(A) = TA^tT^*$$

for every $A \in H_n$.

There is an essential difference between the above theorems. Namely, Theorem 1.2 cannot be extended to the infinite-dimensional case. Indeed, if H is an infinite-dimensional Hilbert space, then H is isometrically isomorphic to $H \oplus H$ and hence $S(H)$, the set of all self-adjoint bounded linear operators on H, may be identified with $S(H \oplus H)$. Elements of $S(H \oplus H)$ can be resprented as 2×2 operator matrices

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

where $A, C : H \to H$ are bounded self-adjoint linear operators and $B : H \to H$ is a bounded linear operator. Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous increasing function satisfying $f(0) = 0$, $x \in H$, $R \in S(H)$ a positive operator, and define $\phi : S(H) \to S(H) \equiv S(H \oplus H)$ by

$$\phi(A) = \begin{bmatrix} A & 0 \\ 0 & f(\langle Ax, x\rangle)R \end{bmatrix}, \quad A \in S(H).$$

It is easy to see that ϕ is continuous and preserves order in both directions. Moreover, $\phi(0) = 0$. However, ϕ is not linear in general.

In Theorem 1.2 the assumption that ϕ preserves order in both directions cannot be weakened to the assumption that ϕ preserves order in one direction only. Namely, even the structure of real-linear maps $\phi : H_n \to H_n$ satisfying $A \leq B \Rightarrow \phi(A) \leq \phi(B)$ is not well understood (of course, linear maps are automatically continuous and satisfy $\phi(0) = 0$). However, we do not know whether the same conclusion as in Theorem 1.2 holds without the continuity assumption.

2. Proof

Two matrices $A, B \in H_n$ are said to be adjacent if rank $(A - B) = 1$. The main tool in the proof of our main result is the following characterization of adjacency preserving maps on H_n [2].

THEOREM 2.1. *Let $\phi : H_n \to H_n$, $n \geq 2$, be a map such that $\phi(A)$ and $\phi(B)$ are adjacent whenever A and B are adjacent, $A, B \in H_n$. Suppose that $\phi(0) = 0$. Then one of the following holds.*

- *There exists a rank-one matrix $R \in H_n$ and a function $\rho : H_n \to \mathbb{R}$ such that*

$$\phi(A) = \rho(A)R, \quad A \in H_n.$$

- *There exist $c \in \{-1, 1\}$ and an invertible $n \times n$ complex matrix T such that either*

$$\phi(A) = cTAT^*$$

for every $A \in H_n$, or

$$\phi(A) = cTA^tT^*$$

for every $A \in H_n$.

PROOF OF THEOREM 1.2. In the proof we will identify $n \times n$ matrices with linear operators acting on $\mathbb{C}^n$.

It is easy to check that ϕ is injective. Indeed, if $\phi(A) = \phi(B)$ then $A \leq B$ and $B \leq A$, and therefore, $A = B$.

Our next goal is to prove that for each $A \geq 0$ we have rank $\phi(A) =$ rank A. We will first prove that rank $\phi(A) \geq$ rank A. Let rank $A = p$ and rank $\phi(A) = q$. Then there exist invertible $n \times n$ complex matrices T and S such that

$$TAT^* = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S\phi(A)S^* = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix},$$

where I_p and I_q are the $p \times p$ identity matrix and the $q \times q$ identity matrix, respectively.

Denote by U the set of all positive invertible Hermitian $p \times p$ matrices B such that $I_p - B$ is positive and invertible. This is an open subset of H_p. The map ψ defined by

$$\psi(B) = S\phi\left(T^{-1}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}(T^{-1})^*\right)S^*$$

is obviously an injective continuous order preserving map from U into H_n. Our next aim is to show that the image of ψ is included in the set of Hermitian matrices of the form $\left[\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $*$ stands for any $q \times q$ Hermitian matrix. Toward this end, we observe that $\psi(0) = 0$ and by the choice of S and T, we have $\psi(I_p) = J_q$, where $J_q = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right]$. If $B \in U$, then $0 \leq B \leq I_p$ and hence $0 \leq \psi(B) \leq J_q$. If we write $C := \psi(B)$ as $\left[\begin{smallmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{smallmatrix}\right]$, where C_{11} is a $q \times q$ matrix, then the inequalities $0 \leq C \leq J_q$ easily imply first that $C_{22} = 0$ and then that also $C_{12} = 0$ and $C_{21} = 0$. This proves that ψ maps the set U into $H_q \oplus 0$. Now, by the invariance of domain theorem [1, p. 344], we have $q \geq p$, as desired.

Thus, we know that rank $\phi(A) \geq \operatorname{rank} A$ for each $A \geq 0$. In particular, rank $\phi(A) =$ rank A whenever rank $A = n$. We continue inductively. Assume that rank $\phi(B) =$ rank B whenever $B \geq 0$ and rank $B = n, n-1, \ldots, k+1$. Let $A \geq 0$ and rank $A = k$, $1 \leq k < n$. We want to prove that rank $\phi(A) = k$. Assume that this is not true. Then rank $\phi(A) = q > k$. We can find $B \in H_n$ of rank q such that $A \leq B$. It follows that $0 \leq \phi(A) \leq \phi(B)$. Both $\phi(A)$ and $\phi(B)$ are of rank q and, consequently, Im $\phi(A) =$ Im $\phi(B)$. Moreover, as $0 \leq \phi(tB) \leq \phi(B)$ for every real t, $0 < t < 1$, and since rank $\phi(tB) = q$, $0 < t < 1$, we have also Im $\phi(A) =$ Im $\phi(tB)$, $0 < t < 1$. By continuity, $\phi(tB)$ tends to zero as $t \to 0$, and thus, we can find a positive real number t such that $\phi(tB) \leq \phi(A)$ implying that $0 \leq tB \leq A$, which is impossible as rank $tB >$ rank A.

Thus, we know now that rank $\phi(A) =$ rank A for every $A \geq 0$. In the same way we see that rank $\phi(A) =$ rank A for every $A \leq 0$.

Let A be any Hermitian matrix of rank one. Then $A = sP$ for some nonzero real number s and some projection of rank one. It follows that either $A \geq 0$ or $A \leq 0$. Consequently, rank $\phi(A) = 1$ for every $A \in H_n$ of rank one.

Let $A, B \in H_n$ be adjacent. The map $\psi : H_n \to H_n$ defined by

$$\psi(C) = \phi(C + A) - \phi(A), \quad C \in H_n,$$

is a continuous map preserving order in both directions and satisfying $\psi(0) = 0$. By the previous step it maps rank-one matrices into rank-one matrices. In particular, $\psi(B - A)$ is of rank one and this yields that $\phi(A)$ and $\phi(B)$ are adjacent.

As ϕ preserves adjacency we can apply Theorem 2.1. By the invariance domain theorem we see that the first possibility cannot occur. So, we have one of the remaining two possibilities. Since ϕ preserves order we necessarily have $c = 1$. This completes the proof. □

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