Glasgow Mathematical Journal (2023), **65**, pp. 288–309
doi:10.1017/S0017089522000301

Glasgow
Mathematical
Journal

RESEARCH ARTICLE

On canonical Fano intrinsic quadrics

Christoff Hische

Mathematisches Institut, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: hische@math.uni-tuebingen.de

Received: 2 June 2020; **Revised:** 16 June 2022; **Accepted:** 20 September 2022; **First published online:** 7 November 2022

Keywords: Fano varieties, 3-folds, singularities

2020 Mathematics Subject Classification: *Primary* - 14J45, *Secondary* - 14J30, 14L30

Abstract
We classify all $\mathbb{Q}$-factorial Fano intrinsic quadrics of dimension three and Picard number one having at most canonical singularities.

1. Introduction

This article contributes to the classification of Fano 3-folds, i.e. normal complex projective varieties of dimension three with an ample anticanonical divisor. For the smooth Fano 3-folds, the work of Iskovskih [11, 12] and Mori and Mukai [16] provides a detailed picture. The singular case, in contrast, is widely open in general. Toric Fano 3-folds with at most canonical singularities have been completely classified by Kasprzyk in [13, 14].

In the present article, we consider *intrinsic quadrics*. These are normal, projective varieties with a Cox ring defined by a single quadratic relation q, see [5]. Intrinsic quadrics have been used i.a. by Bourqui in [6] as a testing ground for Manin's conjecture. Moreover, Fahrner and Hausen [8] give concrete descriptions of all smooth intrinsic quadrics in the Picard numbers one and two. Every Fano intrinsic quadric X is completely determined by its Cox ring

$$\mathcal{R}(X) = \bigoplus_{w \in \mathrm{Cl}(X)} \mathcal{R}(X)_w = \mathbb{C}[T_1, \ldots, T_r]/\langle q \rangle.$$

In particular, if X is $\mathbb{Q}$-factorial and of Picard number one, then we regain X from its Cox ring as follows: The quasitorus H with character group $\mathbb{X}(H) \cong \mathrm{Cl}(X)$ acts diagonally on $\mathbb{C}^r$ via the characters corresponding to the degrees $w_1, \ldots, w_r \in \mathrm{Cl}(X)$ of the generators $T_1, \ldots, T_r$. Our variety X equals the good quotient $(V(q) \setminus \{0\}) /\!\!/ H$.

The description of X via its Cox ring allows us to explicitly compute certain invariants of X. In particular, we obtain its anticanonical self intersection number $-\mathcal{K}_X^3$ and its Fano index, i.e. the largest integer $q(X)$ such that $-\mathcal{K}_X = q(X) \cdot w$ holds for some $w \in \mathrm{Cl}(X)$.

Theorem 1.1. *Every $\mathbb{Q}$-factorial Fano intrinsic quadric of dimension three and Picard number one with at most canonical singularities is isomorphic to precisely one of the varieties X in the list below, specified by its Cox ring $\mathbb{C}[T_1, \ldots, T_r]/\langle q \rangle$ and the matrix Q having the $\mathrm{Cl}(X)$-degrees w_i of the generators T_i as its columns.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q=[w_1,\ldots,w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
1	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1&1&1&1&1\end{bmatrix}$	$[3]$	3	54
2	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}2&2&1&3&2\end{bmatrix}$	$[6]$	6	36
3	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}1&3&1&3&2\end{bmatrix}$	$[6]$	6	48
4	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}2&4&1&5&3\end{bmatrix}$	$[9]$	9	$\frac{729}{20}$
5	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}2&6&3&5&4\end{bmatrix}$	$[12]$	12	$\frac{96}{5}$
6	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3&5&1&7&4\end{bmatrix}$	$[12]$	12	$\frac{1152}{35}$
7	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix}3&7&2&8&5\end{bmatrix}$	$[15]$	15	$\frac{1125}{56}$
8	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}4&2&3&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}8\\ \bar{1}\end{bmatrix}$	1	$\frac{32}{3}$
9	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}6&4&5&5&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}12\\ \bar{1}\end{bmatrix}$	3	$\frac{36}{5}$
10	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}4&2&3&3&6\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}12\\ \bar{1}\end{bmatrix}$	3	12
11	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}1&3&1&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{1}\end{bmatrix}$	3	24
12	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}1&1&1&1&1\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{1}\end{bmatrix}$	3	27
13	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}1&1&1&1&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	54
14	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}1&1&1&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\end{bmatrix}$	4	32
15	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}3&1&2&2&1\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}5\\ \bar{1}\end{bmatrix}$	5	$\frac{125}{6}$
16	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}1&3&2&2&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\end{bmatrix}$	6	18
17	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix}2&2&1&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\end{bmatrix}$	6	18

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q=[w_1,\dots,w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
18	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	24
19	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 7 \\ \bar{1} \end{bmatrix}$	7	$\frac{343}{48}$
20	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{32}{3}$
21	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{32}{3}$
22	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5 & 1 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{256}{15}$
23	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{64}{3}$
24	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5 & 3 & 4 & 4 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 9 \\ \bar{1} \end{bmatrix}$	9	$\frac{243}{20}$
25	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 3 & 5 & 4 & 4 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 10 \\ \bar{0} \end{bmatrix}$	10	$\frac{25}{3}$
26	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5 & 1 & 3 & 3 & 4 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 10 \\ \bar{0} \end{bmatrix}$	10	$\frac{50}{3}$
27	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{36}{5}$
28	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{36}{5}$
29	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 7 & 3 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{288}{35}$
30	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 6 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	12
31	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 6 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	12
32	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 7 & 3 & 5 & 5 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 14 \\ \bar{0} \end{bmatrix}$	14	$\frac{98}{15}$
33	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	18

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \dots, w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
34	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, T_5]}{\langle T_1T_2 + T_3T_4 + T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	18
35	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, T_5]}{\langle T_1T_2 + T_3T_4 + T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_3$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{0} & \bar{2} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	12
36	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, T_5]}{\langle T_1T_2 + T_3T_4 + T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_3$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{2} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	12
37	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, T_5]}{\langle T_1T_2 + T_3T_4 + T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_3$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	16
38	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{1} \end{bmatrix}$	1	19
39	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 2 \\ \bar{3} & \bar{1} & \bar{0} & \bar{2} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{2} \end{bmatrix}$	2	$\dfrac{16}{3}$
40	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{2} \end{bmatrix}$	2	16
41	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, T_5]}{\langle T_1T_2 + T_3T_4 + T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\dfrac{27}{2}$
42	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{3} \end{bmatrix}$	3	$\dfrac{27}{2}$
43	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\dfrac{27}{2}$
44	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	16
45	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 5 \\ \bar{0} \end{bmatrix}$	5	$\dfrac{49}{6}$
46	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	9
47	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	9
48	$\dfrac{\mathbb{C}[T_1, T_2, T_3, T_4, S_1]}{\langle T_1T_2 + T_3^2 + T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_4$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 2 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\dfrac{16}{3}$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

No.	$\mathcal{R}(X)$	Cl(X)	$Q = [w_1, \dots, w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
49	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_5$	$\begin{bmatrix}1&1&1&1&1\\ \bar{2}&\bar{3}&\bar{1}&\bar{4}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	$\frac{54}{5}$
50	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&1\\ \bar{1}&\bar{5}&\bar{2}&\bar{4}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	9
51	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&1\\ \bar{2}&\bar{4}&\bar{3}&\bar{3}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	9
52	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&1\\ \bar{4}&\bar{0}&\bar{2}&\bar{5}&\bar{5}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	9
53	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&2\\ \bar{4}&\bar{0}&\bar{5}&\bar{2}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{2}\end{bmatrix}$	4	$\frac{32}{3}$
54	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix}1&1&1&1&1\\ \bar{2}&\bar{0}&\bar{5}&\bar{1}&\bar{3}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{1}\end{bmatrix}$	3	$\frac{27}{4}$
55	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix}1&1&1&1&2\\ \bar{2}&\bar{0}&\bar{5}&\bar{1}&\bar{2}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\end{bmatrix}$	4	8
56	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix}1&1&1&1&2\\ \bar{2}&\bar{0}&\bar{5}&\bar{1}&\bar{6}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{4}\end{bmatrix}$	4	8
57	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_9$	$\begin{bmatrix}1&1&1&1&1\\ \bar{4}&\bar{5}&\bar{3}&\bar{6}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	6
58	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\begin{bmatrix}1&1&1&1&1\\ \bar{2}&\bar{0}&\bar{7}&\bar{1}&\bar{4}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	3	$\frac{9}{2}$
59	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}2&2&2&1&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{0}&\bar{1}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{1}\end{bmatrix}$	1	$\frac{19}{2}$
60	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&1&1&1&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{1}\\ \bar{1}\end{bmatrix}$	1	16
61	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}2&2&2&1&3\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{1}\\ \bar{1}\end{bmatrix}$	3	$\frac{45}{4}$
62	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&1&1&1&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{0}\end{bmatrix}$	3	$\frac{27}{2}$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q=[w_1,\dots,w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
63	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{1}\end{bmatrix}$	3	$\frac{27}{2}$
64	$\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{1}\end{bmatrix}$	3	$\frac{27}{2}$
65	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&1&1&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{0}\end{bmatrix}$	4	16
66	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}3&3&3&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	6
67	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}2&2&2&1&3\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	9
68	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}3&1&2&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	9
69	$\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&3&2&2&2\\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0}\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	9
70	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}1&1&1&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{1}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	18
71	$\frac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix}2&4&3&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}8\\ \bar{0}\\ \bar{0}\end{bmatrix}$	8	$\frac{16}{3}$
72	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix}2&2&2&1&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\\ \bar{1}&\bar{3}&\bar{3}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{2}\end{bmatrix}$	2	4
73	$\frac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix}1&1&1&2&1\\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\\ \bar{1}&\bar{3}&\bar{3}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{2}\end{bmatrix}$	2	8

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

No.	$\mathcal{R}(X)$	Cl(X)	$Q = [w_1, \dots, w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
74	$\dfrac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix}1&1&1&1&2\\ \bar 0&\bar 0&\bar 0&\bar 1&\bar 1\\ \bar 2&\bar 0&\bar 1&\bar 3&\bar 0\end{bmatrix}$	$\begin{bmatrix}4\\ \bar 0\\ \bar 0\end{bmatrix}$	4	8
75	$\dfrac{\mathbb{C}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&1\\ \bar 1&\bar 0&\bar 1&\bar 0&\bar 0\\ \bar 4&\bar 1&\bar 1&\bar 5&\bar 0\end{bmatrix}$	$\begin{bmatrix}3\\ \bar 0\\ \bar 3\end{bmatrix}$	3	$\frac{9}{2}$
76	$\dfrac{\mathbb{C}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&1\\ \bar 0&\bar 0&\bar 1&\bar 0&\bar 1\\ \bar 2&\bar 0&\bar 4&\bar 1&\bar 1\end{bmatrix}$	$\begin{bmatrix}3\\ \bar 0\\ \bar 0\end{bmatrix}$	3	$\frac{9}{2}$
77	$\dfrac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix}1&1&1&1&1\\ \bar 1&\bar 1&\bar 1&\bar 0&\bar 0\\ \bar 2&\bar 4&\bar 3&\bar 3&\bar 0\end{bmatrix}$	$\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}$	3	$\frac{9}{2}$
78	$\dfrac{\mathbb{C}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_3)^2$	$\begin{bmatrix}1&1&1&1&1\\ \bar 1&\bar 2&\bar 1&\bar 2&\bar 0\\ \bar 2&\bar 1&\bar 1&\bar 2&\bar 0\end{bmatrix}$	$\begin{bmatrix}3\\ \bar 0\\ \bar 0\end{bmatrix}$	3	6
79	$\dfrac{\mathbb{C}[T_1,\dots,T_4,S_1]}{\langle T_1^2+T_2^2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^3$	$\begin{bmatrix}1&1&1&1&2\\ \bar 1&\bar 1&\bar 1&\bar 0&\bar 1\\ \bar 0&\bar 0&\bar 1&\bar 0&\bar 1\\ \bar 1&\bar 0&\bar 0&\bar 0&\bar 1\end{bmatrix}$	$\begin{bmatrix}4\\ \bar 0\\ \bar 0\\ \bar 0\end{bmatrix}$	4	8

To prove this result, we make use of the so called anticanonical complex as firstly introduced in [4] for Fano varieties with a torus action of complexity one, i.e. the general torus orbit is of codimension one. There, the authors have classified all $\mathbb{Q}$-factorial Fano 3-folds with Picard number one having at most terminal singularities and admitting a torus action of complexity one. Note that in our list all varieties defined by a trinomial quadric admit a torus action of complexity one. In particular, varieties Nos. 1, 4, 19 and 49 appear in the classification list of [4]: all of them are terminal and No. 1 is even smooth. In [10], the anticanonical complex has been made accessible for a broader class of varieties, comprising i.a. the intrinsic quadrics. There, all $\mathbb{Q}$-factorial Fano intrinsic quadrics of dimension three having at most canonical singularities and a torus action of complexity two have been classified. These show up as Nos. 64, 69, 77 and 79 in our classification list.

2. Background on intrinsic quadrics

In this section, we recall the basic facts about intrinsic quadrics from [8] and adapt the methods developed in [9, 10] to prove our main result in the subsequent section. Our main tool is the *Cox ring* $\mathcal{R}(X)$, which can be assigned to any normal projective variety X with finitely generated divisor class group Cl(X)

$$\mathcal{R}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

We refer to [1] for a precise definition and background on Cox rings.

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

An *intrinsic quadric* is a normal projective variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$ admitting homogeneous generators $f_1, \ldots, f_r$ such that the ideal of relations is generated by a single, purely quadratic homogeneous relation q. In particular, we have a graded isomorphism

$$\mathcal{R}(X) \cong \mathbb{C}[f_1, \ldots, f_r]/\langle q \rangle.$$

Proposition 2.1 ([8, Proposition 2.1]). *Let X be an intrinsic quadric. Then there exists a graded isomorphism*

$$\mathcal{R}(X) \cong \mathbb{C}[T_1, \ldots, T_n, S_1, \ldots, S_m]/\langle q_{s,t} \rangle, \tag{2.1}$$

for some s,t, where $q_{s,t} := T_1T_2 + \ldots + T_{s-1}T_s + T_{s+1}^2 + \ldots + T_{s+t}^2$ *and the* $\mathrm{Cl}(X)$*-grading on the ring on the right hand side fulfills the following: The variables* T_i, S_k *and the polynomial* $q_{s,t}$ *are homogeneous and we have* $\deg(T_{s+k}) \neq \deg(T_{s+l})$ *for* $1 \leq k < l \leq t$.

The polynomial $q_{s,t}$ is called a *standard* $\mathrm{Cl}(X)$*-homogeneous quadric* and the representation of $\mathcal{R}(X)$ in (2.1) is called the *homogeneous normal form*.

The homogeneous normal form enables us to work in the flexible language introduced in [9]. We adapt the basic constructions presented there to intrinsic quadrics and recall the major results.

Construction 2.2. Fix integers $r, m \geq 0$ and $n_0, \ldots, n_r > 0$, such that $2 \geq n_0 \geq \ldots \geq n_r \geq 1$ holds. Set $\mathfrak{n} := (n_0, \ldots, n_r)$, $n := n_0 + \ldots + n_r$ and define an integral $r \times (n+m)$ matrix P_0 built up from tuples $l_0, \ldots, l_r$ as follows:

$$P_0 := \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}, \qquad l_i := \begin{cases} (1,1) & \text{if } n_i = 2 \text{ holds}, \\ (2) & \text{else.} \end{cases}$$

We will write $\mathbb{C}[T_{ij}, S_k]$ for the polynomial ring in the variables T_{ij} and S_k, where $0 \leq i \leq r$, $1 \leq j \leq n_i$ and $1 \leq k \leq m$ holds. To the matrix P_0, we associate a quadratic relation

$$q = T_0^{l_0} + \ldots + T_r^{l_r}, \qquad \text{with } T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{C}[T_{ij}, S_k].$$

Note that, by the choice of the n_i and l_i, each term of q is in fact either of the form $T_{i1}T_{i2}$ or of the form T_{i1}^2 and there are m free variable $S_1, \ldots S_m$ not occurring in q. Now, let $e_{ij} \in \mathbb{Z}^n$ and $e_k \in \mathbb{Z}^m$ denote the canonical basis vectors regarded as vectors in $\mathbb{Z}^{n+m}$ and consider the projection

$$Q_0 \colon \mathbb{Z}^{n+m} \to K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*),$$

onto the factor group by the row lattice of P_0. We define the K_0*-graded* $\mathbb{C}$*-algebra*

$$R(\mathfrak{n}, P_0) := \mathbb{C}[T_{ij}, S_k]/\langle q \rangle,$$

$$\deg(T_{ij}) := Q_0(e_{ij}), \qquad \deg(S_k) := Q_0(e_k).$$

Example 2.3. *In the notation of Construction 2.2 let* $r = 2$, $m = 0$, $n_0 = n_1 = 2$, $n_2 = 1$ *and* $\mathfrak{n} = (2, 2, 1)$. *Then we obtain a* 2×5 *matrix*

$$P_0 = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix},$$

and the corresponding relation is

$$q = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 \in \mathbb{C}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}],$$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

Concerning the grading on $R(\mathfrak{n}, P_0)$*, we have* $K_0 \cong \mathbb{Z}^3$ *and setting* $w^0_{ij} := \deg(T_{ij}) \in \mathbb{Z}^3$ *we obtain*

$$\left[w^0_{01}, w^0_{02}, w^0_{11}, w^0_{12}, w^0_{21}\right] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Remark 2.4. *Let* $R := R(\mathfrak{n}, P_0)$ *be a* K_0*-graded* $\mathbb{C}$*-algebra as in Construction 2.2. Then* R *is integral and normal if* $r \geq 2$ *holds. Moreover, the* K_0*-grading is the finest possible grading leaving the variables* T_{ij} *and* S_k *and the relation* q *homogeneous.*

Construction 2.5. Let $R(\mathfrak{n}, P_0)$ be a K_0-graded $\mathbb{C}$-algebra as in Construction 2.2. Choose any integral $s \times (n+m)$ matrix D with $r + s \leq n + m$, such that the stack matrix

$$P := \begin{bmatrix} P_0 \\ D \end{bmatrix},$$

has pairwise different and primitive columns generating $\mathbb{Q}^{r+s}$ as a cone. Now, similar to Construction 2.2, consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. Then, we define the *$K$-graded $\mathbb{C}$-algebra*

$$R(\mathfrak{n}, P) := \mathbb{C}[T_{ij}, S_k]/\langle q \rangle,$$

$$\deg(T_{ij}) := Q(e_{ij}), \qquad \deg(S_k) := Q(e_k).$$

Example 2.6. (Example 2.3 continued). *Let* $R(\mathfrak{n}, P_0)$ *be as in Example 2.3. We build up a stack matrix*

$$P = \begin{bmatrix} P_0 \\ \hline D \end{bmatrix} = \left[\begin{array}{ccccc} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \hline -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{array}\right].$$

We directly verify that P *has indeed pairwise different, primitive columns generating* $\mathbb{Q}^4$ *as a cone. We obtain* $K \cong \mathbb{Z}$ *and writing* $w_{ij} := \deg(T_{ij}) \in \mathbb{Z}$ *we obtain*

$$[w_{01}, w_{02}, w_{11}, w_{12}, w_{21}] = [1, 1, 1, 1, 1].$$

In particular $R(\mathfrak{n}, P)$ *is the Cox ring of the smooth three-dimensional projective quadric that occurs as No.*1 *in the list of Theorem 1.1.*

Remark 2.7. *Let* $R(\mathfrak{n}, P)$ *be a* K*-graded* $\mathbb{C}$*-algebra as in Construction 2.5. Then the natural homomorphism* $K_0 \mapsto K$, $[v] \mapsto [v]$ *defines a downgrading from the* K_0*-graded* $\mathbb{C}$*-algebra* $R(\mathfrak{n}, P_0)$ *to the* K*-graded* $\mathbb{C}$*-algebra* $R(\mathfrak{n}, P)$*.*

Proposition 2.8 ([8, Proposition 2.1], compare also [9, Ex. 7.1]). *Let* X *be an intrinsic quadric. Then the* $\mathrm{Cl}(X)$*-graded Cox ring* $\mathcal{R}(X)$ *is isomorphic to a* K*-graded* $\mathbb{C}$*-algebra* $R(\mathfrak{n}, P)$ *as in Construction 2.5.*

We will now use the rings $R(\mathfrak{n}, P)$ to construct intrinsic quadrics, suitably embedded inside toric varieties; we refer to [7] for background on toric geometry. For this, let $R := R(\mathfrak{n}, P)$ be a K-graded $\mathbb{C}$-algebra from Construction 2.5 and denote by γ the positive orthant $\mathbb{Q}^{n+m}_{\geq 0}$. For any face $\gamma_0 \preceq \gamma$, we denote by γ_0^* its *complementary face*, i.e. $\gamma_0^* := \mathrm{cone}(e_i;\, e_i \notin \gamma_0) \preceq \gamma$. Moreover, for a homomorphism

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

of finitely generated abelian groups $A\colon K \to K'$ we denote its unique extension to the $\mathbb{Q}$ vector spaces $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$, resp. $K'_{\mathbb{Q}}$ as well with $A\colon K_{\mathbb{Q}} \to K'_{\mathbb{Q}}$. Finally, we define a polyhedral cone

$$\mathrm{Mov}(R) := \bigcap_{\gamma_0 \preceq \gamma \text{ facet}} Q(\gamma_0) \subseteq K_{\mathbb{Q}}.$$

Construction 2.9. Consider an integral K-graded $\mathbb{C}$-algebra $R := R(\mathfrak{n}, P)$ as in Construction 2.5. Then the K-grading on the polynomial ring $\mathbb{C}[T_{ij}, S_k]$ defines an action of the quasitorus $H := \mathrm{Spec}\, \mathbb{C}[K]$ on $\overline{Z} := \mathbb{C}^{n+m}$ that leaves $\overline{X} := V(q) \subseteq \overline{Z}$ invariant. Now, choose any element u inside the relative interior $\mathrm{Mov}(R)^\circ$ and define fans

$$\Sigma(u) := \{P(\gamma_0^*);\ \gamma_0 \preceq \gamma, u \in Q(\gamma_0)^\circ\}, \qquad \widehat{\Sigma}(u) := \{\gamma_0 \preceq \gamma;\ P(\gamma_0) \in \Sigma(u)\}.$$

This gives rise to the following commutative diagram

$$\begin{array}{ccccccc} V(q) & = & \bar{X} & \subseteq & \bar{Z} & = & \mathbb{C}^{n+m} \\ & & \cup\!\shortmid & & \cup\!\shortmid & & \\ & & \hat{X} & \subseteq & \hat{Z} & & \\ & & \big\downarrow {\scriptstyle /\!\!/ H} & & \big\downarrow {\scriptstyle /\!\!/ H} & & \\ & & X & \subseteq & Z & & \end{array}$$

where Z and $\widehat{Z}$ are the toric varieties defined by $\Sigma(u)$ and $\widehat{\Sigma}(u)$ respectively, $\hat{Z} \to Z$ is a toric characteristic space for the quasitorus action of H on $\widehat{Z}$ and $\widehat{X} := \bar{X} \cap \hat{Z}$. The resulting variety $X(\mathfrak{n}, P, u) := X := \widehat{X} /\!\!/ H$ is projective, irreducible and normal with dimension, divisor class group and Cox ring

$$\dim(X) = s + r - 1, \qquad \mathrm{Cl}(X) = K, \qquad \mathcal{R}(X) = R(\mathfrak{n}, P).$$

In particular, the variety $X(\mathfrak{n}, P, u)$ is an intrinsic quadric with Cox ring $R(\mathfrak{n}, P)$ in homogeneous normal form. Note that $\widehat{X} \subseteq \overline{X}$ and $\widehat{Z} \subseteq \overline{Z}$ are precisely the sets of H-semistable points with respect to the weight u.

Example 2.10 (Example 2.6 continued) *Let $R(\mathfrak{n}, P)$ be as in Example 2.6 and choose $u = 1 \in \mathbb{Z} = K$. Then the resulting variety $X(\mathfrak{n}, P, u)$ is the smooth three-dimensional projective quadric that occurs as No. 1 in the list of Theorem 1.1.*

Theorem 2.11 ([8, Proposition 2.3]) *Any intrinsic quadric is isomorphic to a variety $X(\mathfrak{n}, P, u)$ from Construction 2.9.*

By construction any intrinsic quadric $X = X(\mathfrak{n}, P, u)$ comes embedded inside a toric variety Z defined by a fan $\Sigma(u)$. We turn to the description of the cones $\sigma \in \Sigma(u)$ defining torus orbits $\mathbb{T}^{r+s} \cdot z_\sigma$ in Z, that intersect X non-trivially: Let us denote the columns of P by $v_{ij} := P(e_{ij})$ and $v_k := P(e_k)$ respectively. We call a cone $\sigma \in \Sigma(u)$ *big (elementary big)*, if its set of primitive ray generators contains for every $i = 0, \dots, r$ at least (precisely) one of the vectors v_{ij}. Moreover, we call σ a *leaf cone*, if there exists a set of indices $I_\sigma := \{i_1, \dots i_{r-1}\}$ such that, whenever v_{ij} is a primitive ray generator of σ, then $i \in I_\sigma$ holds. Finally, we call a face $\gamma_0 \preceq \gamma$ an *X-face*, if the torus orbit $\mathbb{T}^{n+m} \cdot z_{\gamma_0^*} \subseteq \mathbb{C}^{n+m}$ defined by the complementary face of γ_0 intersects $\overline{X} \subseteq \mathbb{C}^{n+m}$ non-trivially and $P(\gamma_0^*) \in \Sigma(u)$ holds.

Proposition 2.12 ([9, Proposition 7.8]) *Let $X = X(\mathfrak{n}, P, u)$ be an intrinsic quadric. Then for any cone $\sigma \in \Sigma(u)$, the following statements are equivalent:*

(i) *The torus orbit defined by σ intersects X non-trivially.*
(ii) *We have $\sigma = P(\gamma_0^*)$ for an X-face $\gamma_0 \preceq \gamma$.*
(iii) *The cone σ is a big cone or a leaf cone.*

Remark 2.13. *Let $X = X(n, P, u)$ be a $\mathbb{Q}$-factorial intrinsic quadric with Picard number $\varrho(X) = 1$. Then every face $0 \neq \gamma_0 \preceq \gamma$ that defines a torus orbit $\mathbb{T}^{n+m} \cdot z_{\gamma_0^*} \subseteq \mathbb{C}^{n+m}$ intersecting $\overline{X}$ non-trivially is an X-face. In particular, if all entries of $\mathfrak{n}$ equals one and $m > 0$ holds, then we have precisely one elementary big cone in $\Sigma(u)$. Moreover, if $\mathfrak{n}$ contains an index $n_i = 2$, we obtain at least two elementary big cones in $\Sigma(u)$.*

We turn to the description of the various cones of divisor classes inside the rational divisor class group of an intrinsic quadric.

Remark 2.14. *Let $X = X(\mathfrak{n}, P, u)$ be an intrinsic quadric. Then the cones of effective, movable, semiample and ample divisor classes inside $\mathrm{Cl}(X)_{\mathbb{Q}} = K_{\mathbb{Q}}$ are given as*

$$\mathrm{Eff}(X) = Q(\gamma), \qquad \mathrm{Mov}(X) = \bigcap_{\substack{\gamma_0 \preceq \gamma \\ \text{facet}}} Q(\gamma_0)$$

$$\mathrm{SAmple}(X) = \bigcap_{\substack{\gamma_0 \preceq \gamma \\ X\text{-face}}} Q(\gamma_0), \qquad \mathrm{Ample}(X) = \bigcap_{\substack{\gamma_0 \preceq \gamma \\ X\text{-face}}} Q(\gamma_0)^{\circ}.$$

Note, that due to the projectivity of X, the effective cone $\mathrm{Eff}(X)$ *is pointed.*

Remark 2.15. *Let $X = X(\mathfrak{n}, P, u)$ be an intrinsic quadric, then $u \in \mathrm{Ample}(X)$ holds. Moreover, let $u \neq u' \in \mathrm{Ample}(X)$. Then $X(\mathfrak{n}, P, u) = X(\mathfrak{n}, P, u')$ holds.*

We turn to the explicit description of the anticanonical divisor class of an intrinsic quadric and connected with it, its Fano property. For this, let $R := R(\mathfrak{n}, P)$ be a K-graded $\mathbb{C}$-algebra as in Construction 2.5. We set

$$-\kappa(R) := \sum \deg(T_{ij}) + \sum \deg(S_k) - \deg(q) \in K.$$

Proposition 2.16 ([1, Proposition 3.3.3.2]) *Let $X := X(\mathfrak{n}, P, u)$ be an intrinsic quadric with Cox ring $R := R(\mathfrak{n}, P)$. Then its anticanonical divisor class is given by $-\mathcal{K}_X = -\kappa(R)$. In particular, if $R := R(\mathfrak{n}, P)$ is a K-graded $\mathbb{C}$-algebra as in Construction 2.5 with $-\kappa(R) \in \mathrm{Mov}(R)^{\circ}$, then the intrinsic quadric $X(\mathfrak{n}, P, -\kappa(R))$ is Fano.*

We turn to singularity types of Fano varieties. For this, let X be an arbitrary Fano variety and $\pi : X' \to X$ a resolution of singularities, i.e. π is proper and birational and X' is smooth and let K_X denote any canonical divisor on X. Then, due to the ramification formula, we have

$$K_{X'} = \pi^* K_X + \sum a_i E_i,$$

where $K_{X'}$ is a canonical divisor on X', the E_i are prime divisors located in the exceptional locus $\mathrm{Exc}(\pi)$ and the a_i are rational numbers, the so called *discrepancies*. Note that the discrepancies of a Fano variety are independent of the chosen resolution of singularities. We call X *terminal (canonical, log-terminal)* if all discrepancies a_i are strictly positive (non-negative, strictly greater then -1).

Our central tool to characterize and control these singularity types is the anticanonical complex as introduced in [4] and developed further in [10]. We recall the necessary definitions and results from [10]. For this, let $X = X(\mathfrak{n}, P, u)$ be a Fano intrinsic quadric. Then, by construction, X is embedded inside a toric variety Z. Intersecting X with the open torus $\mathbb{T}^{r+s} \subseteq Z$, we obtain its *tropical variety* as the support of the quasifan

$$\mathrm{trop}(X \cap \mathbb{T}^{r+s}) = |\Sigma_{\mathbb{P}_r}^{\leq r-1} \times \mathbb{Q}^s|,$$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

where the first factor is the $(r-1)$-skeleton of the standard fan of the r-dimensional projective space with primitive ray generators $e_1, \dots, e_r \in \mathbb{C}^r$ and $e_0 := -\sum e_i$. We denote the tropical variety of X with trop(X) and call its maximal linear subspace the *lineality space* $\mathrm{trop}(X)^{\mathrm{lin}} := \{0\} \times \mathbb{Q}^s$.

Construction 2.17. Let $X = X(\mathfrak{n}, P, u)$ be a Fano intrinsic quadric. For every elementary big cone $\sigma = \mathrm{cone}(v_{0j_0}, \dots, v_{rj_r}) \in \Sigma(u)$ define numbers

$$\ell_{\sigma,i} := \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \text{ for } i = 0, \dots, r \quad \text{and} \quad \ell_\sigma := \sum_{i=0}^{r} \ell_{\sigma,i} - (l_{0j_0} \cdots l_{rj_r}).$$

Theorem 2.18 ([10, Cor. 6.5]) *Let $X = X(\mathfrak{n}, P, u)$ be a Fano intrinsic quadric. Then X is log-terminal if and only if $l_\sigma > 0$ holds for all elementary big cones $\sigma \in \Sigma(u)$.*

Construction 2.19. Let $X = X(\mathfrak{n}, P, u)$ be a log-terminal Fano intrinsic quadric. For every elementary big cone $\sigma = \mathrm{cone}(v_{0j_0}, \dots, v_{rj_r}) \in \Sigma(u)$ define points inside the lineality space $\mathrm{trop}(X)^{\mathrm{lin}}$:

$$v_\sigma := \ell_{\sigma,0} v_{0j_0} + \dots + \ell_{\sigma,r} v_{rj_r} \in \mathbb{Z}^{r+s} \quad \text{and} \quad v'_\sigma := \frac{v_\sigma}{\ell_\sigma} \in \mathbb{Q}^{r+s}.$$

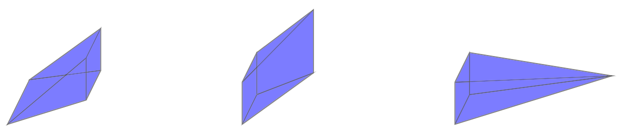
Then, $v'_\sigma \in \sigma$ holds. Now, the *anticanonical complex* $\mathcal{A}$ of X is defined as the polytopal complex obtained as the intersection of the convex hull over the primitive ray generators of $\Sigma(u)$ and the v'_σ, where $\sigma \in \Sigma(u)$ is elementary big, with the tropical variety trop(X).

Example 2.20. *Let $X(\mathfrak{n}, P, u)$ be as in Example 2.10. Then $\Sigma(u)$ contains the following four elementary big cones:*

$$\sigma_1 = \mathrm{cone}(v_{01}, v_{11}, v_{21}), \quad \sigma_2 = \mathrm{cone}(v_{01}, v_{12}, v_{21}),$$
$$\sigma_3 = \mathrm{cone}(v_{02}, v_{11}, v_{21}), \quad \sigma_4 = \mathrm{cone}(v_{02}, v_{12}, v_{21}).$$

A direct calculation gives $v'_{\sigma_1} = (0, 0, -2/3, -1/3)$, $v'_{\sigma_2} = (0, 0, 0, -1/3)$, $v'_{\sigma_3} = (0, 0, 0, 1/3)$ and $v'_{\sigma_4} = (0, 0, 2/3, 1/3)$. In particular, the anticanonical complex is the union of the following three three-dimensional polytopes.

$$\mathrm{conv}\left(v_{01}, v_{02}, v'_{\sigma_1}, \dots, v'_{\sigma_4}\right), \qquad \mathrm{conv}\left(v_{11}, v_{12}, v'_{\sigma_1}, \dots, v'_{\sigma_4}\right), \qquad \mathrm{conv}\left(v_{21}, v'_{\sigma_1}, \dots, v'_{\sigma_4}\right)$$



These polytopes are glued along their common face $\mathrm{conv}(v'_{\sigma_1}, \dots, v'_{\sigma_4})$.

Remark 2.21. *Let $X = X(\mathfrak{n}, P, u)$ be a log-terminal Fano intrinsic quadric. Then $0 \in \mathcal{A}^\circ$ holds.*

Theorem 2.22 ([10, Thm. 1.1]) *Let $X(\mathfrak{n}, P, u)$ be a log-terminal Fano intrinsic quadric. Then the following holds:*

(i) *X is terminal if and only if the only lattice points of the anticanonical complex are the primitive ray generators of $\Sigma(u)$ and the origin.*
(ii) *X is canonical if and only if the only interior lattice point of the anticanonical complex is the origin.*

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

3. Proof of Theorem 1.1

This section is dedicated to the proof of Theorem 1.1. In a first step, we show that in our situation any intrinsic quadric $X(\mathfrak{n}, P, u)$ is defined via a trinomial or a quadrinomial relation q in its Cox ring $R(\mathfrak{n}, P)$. Note that the quadrinomial case is part of [10], where torus actions of higher complexity on singular varieties are investigated, see Remark 3.2. Therefore, we turn to the trinomial case and go through any possible configuration for the defining data $\mathfrak{n}$ and P to create the classification list. Finally, we prove that all of the varieties stated in Theorem 1.1 are pairwise non-isomorphic.

Lemma 3.1. *Let $X = X(\mathfrak{n}, P, u)$ be a $\mathbb{Q}$-factorial Fano intrinsic quadric of dimension three and Picard number one. Then q is either a trinomial or a quadrinomial.*

Proof. We consider the Cox ring $R(\mathfrak{n}, P)$ of X. By assumption we have $n + m = 5$ for the number of variables in $R(\mathfrak{n}, P)$. Thus, by renaming the variables, we may assume that $R(\mathfrak{n}, P) = \mathbb{C}[T_1, \ldots, T_5]/\langle q\rangle$ holds, where q is a quadratic polynomial contained in the following list:

(i) T_1^2, T_1T_2 or $T_1^2 + T_2^2$,
(ii) $T_1T_2 + T_3^2$ or $T_1T_2 + T_3T_4$,
(iii) any square polynomial with three or four terms,
(iv) $T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2$.

If q is one of the polynomials in (i), then $R(\mathfrak{n}, P)$ is not integral; a contradiction. Now assume q is one of the polynomials in (ii). Then the K_0-grading on $R(\mathfrak{n}, P_0)$ turns the total coordinate space $\overline{X}$ into a toric variety and thus X is toric. This implies, that the Cox ring of X is isomorphic to a polynomial ring; a contradiction to the fact that $\overline{X}$ has a singularity at the origin. Finally, assume $q = T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2$ holds. Then we obtain

$$P_0 = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Therefore in order to produce a matrix P with primitive columns as in Construction 2.5, the matrix P has to be quadratic; a contradiction to $\varrho(X) = 1$. Now, the only case left is (*iii*) which proves the assertion. □

For the sake of completeness, we extract the quadrinomial case from [10].

Remark 3.2 (Compare [10, Thm. 1.5]) *Every $\mathbb{Q}$-factorial Fano intrinsic quadric of dimension three and Picard number one that has at most canonical singularities and a Cox ring $R(\mathfrak{n}, P)$, where the defining relation q is a quadrinomial, is isomorphic to precisely one of the varieties X, specified by its* Cl(X)*-graded Cox ring $\mathcal{R}(X)$, its matrix of generator degrees $Q = [w_1, \ldots, w_r]$ and its anticanonical divisor class $-\mathcal{K}_X \in$* Ample(X) *as follows:*

No.	$\mathcal{R}(X)$	Cl(X)	$Q = [w_1, \ldots, w_r]$	$-\mathcal{K}_X$
1	$\dfrac{\mathbb{C}[T_1, \ldots, T_4, S_1]}{\langle T_1^2 + T_2^2 + T_3^2 + T_4^2\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

No.	$\mathcal{R}(X)$	Cl(X)	$Q = [w_1, \ldots, w_r]$	$-\mathcal{K}_X$
2	$\dfrac{\mathbb{C}[T_1, \ldots, T_5]}{\langle T_1T_2 + T_3^2 + T_4^2 + T_5^2 \rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
3	$\dfrac{\mathbb{C}[T_1, \ldots, T_5]}{\langle T_1T_2 + T_3^2 + T_4^2 + T_5^2 \rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
4	$\dfrac{\mathbb{C}[T_1, \ldots, T_5]}{\langle T_1T_2 + T_3^2 + T_4^2 + T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{2} & \bar{4} & \bar{3} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

Note, that these varieties appear as Nos. 64, 69, 77 *and* 79 *in Theorem* 1.1.

Let us turn to the trinomial case. In a first step, we list the possible choices of the data $\mathfrak{n}$ and m. Then, in Propositions 3.5, 3.7 and 3.8, we show that in all cases there are only finitely many possible choices for the matrix P.

Remark 3.3. *Let $X := X(\mathfrak{n}, P, u)$ be a $\mathbb{Q}$-factorial intrinsic quadric of dimension three and Picard number one with trinomial Cox ring $R(\mathfrak{n}, P)$. Then we have $n + m = 5$ for the defining data n and m. Moreover, as the defining relation $q_{s,t}$ is a trinomial, we obtain $r = 2$ and as the Picard number of X is one, we conclude $s = 2$. We end up in one of the following situations.*

(i) $\mathfrak{n} = (1, 1, 1)$ *and* $m = 2$.
(ii) $\mathfrak{n} = (2, 1, 1)$ *and* $m = 1$.
(iii) $\mathfrak{n} = (2, 2, 1)$ *and* $m = 0$.

Remark 3.4. *Let $R(\mathfrak{n}, P)$ be a K-graded $\mathbb{C}$-algebra as in Construction* 2.5. *We call the following admissible operations on P:*

(i) *Add a multiple of one of the first r-rows to one of the last s-rows.*
(ii) *Any elementary row operation between the last s-rows.*
(iii) *Swap two columns v_{i1} and v_{i2}.*
(iv) *Swap two columns of the last m columns.*

The operations of type (i) and (ii) does not effect the ring $R(\mathfrak{n}, P)$. Types iii) and iv) leaves the graded isomorphy type of $R(\mathfrak{n}, P)$ invariant.

Proposition 3.5. *Let $X := X(\mathfrak{n}, P, u)$ be a $\mathbb{Q}$-factorial Fano intrinsic quadric of dimension three and Picard number one, having at most canonical singularities and Cox ring $R(\mathfrak{n}, P)$ with $\mathfrak{n} = (1, 1, 1)$ and $m = 2$. Then X is isomorphic to a variety $X(\mathfrak{n}, P', u)$, where P' is an integral (4×5)-matrix of the following form*

$$P' = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix},$$

fulfilling the following constraints:

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

The set $\{[x_4, y_4], [x_5, y_5], [z_1, z_2]\}$, *where* $[z_1, z_2] := [x_1 + x_2 + x_3, y_1 + y_2 + y_3]$ *equals one of the following sets*

$$\{[1, 0], [0, 1], [-1, -1]\}, \{[1, 1], [-1, 1], [0, -1]\}, \{[1, 1], [-1, 1], [-1, -2]\},$$
$$\{[1, 1], [-1, 1], [-1, -3]\}, \{[2, 1], [1, 1], [-1, -2]\},$$

and we have $x_2, x_3, y_2, y_3 \in \{0, 1\}$. *In particular, we are left with finitely many possibilities for* P'.

Proof. As X has Picard number one and by definition of $\Sigma(u)$, we obtain the following big cone and its associated vertex of the anticanonical complex of X:

$$\sigma = \mathrm{cone}(v_{01}, v_{11}, v_{21}) \in \Sigma, \qquad v'_\sigma = [0, 0, x_1 + x_2 + x_3, y_1 + y_2 + y_3].$$

In particular, forgetting about the first two coordinates, the anticanonical complex of X intersected with the lineality space $\mathrm{trop}(X)^{\mathrm{lin}}$ is the two-dimensional lattice polytope

$$\Delta := \mathrm{conv}([x_4, y_4], [x_5, y_5], [z_1, z_2]), \qquad [z_1, z_2] := [x_1 + x_2 + x_3, y_1 + y_2 + y_3].$$

As X has at most canonical singularities, the origin is the only interior lattice point of Δ. Thus, by applying admissible operations on the last two rows of P, we may assume that Δ is one of the 16 two-dimensional reflexive polytopes [3, 15, 17]. In particular, as Δ has three vertices, we may assume that it is one of the following:

$$\mathrm{conv}([1, 0], [0, 1], [-1, -1]),\ \mathrm{conv}([1, 1], [-1, 1], [0, -1]),\ \mathrm{conv}([1, 1], [-1, 1], [-1, -2]),$$

$$\mathrm{conv}([1, 1], [-1, 1], [-1, -3]),\ \mathrm{conv}([2, 1], [-1, 1], [-1, -2]).$$

By definition, the vertices of Δ are invariant under adding a multiple of the first two rows of P to one of the last two rows. Thus, applying suitable admissible operations, we can achieve in addition, that we have $x_2, x_3, y_2, y_3 \in \{0, 1\}$. □

Remark 3.6. *Let* $X = X(\mathfrak{n}, P, u)$ *be a* $\mathbb{Q}$*-factorial intrinsic quadric of Picard number one and consider the rational degree-vector*

$$d := (\deg_{\mathbb{Q}}(T_{01}), \ldots, \deg_{\mathbb{Q}}(T_{rn_r}), \deg_{\mathbb{Q}}(T_1), \ldots, \deg_{\mathbb{Q}}(T_m)) \in \mathbb{Z}^{n+m}.$$

By construction, we have $\mathrm{Lin}_{\mathbb{Q}}(d) = \ker_{\mathbb{Q}}(P)$ *and as the effective cone of X is pointed, we may assume* $d \in \mathbb{Q}^{n+m}_{>0}$. *Now, denote with* P_{ij} *resp.* P_k *the submatrices of P arising by deleting the ij-th resp. k-th column and set* $w_{ij} := (-1)^{c_{ij}}\det(P_{ij})$, $w_k := (-1)^{c_k}\det(P_k)$, *where* c_{ij} *resp.* c_k *denotes the number of the column* $P_{*,ij}$ *resp.* $P_{*,k}$. *Then we obtain a non-zero vector*

$$(w_{01}, \ldots, w_{rn_r}, w_1, \ldots w_m) \in \ker_{\mathbb{Q}}(P).$$

This vector is a scalar multiple of d. In particular, the w_{ij} *and* w_k *are either all positive or negative. We call them the rational weights.*

Proposition 3.7. *Let* $X := X(\mathfrak{n}, P, u)$ *be a* $\mathbb{Q}$*-factorial Fano intrinsic quadric of dimension three and Picard number one, having at most canonical singularities and Cox ring* $R(\mathfrak{n}, P)$ *with* $\mathfrak{n} = (2, 1, 1)$ *and* $m = 1$. *Then X is isomorphic to a variety* $X(\mathfrak{n}, P', u)$, *where* P' *is an integral* (4×5)*-matrix of the following form*

$$P' = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix},$$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

fulfilling the following constraints: We have $x_1 = y_1 = y_2 = 0$ *and*

$$0 < x_2 \leq 2 - (y_3 + y_4), \qquad 0 \leq x_4 \leq 1, \qquad 0 < x_5 \leq |y_5|,$$

$$-18 - y_4 \leq y_3 < -y_4, \qquad -1 \leq y_4 \leq 1$$

$$\frac{2x_5y_3 + 2x_5y_4 - 2x_4y_5 - 4x_2y_5}{2y_5} < x_3 < -x_4, \quad 0 < y_5 \leq \begin{cases} 9 & x_2 = 1 \\ \dfrac{x_2 - \frac{1}{2}(y_3 + y_4)}{x_2 - 1} & \text{else.} \end{cases}$$

In particular, we are left with finitely many possibilities for P'.

Proof. By applying suitable admissible operations on P we may assume $x_1 = y_1 = y_2 = 0$ and

$$x_2 > 0, \quad 0 < x_5 \leq |y_5|, \quad x_4, y_4 \in \{0, 1\}.$$

Moreover, by multiplying the last row with (-1), if necessary, we may assume that we have positive weights:

$$\begin{aligned} w_{01} &= 4x_2y_5 + 2x_3y_5 - 2x_5y_3 + 2x_4y_5 - 2x_5y_4, \\ w_{02} &= -2x_3y_5 + 2x_5y_3 - 2x_4y_5 + 2x_5y_4, \\ w_{11} &= 2x_2y_5, \\ w_{21} &= 2x_2y_5, \\ w_1 &= -2x_2y_3 - 2x_2y_4. \end{aligned}$$

Note that the last row operation possibly changes the sign of y_4. Thus, we may only assume that $y_4 \in \{-1, 0, 1\}$ holds. Now, positivity of the weights w_{01}, w_{11} and w_1 imply

$$\frac{2x_5y_3 + 2x_5y_4 - 2x_4y_5 - 4x_2y_5}{2y_5} < x_3, \qquad 0 < y_5 \qquad \text{and} \qquad y_3 < -y_4.$$

We investigate the anticanonical complex of X. As X has Picard number one, we obtain the following two big cones with their respective associated vertices of the anticanonical complex of X:

$$\sigma_1 = \mathrm{cone}(v_{01}, v_{11}, v_{21}), \quad v'_{\sigma_1} = \left[0, 0, \frac{1}{2}(x_3 + x_4), \frac{1}{2}(y_3 + y_4)\right],$$

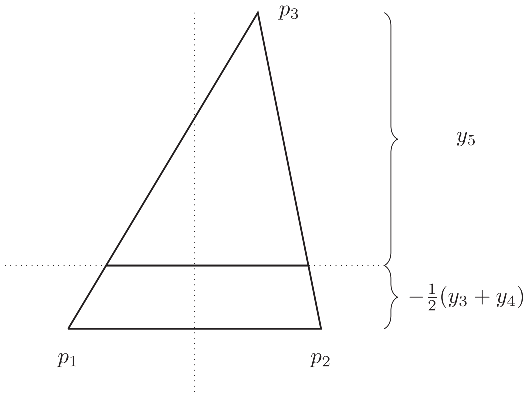
$$\sigma_2 = \mathrm{cone}(v_{02}, v_{11}, v_{21}), \quad v'_{\sigma_2} = \left[0, 0, \frac{1}{2}(x_3 + x_4) + x_2, \frac{1}{2}(y_3 + y_4)\right].$$

In particular, forgetting about the first coordinates, the anticanonical complex of X intersected with the lineality space $\mathrm{trop}(X)^{\mathrm{lin}}$ is a triangle $\Delta = \mathrm{conv}(p_1, p_2, p_3)$ with

$$p_1 = \left[\frac{1}{2}(x_3 + x_4), \frac{1}{2}(y_3 + y_4)\right], \ p_2 = \left[\frac{1}{2}(x_3 + x_4) + x_2, \frac{1}{2}(y_3 + y_4)\right], \ p_3 = [x_5, y_5].$$

We proceed by investigating the polytope Δ. As $x_5 > 0$ and x_2 and $w_{11} = 2x_2y_5$ are positive, we obtain $y_5 > 0$. In particular, the vertex p_3 is contained in the positive orthant. Moreover, as w_1 is positive, we conclude $y_3 + y_4 < 0$ and thus the points p_1 and p_2 are contained in the lower half plane. Note that the line segment $\overline{p_1p_2}$ is parallel to the x-axis. As X is Fano, we have $0 \in \Delta^\circ$ and conclude $x_3 + x_4 < 0$, as x_2 is positive. We sketch the situation:

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press



Note, that in this situation we cannot determine the position of p_2 with respect to the y-axis. We now investigate slices of the polytope Δ: Due to the singularity type of X, we have

$$|\Delta \cap \{y=0\}| = \frac{x_2 y_5}{y_5 - \frac{1}{2}(y_3+y_4)} \leq 2.$$

Thus, reordering suitably and using $y_5 > 0$ yields

$$x_2 \leq 2 - \frac{(y_3+y_4)}{y_5} \leq 2 - (y_3+y_4).$$

Similarly, we have

$$|\Delta \cap \{y=1\}| = \frac{x_2(y_5-1)}{y_5 - \frac{1}{2}(y_3+y_4)} \leq 1.$$

In particular, if $x_2 \neq 1$ holds, this implies

$$y_5 \leq \frac{x_2 - \frac{1}{2}(y_3+y_4)}{x_2 - 1}.$$

We proceed by investigating the tetrahedron Δ' defined by the following vertices:

$$[0,0,x_5,y_5], \qquad [-1,-1,x_2,0],$$

$$\left[0,0,\frac{1}{2}(x_3+x_4),\frac{1}{2}(y_3+y_4)\right], \qquad \left[0,0,\frac{1}{2}(x_3+x_4)+x_2,\frac{1}{2}(y_3+y_4)\right].$$

Note that by construction Δ' is contained in the anticanonical complex of X and thus has the origin as its unique interior lattice point. The polytope Δ' is living inside the linear space spanned by [1, 1, 0, 0], [0, 0, 1, 0] and [0, 0, 0, 1]. In particular, we may regard Δ' as a polytope in $\mathbb{Q}^3$ by forgetting about the first coordinate. Now, Δ' is contained in the lattice polytope Δ'' defined by the following vertices:

$$[-1,x_2,0], \quad [1,x_3+x_4-x_2,y_3+y_4], \quad [1,x_3+x_4+x_2,y_3+y_4], \quad [1,2x_5-x_2,2y_5].$$

Note that by construction Δ'' is a lattice polytope having the origin as its unique interior lattice point. Thus, due to [2, Thm 2.2], its standard $\mathbb{Q}^3$-volume is bounded by 12 which gives

$$\frac{4}{3}x_2y_5 - \frac{2}{3}x_2(y_3+y_4) \leq 12. \tag{3.1}$$

Now, reordering and using $x_2 > 0$ we obtain

$$-\frac{18}{x_2} + 2y_5 - y_4 \leq y_3,$$

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

and as $x_2 > 0$ and $y_5 > 0$ hold, we obtain $-18 - y_4 \leq y_3$. Moreover, reordering Equation 3.1 once more, we arrive at

$$\frac{4}{3}x_2y_5 \leq 12 + \frac{2}{3}x_2(y_3 + y_4).$$

Using negativity of $x_2(y_3 + y_4)$ and $x_2 > 0$, we conclude $y_5 \leq 9$. □

Proposition 3.8. *Let $X := X(\mathfrak{n}, P, u)$ be a $\mathbb{Q}$-factorial Fano intrinsic quadric of dimension three and Picard number one, having at most canonical singularities and Cox ring $R(\mathfrak{n}, P)$ with $\mathfrak{n} = (2, 2, 1)$ and $m = 0$. Then X is isomorphic to a variety $X(\mathfrak{n}, P', u)$, where P' is an integral (4×5)-matrix of the following form*

$$P' = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix},$$

fulfilling the following constraints: We have $x_1 = y_1 = y_2 = x_4 = y_4 = 0$ and

$$0 < x_2 \leq 3, \qquad \frac{2x_2y_3 + x_5y_3}{y_5} < x_3 < \frac{x_5y_3}{y_5}, \qquad 0 < x_5 \leq |y_5|,$$

$$0 < y_3 \leq \frac{12}{x_2}, \qquad -2y_3 < y_5 < 0.$$

In particular, we are left with finitely many possibilities for P'.

Proof. By applying suitable admissible operations on P, we may assume $x_1 = y_1 = y_2 = x_4 = y_4 = 0$, $x_2 > 0$, and $0 < x_5 \leq |y_5|$. Moreover, by multiplying the last row with (-1), if necessary, we may assume, that we have positive weights:

$$\begin{aligned} w_{01} &= 2x_2y_3 - x_3y_5 + x_5y_3, \\ w_{02} &= x_3y_5 - x_5y_3, \\ w_{11} &= -x_2y_5, \\ w_{12} &= 2x_2y_3 + x_2y_5, \\ w_{21} &= x_2y_3. \end{aligned}$$

Now, positivity of the weights w_{21}, w_{11} and w_{12} imply

$$y_3 > 0, \qquad y_5 < 0 \qquad \text{and} \qquad -2y_3 < y_5.$$

Thus, using positivity of w_{01} and w_{02}, we conclude

$$x_3 > \frac{2x_2y_3 + x_5y_3}{y_5} \qquad \text{and} \qquad x_3 < \frac{x_5y_3}{y_5}.$$

We investigate the anticanonical complex of X. As X has Picard number one, we obtain the following four big cones with their respective associated vertices of the anticanonical complex

$$\begin{aligned} \sigma_1 &= \operatorname{cone}(v_{01}, v_{11}, v_{21}), & v'_{\sigma_1} &= \left[0, 0, \frac{1}{3}x_5 + \frac{2}{3}x_3, \frac{1}{3}y_5 + \frac{2}{3}y_3\right], \\ \sigma_2 &= \operatorname{cone}(v_{01}, v_{12}, v_{21}), & v'_{\sigma_2} &= \left[0, 0, \frac{1}{3}x_5, \frac{1}{3}y_5\right], \\ \sigma_3 &= \operatorname{cone}(v_{02}, v_{11}, v_{21}), & v'_{\sigma_3} &= \left[0, 0, \frac{1}{3}x_5 + \frac{2}{3}x_3 + \frac{2}{3}x_2, \frac{1}{3}y_5 + \frac{2}{3}y_3\right], \\ \sigma_4 &= \operatorname{cone}(v_{02}, v_{12}, v_{21}), & v'_{\sigma_4} &= \left[0, 0, \frac{1}{3}x_5 + \frac{2}{3}x_2, \frac{1}{3}y_5\right]. \end{aligned}$$

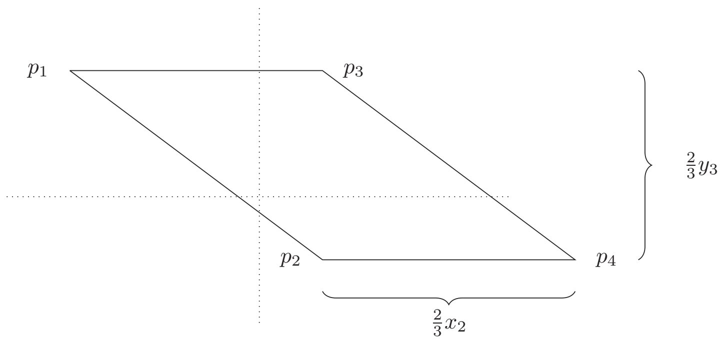
https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

In particular, forgetting about the first coordinates, the anticanonical complex of X intersected with the lineality space $\operatorname{trop}(X)^{\text{lin}}$ is a trapezoid $\Delta = \operatorname{conv}(p_1, p_2, p_3, p_4)$, with

$$p_1 = \left[\frac{1}{3}x_5 + \frac{2}{3}x_3, \frac{1}{3}y_5 + \frac{2}{3}y_3\right], \quad p_2 = \left[\frac{1}{3}x_5, \frac{1}{3}y_5\right],$$

$$p_3 = \left[\frac{1}{3}x_5 + \frac{2}{3}x_3 + \frac{2}{3}x_2, \frac{1}{3}y_5 + \frac{2}{3}y_3\right], \quad p_4 = \left[\frac{1}{3}x_5 + \frac{2}{3}x_2, \frac{1}{3}y_5\right].$$

We proceed by investigating the polytope Δ. In a first step, we determine the position of its vertices relative to the x- and y-axis. First note that by assumption x_2 and x_5 are positive and by positivity of the weight $w_{11} = -x_2y_5$ we obtain $y_5 < 0$. In particular, we have $(p_2)_1, (p_4)_1 > 0$ and $(p_2)_2, (p_4)_2 < 0$. Moreover, as X is Fano, we obtain $0 \in \Delta^\circ$ and thus $(p_1)_1 < 0$ and $(p_1)_2 > 0$ due to the positivity of x_2. We sketch the situation:



Note that we cannot determine the position of p_3 with respect to the y-axis. Now, due to the singularity type of X, the $y = 0$ slice of Δ implies

$$|\Delta \cap \{y = 0\}| = \frac{2}{3}x_2 \leq 2,$$

and thus $x_2 \leq 3$ holds. We proceed by investigating the pyramid

$$\Delta' := \operatorname{conv}\left([0, 2, x_5, y_5], v'_{\sigma_1}, \ldots, v'_{\sigma_4}\right),$$

By construction Δ' is contained in the anticanonical complex of X and by deleting the first coordinate, we may regard Δ' as a polytope inside $\mathbb{Q}^3$ having the origin as its unique interior lattice point, due to the singularity type of X. We proceed by modifying $\Delta' \subseteq \mathbb{Q}^3$. By extending the edges starting in $[2, x_5, y_5]$, we enlarge Δ' to the lattice polytope Δ'' having the following vertices:

$$[2, x_5, y_5], \qquad [-1, x_3, y_3], \qquad [-1, 0, 0], \qquad [-1, x_2 + x_3, y_3], \qquad [-1, x_2, 0].$$

Note that by construction Δ'' still has the origin as its unique interior lattice point. Thus, due to [2, Thm. 2.2], its standard $\mathbb{Q}^3$-volume is bounded by 12 and we conclude

$$y_3 \leq \frac{12}{x_2}.$$

□

Proof of Theorem 1.1 *(the classification list).* Due to Lemma 3.1 and Remark 3.2, we only need to consider the trinomial case. Then, due to Propositions 3.5, 3.7 and 3.8 we only have finitely many possible Fano varieties $X(\mathfrak{n}, P, u)$ to check. Computing the anticanonical complex for all possible configurations, the resulting canonical Fano varieties are listed in Theorem 1.1 with Nos. 1 - 63, 65 - 68, 70 - 76 and 78. The missing varieties are directly imported from [10, Thm. 1.5] as Nos. 64, 69, 77 and 79. □

Remark 3.9. *In order to calculate the anticanonical self-intersection numbers $-\mathcal{K}_X^3$ for the varieties listed in Theorem 1.1, we proceed as follows (compare [1, Proposition 5.4.2.1]). By construction, any $\mathbb{Q}$-factorial intrinsic quadric $X = X(\mathfrak{n}, P, u)$ is a hypersurface in its ambient toric variety Z, which we may assume to be projective and $\mathbb{Q}$-factorial as X is so. In particular, the defining fan of Z is simplicial. Thus, in our three-dimensional case, we can calculate the intersection numbers of divisors $C_X = (D_Z)|_X$, $C'_X = (D'_Z)|_X$ and $C''_X = (D''_Z)|_X$ obtained by restricting toric divisors as*

$$C_X \cdot C'_X \cdot C''_X = D_Z \cdot D'_Z \cdot D''_Z \cdot E_1,$$

where E_1 is any toric representative of the degree of the defining relation q of $R(\mathfrak{n}, P)$. This means in particular that the anticanonical self-intersection number $-\mathcal{K}_X^3$ can be computed using the following fact: Let Z be an n-dimensional complete toric variety with simplicial defining fan Σ_Z. For toric prime divisors $D_{\varrho_1}, \dots, D_{\varrho_n}$ set $\sigma :=$ cone$(\varrho_1, \dots, \varrho_n)$ and let v_i denote the primitive ray generator of ϱ_i. Then we have

$$D_{\varrho_1} \cdot \ldots \cdot D_{\varrho_n} = \begin{cases} [N \cap \mathrm{lin}_{\mathbb{Q}}(\sigma) : \mathrm{lin}_{\mathbb{Z}}(v_1, \dots, v_n)]^{-1}, & \text{if } \sigma \in \Sigma_Z \\ 0, & \text{else.} \end{cases}$$

We exemplarily calculate the anticanonical self-intersection number for No. 9 in our list:

Example 3.10. *We have*

$$\mathcal{R}(X) = \mathbb{C}[T_1, T_2, T_3, T_4, S_1]/\langle T_1T_2 + T_3^2 + T_4^2 \rangle, \quad \mathrm{Cl}(X) = \mathbb{Z} \times \mathbb{Z}_2, \quad Q = \begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix},$$

$-\mathcal{K}_X = (12, \bar{1})$ *and* $\deg(q) = (10, \bar{0})$. *Let $D_1, \dots, D_5$ denote the toric prime divisors on Z with $[D_i] = w_i \in \mathrm{Cl}(Z) = \mathrm{Cl}(X)$. We have $(3D_2)|_X \in -\mathcal{K}_X$. Thus, using $[3D_2] = [D_1 + 3D_5] \in \mathrm{Cl}(Z)$, we obtain*

$$\begin{aligned} -\mathcal{K}_X^3 &= 3D_2 \cdot (D_1 + 3D_5) \cdot 3D_2 \cdot 2D_3 = 18D_1 \cdot D_2 \cdot D_2 \cdot D_3 + 54D_2 \cdot D_2 \cdot D_3 \cdot D_5 \\ &= 36D_1 \cdot D_2 \cdot D_3 \cdot D_5 + 36D_1 \cdot D_2 \cdot D_3 \cdot D_5 = \frac{36}{5}, \end{aligned}$$

where, for the second equality, we use $[2D_2] = [4D_5]$ *and* $[3D_2] = [2D_1]$.

Now we turn to the irredundancy of the classification list.

Remark 3.11. *Let $X = X(\mathfrak{n}, P, u)$ be an n-dimensional intrinsic quadric. Then the following numbers are invariants of X:*

(i) *The anticanonical self-intersection number $-\mathcal{K}_X^n$, which can be directly computed as indicated in Remark 3.9.*
(ii) *The Fano index q(X), which is defined as the largest integer q(X), such that $-\mathcal{K}_X = q(X) \cdot w$ holds with some $w \in \mathrm{Cl}(X)$.*
(iii) *The Picard index p(X), which is defined as the index of the Picard group inside the divisor class group. Note that, in our situation, the Picard group is given as*

$$\mathrm{Pic}(X) = \bigcap_{\substack{\gamma_0 \preceq \gamma \\ X\text{-face}}} Q(\gamma_0 \cap \mathbb{Z}^{n+m}) \subseteq \mathrm{Cl}(X).$$

(iv) *The dimension of the automorphism group* dim(Aut(X)).

Moreover, if X is isomorphic to another intrinsic quadric $X' = X(\mathfrak{n}', P', u')$, then $\mathcal{R}(X)$ and $\mathcal{R}(X')$ are isomorphic as graded rings. In this case, the following holds:

(i) We have $\dim\left(\overline{X}^{\,\mathrm{sing}}\right) = \dim\left(\overline{X'}^{\,\mathrm{sing}}\right)$.
(ii) There is a bijection between the set of generator degrees Ω_X and $\Omega_{X'}$.
(iii) The sets $\Omega_X^{\dim} := \{\dim(\mathcal{R}(X)_w);\ w \in \Omega_X\}$ and $\Omega_{X'}^{\dim}$ coincide.

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

Proposition 3.12. *The varieties defined by the data in Theorem 1.1 are pairwise non-isomorphic.*

Proof. We denote by X_i the Fano variety defined by the i-th datum in Theorem 1.1, by $\mathcal{R}_i$ its Cox ring, by $\overline{X_i}$ its total coordinate space and by $\Omega_i = \{w_1, \ldots, w_r\}$ its set of generator degrees. As the divisor class group, the Fano index and the anticanonical self-intersection number presented in Theorem 1.1 are invariants, we only need to compare those varieties X_i and X_j, where all these data coincide. The next table presents invariants of these varieties, where the cases to compare are divided via horizontal lines:

i	$p(X_i)$	$\dim(\mathrm{Aut}(X_i))$	$\dim\left(\overline{X_i}^{\mathrm{sing}}\right)$
16	24	2	1
17	24	2	0
20	48	2	1
21	24	2	1
27	240	2	1
28	120	2	1
30	24	2	1
31	48	2	1
33	9	2	0
34	9	2	0
35	54	2	0
36	18	2	0
41	16	2	0
42	16	2	1
43	8	2	1
46	48	2	1
47	48	2	1
50	36	2	0
51	36	2	0
52	36	2	1
55	64	2	1
56	64	2	1
62	8	2	2
63	8	2	1
64	8	1	0
67	48	2	2
68	48	2	1
69	48	1	0
75	72	2	2
76	72	2	1
77	72	1	0

There are only four cases left, that cannot be distinguished via the table above. We treat them in the following paragraphs:

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press

X_{33} and X_{34}. In this case, the homogeneous component of $\mathcal{R}_{33}$ of degree $(1, \bar{0}) \in \Omega_{33}$ has dimension three. This is in contrast to R_{34}, where the maximal dimension of the homogeneous components with respect to the generator degrees in Ω_{34} is two.

X_{46} and X_{47}. In this situation, all homogeneous components of $\mathcal{R}_{46}$ with respect to the weights in Ω_{46} are one-dimensional which is in contrast to the two-dimensional homogeneous component of $\mathcal{R}_{47}$ of degree $(2, \bar{0}) \in \Omega_{47}$.

X_{50} and X_{51}. Note, that due to Remark 3.11, we have a bijection $\Omega_{50} \to \Omega_{51}$. Now $|\Omega_{50}| = 5$ which is in contrast to $|\Omega_{51}| = 4$.

X_{55} and X_{56}. Assume there is a graded isomorphism $\mathcal{R}_{55} \to \mathcal{R}_{56}$. Then we have an isomorphism $\mathrm{Cl}(X_{55}) \to \mathrm{Cl}(X_{56})$ mapping Ω_{55} onto Ω_{56}. We go through the possible images of $(1, \bar{1}) \in \Omega_{55}$: Assume that $(1, \bar{1})$ is mapped on either $(1, \bar{1})$ or $(1, \bar{5})$. Then $(2, \bar{2}) \in \Omega_{55}$ is mapped on $(2, \bar{2})$ which is not contained in Ω_{56}; a contradiction. Now assume $(1, \bar{1})$ is mapped on $(1, \bar{0})$ or $(1, \bar{2})$ then $(2, \bar{2})$ is mapped on either $(2, \bar{0})$ or $(2, \bar{4})$ which are not contained in Ω_{56}; a contradiction; Finally assume that $(1, \bar{1})$ is mapped on $(2, \bar{6})$. Then $(2, \bar{2})$ is mapped on $(4, \bar{4})$ which is again not contained in Ω_{56}; a contradiction. This implies that there is no graded isomorphism $\mathcal{R}_{55} \to \mathcal{R}_{56}$ and thus X_{55} and X_{56} cannot be isomorphic. □

Acknowledgements. We would like to thank the anonymous referee for his/her careful reading and the helpful suggestions.

References

[1] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface. *Cox rings*, vol. 144. Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, 2015).

[2] G. Averkov, J. Krümpelmann and B. Nill. Largest integral simplices with one interior integral point: Solution of Hensley's conjecture and related results, *Adv. Math.* **274** (2015), 118–166.

[3] V. V. Batyrev. Higher dimensional toric varieties with ample anticanonical class, *Doctoral Dissertation, Moscow State University*, 1985.

[4] B. Bechtold, J. Hausen, E. Huggenberger, and M. Nicolussi. On terminal Fano 3-folds with 2-torus action, *Int. Math. Res. Not. IMRN* **5** (2016), 1563–1602.

[5] F. Berchtold and J. Hausen. Homogeneous coordinates for algebraic varieties. *J. Algebra* **266**(2) (2003), 636–670.

[6] D. Bourqui. La conjecture de Manin géométrique pour une famille de quadriques intrinsèques. *Manuscripta Math.* **135**(1–2) (2011), 1–41.

[7] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric Varieties*, vol. 124. Graduate Studies in Mathematics. (American Mathematical Society, Providence, RI, 2011).

[8] A. Fahrner and J. Hausen. On intrinsic quadrics. *Canadian J. Math.* **72**(1) (2020), 145–181.

[9] J. Hausen, C. Hische, and M. Wrobel. On torus actions of higher complexity. *Forum Math. Sigma* **7** (2019), 81.

[10] C. Hische and M. Wrobel. On the anticanonical complex. *arXiv e-prints*, page arXiv:1808.01997, August 2018.

[11] V. A. Iskovskih. Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.* **41**(3) (1977), 516–562, 717.

[12] V. A. Iskovskih. Fano threefolds. II. *Izv. Akad. Nauk SSSR Ser. Mat.* **42**(3) (1978), 506–549.

[13] A. M. Kasprzyk. Toric Fano three-folds with terminal singularities. *Tohoku Math. J. (2)* **58**(1) (2006), 101–121.

[14] A. M. Kasprzyk. Canonical toric Fano threefolds. *Canad. J. Math.* **62**(6) (2010), 1293–1309.

[15] R. J. Koelman. The number of moduli of families of curves on toric surfaces. *Doctoral Dissertation, Katholieke Universiteit Nijmegen*, 1991.

[16] S. Mori and S. Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.* **36**(2) (1981/82), 147–162.

[17] S. Rabinowitz. A census of convex lattice polygons with at most one interior lattice point. *Ars Combin.* **28** (1989), 83–96.

https://doi.org/10.1017/S0017089522000301 Published online by Cambridge University Press