This is a ``preproof'' accepted article for *The Bulletin of Symbolic Logic*.
This version may be subject to change during the production process.
DOI: 10.1017/bsl.2025.20

On the Proof-Theoretic Structure of Counterfactual Inference

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April 4, 2025

Abstract

In this paper, a proof-theoretic perspective on counterfactual inference is proposed. On this perspective, proof-theoretic structure is fundamental. We start from a certain primacy of inferential practice and structural proof theory. Models are required neither for the explanation of the meaning of counterfactuals, nor for that of counterfactual inference. Taking a proof-theoretic perspective and an intuitionistic stance on meaning (cf. BHK), we define modal intuitionistic natural deduction systems for drawing conclusions from counterfactual assumptions. These proof systems are modal insofar as derivations in them make use of assumption modes which are sensitive to the factuality status (e.g., factual, counterfactual) of the formula that is to be assumed. This status is determined by a reference proof system on top of which a modal proof system is defined. The rules of a modal system draw on this status.

The main results obtained are preservation, normalization, subexpression (incl. subformula) property, and internal completeness. The systems are applied to the analysis of reasoning with natural language constructions such as 'If A were the case, B would [might] be the case', 'Since A is the case, B is [might be] the case'. A proof-theoretic semantics is provided for them.

MSC 2020: 03A05, 03B65, 03F03.

Keywords: counterfactuals, intuitionistic logic, proof theory, proof-theoretic semantics.

1 Introduction

1.1 Counterfactual inference

We tend to make counterfactual assumptions and to draw conclusions from them. Among the most fundamental constructions that we use to express our steps in counterfactual reasoning are *counterfactual conditionals* of the form:

(1.1) If it were the case that A, then it would be the case that B.
(If A were the case, B would be the case.)

and their past subjunctive versions of the form 'If it had been the case that A, then it would have been the case that B' ('If A had been the case, B would have been the case'). A classic example (due to [28]):

(1.2) If Hoover had been born a Russian, then he would have been a Communist.

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What is characteristic of counterfactual inference is the fact that reasoning with counterfactuals may give rise to so-called *counterfactual fallacies*. These suggest that their logical behaviour differs considerably from that of the material conditional. Firstly, transitivity does not seem to hold in general. For example, it defies our intuitions to conclude (1.4) from (1.2) and (1.3).

(1.3) If he had been a Communist, then he would have been a traitor.

(1.4) If he had been born a Russian, then he would have been a traitor.

Secondly, the logic of counterfactuals seems to be non-monotonic, as the so-called fallacy of strengthening of the antecedent suggests. For example, we are not willing to infer (1.5) from (1.2).

(1.5) If Hoover had been born a Russian and had been a political activist, then he would have been a Communist.

Thirdly, also contraposition seems to fail, as we are reluctant to pass from (1.2) to (1.6).

(1.6) If Hoover had not been a Communist, then he would not have been born a Russian.

The terminology is not uniform. On a *narrow* conception, counterfactuals are contrary-to-fact conditionals (cf. [4]) and are classified as subjunctive conditionals whose antecedents are assumed to be false (e.g., [37]). On a *wide* conception, this assumption is not made, and 'counterfactual conditional' and 'subjunctive conditional' are used interchangeably (e.g., [31], [38]). Accordingly, on the latter but not on the former conception, (1.7) can be classified as a couterfactual, even if it is used to support the thesis that its antecedent is true (see [1]: 37).

(1.7) If Jones had taken arsenic, he would have shown just exactly those symptoms which he does in fact show.

The latter usage seems to be more entrenched (cf. [31]). It seems, though, that (1.7) can be classified as counterfactual in a narrow sense, if we admit a narrow conception on which counterfactuals are subjunctive conditionals whose antecedents are assumed to be not settled. We may then quite naturally regard uses of the likes of (1.7) as supporting the truth of their antecedents. In developing our proposal, we will take it that counterfactuals presuppose that their antecedents are not true, i.e., that they do not express facts.

1.2 The model-theoretic perspective

Counterfactual inference is typically studied from a model-theoretic perspective. On this perspective, model-theoretic structures are methodologically fundamental for the study of the meaning and the logic of counterfactuals (see, e.g., [31] for overview). Their *meaning* is defined, most prominently, either in terms of *truth conditions* by appeal to models which involve sets of possible worlds and suitable relations on them, or dynamically by appeal to their *context change potential* which is technically construed as a function from sets of worlds to sets of worlds.

The most familiar kind of model-theoretic approach to the semantics of counterfactuals is certainly the one which explains their meaning in terms of truth conditions which appeal to the idea of the *similarity* of possible worlds (e.g., [17], [21], [28]). Roughly: (1.1) (in symbols: $A > B$) is true in a possible world w (e.g., the actual world) just in case B is true in all the possible worlds in which A is true that are most similar to w, where the crucial

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assumption is made that there is an ordering of all worlds according to their similarity to w. One way to capture the notion of similarity formally is by means of a *selection function* which takes an antecedent statement A and the actual world and determines the possible worlds in which A is true that are closest to the actual world (see, in particular, [21], [22]). Another way, due to D. Lewis [17], proceeds in terms of *spheres*. Intuitively, each world is assigned a set of nested spheres of possible worlds which surround it. The possible worlds contained in its inner spheres are more similar to it than those contained in its outer surrounding spheres. Counterfactual fallacies can be nicely illustrated by means of this semantics. For example, transitivity fails for the derivation of (1.4) from premisses (1.2) and (1.3), since the closest worlds in which the antecedent of (1.2) is true (i.e., Hoover is born a Russian) are less similar to the actual world, and thus in spheres more remote to it, than the closest worlds in which the antecedent of (1.3) is true (i.e., Hoover is a Communist). A generalization of sphere semantics is *preference semantics*. It has been proposed by J. P. Burgess in [3]. This semantics does not assign a system of spheres to every world. Rather it equips each world w with a ternary preference relation (symbol: $<_w$) which is transitive and irreflexive. Intuitively, $x <_w y$ says that world x is preferred to world y with respect to w. According to this analysis, $A > B$ is true at w in case B is true at all the worlds in which A is true that are most preferred with respect to w. Like sphere semantics this semantics can be construed as a special case of selection-function semantics.

The conception of counterfactual inference as reflecting reasoning about possible worlds and their set-theoretic relations to each other also pertains to the aforementioned *dynamic approaches*. Here (e.g., [35], [36]), very roughly, an information state or context is formally represented as a set of possible worlds, and the meaning of a formula is given by an interpretation function which assigns a context change potential to it. Intuitively, in an update, formulae delete all worlds from contexts in which they are false.

On the model-theoretic perspective, the *logic* of counterfactuals is explained on the basis of model-theoretically defined consequence relations. Roughly, *consequence* is understood according to the general idea that all models of the premisses are models of the conclusion. Correspondingly, *validity* is conceived of as truth in all models. On this perspective, *proof systems* are secondary. Typically, the task is to define a Hilbert-style axiom system for a given consequence relation. And this is to be done in such a way that a completeness theorem can be established for the system. Completeness is here understood as *external completeness*: the systems are to be complete with respect to the models. Usually, the axiom systems extend *classical logic* and the results obtained for them presuppose classical reasoning also in the metatheory.

Structural proof theory is not even secondary, as structural proof systems (e.g., natural deduction, sequent calculi) are usually defined for the axiomatic counterfactual logics that have been defined for the model-theoretic consequence relations. *Labelled (or external)* proof systems (e.g., [12], [19], [24]) exhibit this model-theoretic dependency in a particularly transparent way, since they, in effect, incorporate the model-theoretic semantics into their rules. The calculus *G3V* for Lewis's most basic logic *V* (cf. his chart in [17]: ch. 6) proposed in [12] can serve as a good illustration of such an incorporation. The system uses labels for possible worlds (e.g., x) and spheres (e.g., a). Its rules manipulate labelled formulae such as $x : A$, $a \in S(x)$, and $a \Vdash^{\exists} A$. The first formula means that world x forces the truth of A, the second that sphere a is contained in x's system of spheres, and the last one that sphere a contains a world that forces the truth of A. *Internal* proof systems for counterfactual logics (e.g., [16], [23]), by contrast, do not involve a syntax of labels and labelled formulae that cannot be defined in terms of the object language of these logics. Instead, they make use of structural operators and specific rules which

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directly imitate model-theoretic structures involved in the semantics. For example, in [23] a sequent calculus for Lewis's V is proposed whose sequents contain so-called *blocks*—essentially syntactic structures which represent finite disjunctions of *comparative possibility* formulae of the form $A \preccurlyeq B$ ('it is at least as possible that A as it is that B')—and specific rules for the handling of blocks such as *Com* which, roughly, reflects the nesting of spheres and *Jump* which reflects moving from world to world in the model.[1]

Figuratively speaking, on the model-theoretic perspective we represent counterfactual inference primarily, e.g., by drawing similarity circles around possible worlds (sphere semantics; [17]) or, alternatively, by crossing possible worlds out (update semantics; e.g., [36]). Thus, we construe counterfactual inference, ultimately, as an operation on representations of alternative scenarios. The deductive use of counterfactual assumptions and counterfactuals in natural counterfactual reasoning is not basic on that perspective.

Indeed, one may argue that not even the use of counterfactuals needs to be considered essential on the model-theoretic perspective. As the reader may know, due to the presence of a "long and obscure" ([17]: 133) axiom in his favourite logic of counterfactuals, *VC*, Lewis preferred an axiomatization of that logic, and other counterfactual logics ([17]: 131), that uses the aforementioned notion of comparative possibility ($\preccurlyeq$) rather than the counterfactual (Lewis's symbol: $\Box\!\!\rightarrow$) as an undefined notion. On Lewis's account, these operators are interdefinable. Specifically, the definition of $\Box\!\!\rightarrow$ in terms of primitive $\preccurlyeq$ is as follows: $A \Box\!\!\rightarrow B =_{def} (\bot \preccurlyeq A) \vee \neg((A \& \neg B) \preccurlyeq (A \& B))$. What is interesting about this is that, on this approach, the study of counterfactual inference can, in principle, discard the counterfactual conditional. What, at bottom, matters is classical reasoning about comparative possibility in model-theoretic structures (cf. [17]: sect. 2.5).

1.3 The proof-theoretic perspective

The model-theoretic perspective on semantics and logic is not the only possible one, of course. For example, W. H. Holliday and T. F. Icard III observe in [15]:

> "In this paper, we have assumed that the models come first and the axiomatizations are then to be discovered. Yet in some cases it may seem that the model-theoretic proposals are largely guided by the task of delivering the right entailment predictions. Given this apparent primacy of entailment and inference patterns, one might wonder whether the semanticist ought simply to focus attention on proof systems themselves and eschew model theory altogether. In that case, axiomatic systems, together with a specific deductive apparatus intended to capture natural inferential patterns, would be the main object of study. This kind of project has of course been pursued within linguistic semantics (...), and there is a distinguished tradition of proof theoretic semantics within philosophy of logic and language."[2] ([15]: 93)

Though it is natural to construe counterfactual inference as reasoning from counterfactual assumptions, counterfactual assumptions and the inferential practice of drawing conclusions from them play only a subordinate role, if any, in contemporary research on counterfactual inference.

[1] Translations of internal into labelled systems and back (see [11], [12]) shed light on their relation and suggest that they complement each other methodologically helping to carry over insights (concerning, e.g., syntactic admissibility of cut, complexity bounds) already obtained for one kind of calculus to the other.

[2] Here, the authors refer to work by N. Francez, R. Dyckhoff, A. Szabolcsi on linguistic semantics and to work by, e.g., D. Prawitz and M. Dummett on proof-theoretic semantics. (However, it should be mentioned that axiomatic systems are not necessarily presupposed by "[t]his kind of project".)

On the proof-theoretic perspective on counterfactual inference which I would like to suggest, proof-theoretic structure is fundamental for the semantics and logic of counterfactuals. We start from a certain primacy of inferential practice and proof theory (e.g., [26], [27], [32]). Proof-theoretic structure can be conceived of in terms of trees in which one proceeds from *counterfactual assumptions*, the leaves of the tree, to conclusions, its root, by means of rules of inference. The main formal methods of inquiry are those of structural proof theory (e.g., [20], [25], [33]).

On this perspective, meaning is explained, following the tradition of *proof-theoretic semantics* in terms of derivations in suitable structural proof systems (see [27] for overview). Historically, the guiding idea behind the proof-theoretic approach to meaning has been first formulated by G. Gentzen for the operators of first-order logic in [9]:

> "The introductions represent, as it were, the 'definitions' of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'." ([9]: 80).

We take a structural proof system (more specifically, a natural deduction system) to be suitable, in case it admits this kind of inversion (cf. [25]: 33) and seek to develop such a system for counterfactual inference.

Importantly, we take an *intuitionistic stance*, one that rests on the BHK-perspective on meaning (cf. [34]: 9). Specifically, we take meaning to be determined in terms of *canonical derivations*, i.e., derivations which apply an introduction rule in the last inference step (cf. [6], [26]). Endorsing a BHK-conception, the inference rules of the proof system should capture the constructive meaning of its operators. In particular, the proof systems should admit an explanation of what it takes to infer—constructively—a consequent of a counterfactual conditional from its antecedent. Given this conception, we take *truth* to rest on explicit constructive canonical proof. As a consequence of our intuitionistic stance, we do not take *validity* to be truth in all models (not even in Heyting-algebra-valued models, or in intuitionistic Kripke models), as the model-theoretic approach to intuitionistic logic is alien to the BHK-conception. Rather, we take validity to be determined directly by the principles of constructive proof without reference to external structures (e.g., [18]). This intuitionistic perspective on validity has been adequately captured by J. D. Hamkins as follows:

> "[I]n classical logic, one has a semantic validity concept that is independent of, and perhaps prior to, any proof system; one defines that an assertion is *valid* classically when it is true in all models. The goal of classical logic, then, is to capture this validity concept in a system of formal reasoning, and we judge a proof system by whether or not it does so. Namely, a proof system implements classical logic precisely if it is sound and complete with respect to semantic validity. Intuitionistic logic, in contrast, does not seem to begin with a clear prior semantics or validity concept. Rather, intuitionism begins with general ideas about the nature of constructive reasoning; one then designs a proof system as Heyting did, so as to implement and formalize those guiding ideas. The result is intuitionist logic, provided as a formal proof system. In effect, the proof system itself helps us to clarify and express more fully what intuitionistic logic is in the first place. In particular, one can use the intuitionistic proof system to define the corresponding validity concept: to be valid intuitionistically means

> ultimately to be provable in the intuitionistic system, to be provable according to constructive principles of reasoning." ([14]: 183).

Accordingly, we shall not construe completeness as an external notion, i.e., as completeness with respect to structures (e.g., models) external to the proof system, but as *internal completeness*, i.e., as an internal property of a suitable proof system, specifically, one that admits of normalization (or cut-elimination) and possesses the subformula property. A notion of internal completeness has been characterized by J.-Y. Girard as follows:

> "If we consider cut-free proofs, then all possible proofs are already there, there is no way to produce new ones. In other terms, the calculus is complete—nothing is missing. Observe that this completeness does not refer to any sort of model, it is an internal property of syntax. Such a property cannot be an accident, it should be given its real place, the first: *The subformula property is the actual completeness.*" ([10]: 139-40).[3]

On our proof-theoretic perspective, there is, thus, no semantics/proof system-dichotomy. A proof system, given that it has certain desirable proof-theoretic properties, is itself a semantical framework (cf. [39]). This perspective, thus, differs fundamentally from the model-theoretic one taken in recent work on intuitionistic versions of counterfactual logics (e.g., [5], [40]). In particular, models and their semantic ontology (e.g., possible worlds) are required neither for the formal explanation of the meaning of counterfactuals, nor for that of valid counterfactual inference.

1.4 The proposal

Starting from the conviction that the study of counterfactual inference, i.e., the study of reasoning from counterfactual assumptions, is carried out more adequately as a study of proof-theoretic rather than model-theoretic structure, we define *modal natural deduction systems*, building on previous work in [43], where only the implicational fragment has been considered. Such proof systems are *modal*, since they make use of *modes of assumptions* which are sensitive to the factuality status of the formula that is to be assumed. We shall distinguish three kinds of status: factual, counterfactual, and independent. The rules of a modal system draw on this status. The factuality status is determined by means of a *reference proof system* on top of which a modal proof system is defined. Specifically, the factuality status of atomic sentences is determined by a subatomic system which is integrated into the reference proof system.

Due to the possibility of making assumptions in various modes, modal proof systems can be used for the proof-theoretic modeling of reasoning from both counterfactual and factual assumptions. Quite naturally, then, the availability of these assumption modes will allow us to study not only the inferential behaviour of counterfactuals, but also that of *reason giving* or *causal since-subordinator sentences* (*factuals*, for short) which, intuitively, presuppose that their antecedents are true:

(1.8) Since A is the case, B is the case.

On the simplifying assumption that 'because' behaves inferentially in a way sufficiently similar to causal 'since', we may use modal proof systems also for the analysis of reasoning with sentences of the form:

[3] Conceptions of internal completeness have been around since Gentzen as has been argued by C. Franks in [8].

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(1.9) B is the case, because A is the case.

The structure of this paper is as follows: Section 2 defines reference proof systems and Section 3 modal proof systems. The rules of modal proof systems are considerably more involved than those of natural deduction for intuitionistic logic. This is something one naturally expects, when dealing with counterfactuals. Specifically, before proving normalization, we have to make sure that the conversions for derivations in modal systems do not lead us from derivations to non-derivations. This preservation of the legitimacy of derivations is shown in Section 4 together with the normalization theorem. Presupposing these insights, Section 5 proves the subexpression (incl. subformula) property for modal systems which admits both the formulation of a decision procedure, the method of counter-derivations, and a demonstration of the internal completeness of these systems. It uses the method to discuss counterfactual fallacies and to assess familiar axioms of counterfactual logics. Section 6 defines a proof-theoretic semantics for factuals and counterfactuals and proposes a refinement of modal proof systems which makes them suitable for reasoning with conditional 'might'-constructions of the following forms:

(1.10) If A were the case, B might be the case.

(1.11) Since A is the case, B might be the case.

2 Reference proof systems

The purpose of a reference proof system, an R-system, is to determine what counts as a fact. R-systems are defined for the language $L0$.

Definition 2.1. The language $L0$ comprises the following primitive symbols:

1. denumerably many individual (or nominal) constants (metavariables: α, α_i);
2. denumerably many n-ary predicate constants (metavariables: φ^n, φ^n_i);
3. the two-place connectives $\supset$, &, and $\vee$;
4. the logical constant $\bot$ (absurdity);
5. brackets (,).

$\mathcal{C}$ and $\mathcal{P}$ are the sets of nominal and predicate constants, respectively. Their union $\mathcal{C} \cup \mathcal{P}$ is the set of non-logical constants (metavariables: τ, τ_i). Notation: $\star \in \{\supset, \&, \vee\}$.

Definition 2.2. A *prime formula* of $L0$ is either an atomic sentence ($\varphi^n\alpha_1...\alpha_n$) or $\bot$. The notion of a formula (of $L0$) is defined inductively by:

1. Any prime formula is a formula.
2. If A and B are formulae, then so are $A \supset B$ (implication), $A\&B$ (conjunction), and $A \vee B$ (disjunction).

Atm is the set of atomic sentences. $Atm(\alpha) =_{def} \{A \in Atm : A$ contains at least one occurrence of $\alpha \in \mathcal{C}\}$ and $Atm(\varphi^n) =_{def} \{A \in Atm : A$ contains an occurrence of $\varphi^n \in \mathcal{P}\}$. Fml_0 is the set of formulae of $L0$.

Definition 2.3. Defined operators of $L0$:

1. $\neg A =_{def} A \supset \bot$ (negation);

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2. $A \leftrightarrow B =_{def} (A \supset B) \& (B \supset A)$ (bi-implication).

Definition 2.4. 1. Any non-logical constant and any formula of $L0$ is an *expression* (metavariables: ϵ, ϵ').

2. Let ϵ be an expression (of $L0$). *Subexpressions* of ϵ are defined by:

   (a) ϵ is a subexpression of ϵ.

   (b) If $\varphi^n \alpha_1 ... \alpha_n$ is a subexpression of A, then so are $\varphi^n, \alpha_1, ..., \alpha_n$.

   (c) If $B \star C$ is a subexpression of A, then so are B and C.

A special case of the above definition:

Definition 2.5. Let A be a formula (of $L0$). *Subformulae* of A are defined by:

1. A is a subformula of A.

2. If $B \star C$ is a subformula of A, then so are B and C.

A reference proof system R is a natural deduction system for intuitionistic logic that incorporates a subatomic system. The purpose of a subatomic system is to define the meaning of atomic sentences (e.g., 'Hoover is a Communist') and that of their components in a proof-theoretic way. We, first, recapitulate what a subatomic system is.

Definition 2.6. A *subatomic system* $\mathcal{S}$ is a pair $\langle \mathcal{I}, \mathcal{R} \rangle$, where $\mathcal{I}$ is a subatomic base and $\mathcal{R}$ a set of rules for the introduction and elimination of atomic sentences. These components are defined as follows:

1. A *subatomic base* $\mathcal{I}$ is a 3-tuple $\langle \mathcal{C}, \mathcal{P}, v \rangle$, where v is such that:

   (a) for any $\alpha \in \mathcal{C}$, $v : \mathcal{C} \to \wp(Atm)$, where $v(\alpha) \subseteq Atm(\alpha)$;

   (b) for any $\varphi^n \in \mathcal{P}$, $v : \mathcal{P} \to \wp(Atm)$, where $v(\varphi^n) \subseteq Atm(\varphi^n)$.

   For any $\tau \in \mathcal{C} \cup \mathcal{P}$: $\tau\Gamma =_{def} v(\tau)$. $\tau\Gamma$ is the set of *term assumptions* for τ.

2. $\mathcal{R}$ is a set of *I/E-rules*:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ \varphi_0^n\Gamma\end{array} \quad \begin{array}{c}\mathcal{D}_1\\ \alpha_1\Gamma\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_n\\ \alpha_n\Gamma\end{array}}{\varphi_0^n \alpha_1 ... \alpha_n}\,(as\mathrm{I}) \qquad\qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \varphi_0^n \alpha_1 ... \alpha_n\end{array}}{\tau_i\Gamma}\,(as\mathrm{E}_i)$$

where $\varphi_0^n \alpha_1 ... \alpha_n \in \varphi_0^n\Gamma \cap \alpha_1\Gamma \cap ... \cap \alpha_n\Gamma$ where $i \in \{0, ..., n\}$ and $\tau_i \in \{\varphi_0^n, \alpha_1, ..., \alpha_n\}$

Definition 2.7. *Derivations in $\mathcal{S}$-systems.*

Basic step. Any term assumption $\tau\Gamma$ and any atomic sentence A (i.e., a derivation from the open assumption of A) is an $\mathcal{S}$-derivation.

Induction step. If $\mathcal{D}_i$, for $i \in \{0, ..., n\}$, are $\mathcal{S}$-derivations, then an $\mathcal{S}$-derivation can be constructed by means of the asI/E-rules displayed above.

Definition 2.8. 1. An atomic sentence which is the conclusion of (an application of) asI and at the same time the premiss of asE is a *maximum atomic sentence* (or an *as-maximum*).

2. A successive application of asI and asE is an *as-detour*.

3. Derivations in $\mathcal{S}$-systems which do not contain as-maxima are $\mathcal{S}$-derivations in *normal* form (or *normal* $\mathcal{S}$-derivations).

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Definition 2.9. *Detour conversion for as*:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \varphi_0^n\Gamma\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \alpha_1\Gamma\end{array} \quad ... \quad \begin{array}{c}\mathcal{D}_n\\ \alpha_n\Gamma\end{array}}{\varphi_0^n\alpha_1...\alpha_n}\,(as\mathrm{I})}{\tau_i\Gamma}\,(as\mathrm{E}_i) \quad \text{conv} \quad \begin{array}{c}\mathcal{D}_i\\ \tau_i\Gamma\end{array}$$

Theorem 2.1. Any derivation $\mathcal{D}$ in an $\mathcal{S}$-system can be transformed into a normal $\mathcal{S}$-derivation.

Proof. Immediate. □

Definition 2.10. Let $\mathcal{D}$ be a derivation in an $\mathcal{S}$-system.

1. An $\mathcal{S}$-*unit* in $\mathcal{D}$ is either an occurrence of (i) an atomic sentence or (ii) a term assumption $\tau\Gamma$ in $\mathcal{D}$. We use $U_\mathcal{S}, U'_\mathcal{S}$ (possibly with subscripts) for $\mathcal{S}$-units.

2. In case $U_\mathcal{S}$ is a term assumption $\tau\Gamma$ in $\mathcal{D}$, τ is *the expression in* $U_\mathcal{S}$.

Theorem 2.2. If $\mathcal{D}$ is a normal $\mathcal{S}$-derivation of an $\mathcal{S}$-unit $U_\mathcal{S}$ from a set of $\mathcal{S}$-units Γ, then each $\mathcal{S}$-unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U_\mathcal{S}\}$.

Proof. Immediate. □

These observations allow us to give an elementary proof-theoretic semantics for non-logical constants and atomic sentences of $L0$.

Definition 2.11. The *meaning* of

1. a *non-logical constant* τ is given by the term assumptions $\tau\Gamma$ for it which are determined by the subatomic base $\mathcal{I}$ of the subatomic system $\mathcal{S}$;

2. an *atomic sentence* $\varphi^n\alpha_1...\alpha_n$ is given by the set of its canonical derivations in $\mathcal{S}$, such derivations apply asI in the last inference step.

Having recalled what a subatomic system is, we now define the intended kind of reference proof system. Such systems combine subatomic with superatomic inference.

Definition 2.12. *Derivations in* R*-systems.*

Basic step. Any term assumption $\tau\Gamma$ and any $L0$-formula A (i.e., a derivation from the open assumption of A) is an R-derivation.

Induction step. If $\mathcal{D}_0$, ..., $\mathcal{D}_n$ are R-derivations, then an R-derivation can be constructed by means the asI/E-rules and the following familiar intuitionistic rules, where C is possibly a term assumption:

$$\dfrac{\begin{array}{c}[A]^{(u)}\\ \mathcal{D}_1\\ B\end{array}}{A \supset B}\,(\supset\mathrm{I}),\, u \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \supset B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\,(\supset\mathrm{E}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}}{A\&B}\,(\&\mathrm{I}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\&B\end{array}}{A}\,(\&\mathrm{E1}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\&B\end{array}}{B}\,(\&\mathrm{E2})$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array}}{A \vee B}\,(\vee\mathrm{I1}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ B\end{array}}{A \vee B}\,(\vee\mathrm{I2}) \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}[A]^{(u)}\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[B]^{(v)}\\ \mathcal{D}_3\\ C\end{array}}{C}\,(\vee\mathrm{E}),\, u, v \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \bot\end{array}}{A}\,(\bot\mathrm{i})$$

It can be shown that derivations in R-systems normalize (i.e., contain no detours) and that normal derivations enjoy the subexpression and the subformula property.

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Theorem 2.3. *Normalization (R-systems)*: Any derivation $\mathcal{D}$ in an R-system can be transformed into a normal R-derivation.

Theorem 2.4. *Subexpression property (R-systems)*: If $\mathcal{D}$ is a normal R-derivation of a unit U_{R} from a set of units Γ, then each unit in $\mathcal{D}$ is a subexpression of an expression in $\Gamma \cup \{U_{\mathsf{R}}\}$.

Corollary 2.1. *Subformula property (R-systems)*: If $\mathcal{D}$ is a normal R-derivation of formula A from a set of formulae Γ, then each formula in $\mathcal{D}$ is a subformula of a formula in $\Gamma \cup \{A\}$.

The proofs of these results are adaptations of the proofs in [43]. They are omitted, since they are special cases of the corresponding proofs in Sections 4 and 5.

Remark 2.1. Traditional benefits of the subformula property include the consistency of a proof system and a simplification of proof search in it. Specifically, all theorems of R-systems can be established by means of normal canonical proofs.

Definition 2.13.

1. A derivation $\mathcal{D}$ of a formula A in an R-system is a *canonical derivation* iff it derives A by means of an I-rule in the last step of $\mathcal{D}$.
2. A canonical derivation $\mathcal{D}$ of A in an R-system is a *canonical proof* of A in that system iff there are no applications of *as*-rules and no undischarged (or open) assumptions in $\mathcal{D}$.
3. The conclusions of canonical R-derivations are R-*theses* and the conclusions of R-proofs are also R-*theorems*.

We may now define the notion of a fact.

Definition 2.14. An *established thesis* (or *fact*) of R is an $L0$-formula for which a *canonical* R-derivation has been constructed. Θ_{R} is the set of so far established theses.

3 Modal proof systems

Modal proof systems are modal, because they make use of modes of making assumptions. These modes are sensitive to the factuality status of the formula that is to be assumed. This status is determined by means of a reference proof system. Modal proof systems are defined for the language $L1$ which extends $L0$.

Definition 3.1. The language $L1$ comprises:

1. the primitive symbols of $L0$; and
2. the two-place connectives for factual, counterfactual, and mode-sensitive implication, conjunction, and disjunction: $\supset_f$, $\supset_c$, $\supset_*$, $\&_f$, $\&_c$, $\&_*$, and $\vee_f$, $\vee_c$, $\vee_*$.

Metalinguistic notation: $\star^i \in \{\supset, \supset_f, \supset_c, \supset_*\}$, $\star^c \in \{\&, \&_f, \&_c, \&_*\}$, and $\star^d \in \{\vee, \vee_f, \vee_c, \vee_*\}$.

Definition 3.2. The notion of a formula of $L1$ is defined inductively by:

1. Any formula of $L0$ is a formula (of $L1$).
2. If A, B are formulae, then $A \supset_f B$, $A \supset_c B$, $A \supset_* B$, $A \&_f B$, $A \&_c B$, $A \&_* B$, $A \vee_f B$, $A \vee_c B$, $A \vee_* B$ are formulae.

Fml_1 is the set of formulae of $L1$.

Notational convention: In case a compound $L0$-formula A is used as a formula of $L1$, we write $\supset_*$ [$\&_*$, $\vee_*$] for occurrences of $\supset$ [$\&$, $\vee$] in A.

Definition 3.3. Defined operators of $L1$:

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1. $\neg_* A =_{def} A \supset_* \bot$ (mode-sensitive negation);
2. $A \leftrightarrow_* B =_{def} (A \supset_* B) \&_* (B \supset_* A)$ (mode-sensitive bi-implication).

Remark 3.1. We may also allow for factual and counterfactual versions of negation ($\neg_f$, $\neg_c$) and bi-implication ($\leftrightarrow_f$, $\leftrightarrow_c$).

Definition 3.4. *Expression, subexpression, and subformula*: The definitions of these notions for $L1$ are analogous to those for $L0$.

Assumption modes are characteristic of modal proof systems.

Definition 3.5. *Assumption modes.* There are three modes of making assumptions in a modal proof system:

1. $|A|$ indicates that A is assumed in the *factual mode*, given that $A \in Fml_0$ and $A \in \Theta_{\mathsf{R}}$.
2. $\wr A \wr$ indicates that A is assumed in the *counterfactual mode*, given that $A \in Fml_0$ and $A \in \Theta^c_{\mathsf{R}}$, where $\Theta^c_{\mathsf{R}} =_{def} Fml_0 \setminus \Theta_{\mathsf{R}}$.
3. (A) indicates that A is assumed in the usual *independent (or neutral) mode*, where $A \in Fml_1$. Specifically, in case A is also an $L0$-formula, A is assumed independently of whether it is contained in Θ_{R} or Θ^c_{R}.

$/A/$ indicates that A is assumed in one of the three modes. We call the symbols flanking assumed formulae *status markers.*

Remark 3.2. 1. Let A be a formula of $L0$ and suppose we have established A as a fact. We can, then, make the factual assumption $|A|$. However, we are debarred from making the counterfactual assumption $\wr A \wr$. We may assume counterfactually only $\wr \neg A \wr$, provided that we have no canonical derivation of $\neg A$ so far.

2. The independent mode of assumption is the standard one. It has been used already in reference proof systems. We use the status markers (), in case we make independent assumptions in modal proof systems. In derivations, the context will help one to determine whether the brackets are used as status markers or as auxiliary symbols in formulae: In case a top formula occurrence is enclosed in (), these symbols are status markers.

3. As a consequence of Definition 3.5, formulae which contain a factual or a counterfactual operator can be assumed only in the independent mode. Given the notational convention, formulae containing exclusively mode-sensitive operators can be assumed in any mode.

Definition 3.6. *Negated status markers.* $\not| A \not|$ indicates either $\wr A \wr$ or (A); $\not\wr A \not\wr$ indicates either $|A|$ or (A); $\not(A \not)$ indicates either $|A|$ or $\wr A \wr$.

Definition 3.7. 1. A formula A in a derivation $\mathcal{D}$ in a modal proof system may have three different kinds of *modal status.*

   (a) A has *factual status* in $\mathcal{D}$, if it depends on no counterfactual assumption, and either
      i. A depends on at least one factual assumption (special case: $|A|$), or
      ii. A has been derived by means of term assumptions, or
      iii. A is a conclusion of a canonical derivation in R.
   (b) A has *counterfactual status* in $\mathcal{D}$, if it depends on at least one counterfactual assumption (special case: $\wr A \wr$).
   (c) A has *independent status* in $\mathcal{D}$, if it has no factual or counterfactual status (special case: (A)).

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2. Any term assumption $\tau\Gamma$ in a derivation $\mathcal{D}$ in a modal proof system has factual status.

Definition 3.8. *Status markers (Derivations).* $|\mathcal{D}|$ $[\wr\mathcal{D}\wr, (\mathcal{D})]$ indicates that the conclusion of $\mathcal{D}$ has factual [counterfactual, independent] status. $/\mathcal{D}/$ indicates that the conclusion of $\mathcal{D}$ has one of the three kinds of status. $\wr\!(\mathcal{D})\!\wr$ indicates either $\wr\mathcal{D}\wr$ or $(\mathcal{D})$; $|\!(\mathcal{D})\!|$ indicates either $|\mathcal{D}|$ or $(\mathcal{D})$; $|\!\wr \mathcal{D} \wr\!|$ indicates either $|\mathcal{D}|$ or $\wr\mathcal{D}\wr$.

Definition 3.9. *Assumption classes.* $[/A/]^{(u)}$ denotes either (i) an assumption class, i.e. a set of undischarged assumptions of occurrences of formula A, or (ii) a single undischarged assumption of an occurrence of formula A in a derivation $/\mathcal{D}/$ marked by u. The second alternative is used, in case $/\mathcal{D}/$ is fully explicit. If B is the conclusion and $[/A/]^{(u)}$ an assumption class of $/\mathcal{D}/$, we write:

$$(\dagger)\ \begin{array}{c}[/A/]^{(u)}\\ /\mathcal{D}/\\ B\end{array} \tag{1}$$

Remark 3.3. A derivation of the form (†) cannot take just any shape, since, by Definition 3.7, combinations (b), (g), and (h) are precluded:

$$\begin{array}{ccc}
\text{(a)}\ \begin{array}{c}[|A|]^{(u)}\\ |\mathcal{D}|\\ B\end{array} & \text{(b)}\ \begin{array}{c}[\wr A\wr]^{(u)}\\ |\mathcal{D}|\\ B\end{array} & \text{(c)}\ \begin{array}{c}[(A)]^{(u)}\\ |\mathcal{D}|\\ B\end{array}\\
\text{(d)}\ \begin{array}{c}[|A|]^{(u)}\\ \wr\mathcal{D}\wr\\ B\end{array} & \text{(e)}\ \begin{array}{c}[\wr A\wr]^{(u)}\\ \wr\mathcal{D}\wr\\ B\end{array} & \text{(f)}\ \begin{array}{c}[(A)]^{(u)}\\ \wr\mathcal{D}\wr\\ B\end{array}\\
\text{(g)}\ \begin{array}{c}[|A|]^{(u)}\\ (\mathcal{D})\\ B\end{array} & \text{(h)}\ \begin{array}{c}[\wr A\wr]^{(u)}\\ (\mathcal{D})\\ B\end{array} & \text{(i)}\ \begin{array}{c}[(A)]^{(u)}\\ (\mathcal{D})\\ B\end{array}
\end{array}$$

The construction of new derivations from old ones gives rise to a certain dendrochronology.

Definition 3.10. *Status marker nesting.* If $/\mathcal{D}/$ is a derivation of the form (†), and r a discharging rule for a dummy operator \$ (considered here only for the purpose of illustration) applied to the conclusion B of $/\mathcal{D}/$, we use the following notation for the new derivation $//\mathcal{D}//$ resulting from that application which discharges the members of $[/A/]^{(u)}$:

$$(\ddagger)\ \begin{array}{c}[/A/]^{(u)}\\ //\mathcal{D}//\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array} \tag{2}$$

In $//\mathcal{D}//$ the inner slashes indicate the status of B and the outer ones the status of $A\$B$ after the discharge by r.

Remark 3.4. 1. A derivation of the form (‡) can take any of the following shapes:

$$\begin{array}{ccc}
\text{(a)}\ \begin{array}{c}[|A|]^{(u)}\\ ||\mathcal{D}||\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array} & \text{(b)}\ \begin{array}{c}[\wr A\wr]^{(u)}\\ |\wr\mathcal{D}\wr|\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array} & \text{(c)}\ \begin{array}{c}[(A)]^{(u)}\\ ||\mathcal{D}||\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array}\\
\text{(d)}\ \begin{array}{c}[|A|]^{(u)}\\ \wr\wr\mathcal{D}\wr\wr\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array} & \text{(e)}\ \begin{array}{c}[\wr A\wr]^{(u)}\\ \wr\wr\mathcal{D}\wr\wr\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array} & \text{(f)}\ \begin{array}{c}[(A)]^{(u)}\\ \wr\wr\mathcal{D}\wr\wr\\ \dfrac{B}{A\$B}\ \mathrm{r},\, u\end{array}
\end{array}$$

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$$
\begin{array}{ccc}
[|A|]^{(u)} & [\wr A \wr]^{(u)} & [(A)]^{(u)} \\
(|\mathcal{D}|) & (\wr\mathcal{D}\wr) & ((\mathcal{D})) \\
\text{(g)}\ \dfrac{B}{A\$B}\ \mathrm{r},\ u & \text{(h)}\ \dfrac{B}{A\$B}\ \mathrm{r},\ u & \text{(i)}\ \dfrac{B}{A\$B}\ \mathrm{r},\ u
\end{array}
$$

Specifically, in combination (b) [(h)] the members of $[\wr A \wr]^{(u)}$ are the only undischarged counterfactual assumptions in $\wr\mathcal{D}\wr$ and $A\$B$ has factual [independent] status. In combination (g) the members of $[|A|]^{(u)}$ are the only undischarged factual assumptions in $|\mathcal{D}|$ and $A\$B$ has independent status; in (a), there are either other open factual assumptions besides those in $[|A|]^{(u)}$ or term assumptions in $|\mathcal{D}|$ and $A\$B$ has factual status.

2. Combinations (a), (d), (g) fall under (j), combinations (b), (e), (h) fall under (k), combinations (c), (f), (i) fall under (l), and (j), (k), (l) under (‡):

$$
\begin{array}{ccc}
[|A|]^{(u)} & [\wr A \wr]^{(u)} & [(A)]^{(u)} \\
/\{\!\!\{ \mathcal{D} \}\!\!\}/ & /\wr \mathcal{D} \wr/ & //\mathcal{D}// \\
\text{(j)}\ \dfrac{B}{A\$B}\ \mathrm{r},\ u & \text{(k)}\ \dfrac{B}{A\$B}\ \mathrm{r},\ u & \text{(l)}\ \dfrac{B}{A\$B}\ \mathrm{r},\ u
\end{array}
$$

3. The number of rings of a derivation tree may grow. When considering the structure of derivations, it will often be convenient to indicate only the outermost status markers. For example, we may write $(\mathcal{D})$ instead of $(|\wr\mathcal{D}\wr|)$. Note that, e.g. $\wr|(\mathcal{D})|\wr$, is precluded by Definition 3.7.

4. By Definition 3.7, the conclusion of a derivation without term assumptions in which all assumptions have been discharged that is not an R-theorem has independent status.

Modal proof systems, M-systems, are defined on top of reference proof systems and do justice to the specific needs of counterfactual inference mentioned in the Introduction by means of specific side conditions.

Definition 3.11. *Derivations (M-systems).* The notion of an M-derivation is defined by:

Basic step. Any derivation in the R-system of an M-system, any $L0$-formula A assumed in the factual (resp. counterfactual) mode $|A|$ ($\wr A \wr$), i.e., a derivation from the open factual (counterfactual) assumption of A, and any $L1$-formula A assumed in the independent mode (A), i.e., a derivation from the open independent assumption of A, is a derivation in that M-system.

Induction step. If $\mathcal{D}_0$, ..., $\mathcal{D}_n$ are M-derivations and C possibly a term assumption, then an M-derivation can be constructed by means of the following rules:

Rules for atomic sentences:

$$
\dfrac{\begin{matrix}/\mathcal{D}_0/ & /\mathcal{D}_1/ & & /\mathcal{D}_n/ \\ \varphi_0^n\Gamma & \alpha_1\Gamma & \dots & \alpha_n\Gamma\end{matrix}}{\varphi_0^n\alpha_1\dots\alpha_n}\ (as_*\mathrm{I})
\qquad\qquad
\dfrac{\begin{matrix}/\mathcal{D}_1/ \\ \varphi_0^n\alpha_1\dots\alpha_n\end{matrix}}{\tau_i\Gamma}\ (as_*\mathrm{E}_i)
$$

where $\varphi_0^n\alpha_1\dots\alpha_n \in \varphi_0^n\Gamma \cap \alpha_1\Gamma \cap \dots \cap \alpha_n\Gamma$ where $i \in \{0, \dots, n\}$ and $\tau_i \in \{\varphi_0^n, \alpha_1, \dots, \alpha_n\}$

Rules for implications:

$$
\dfrac{\begin{matrix}[|A|]^{(u)} \\ /\{\!\!\{ \mathcal{D}_1 \}\!\!\}/ \\ B\end{matrix}}{A \supset_f B}\ (\supset_f\mathrm{I}),\ u
\quad
\dfrac{\begin{matrix}/\mathcal{D}_1/ & |\mathcal{D}_2| \\ A \supset_f B & A\end{matrix}}{B}\ (\supset_f\mathrm{E})
\quad
\dfrac{\begin{matrix}[\wr A \wr]^{(u)} \\ /\wr \mathcal{D}_1 \wr/ \\ B\end{matrix}}{A \supset_c B}\ (\supset_c\mathrm{I}),\ u
\quad
\dfrac{\begin{matrix}/\mathcal{D}_1/ & \wr\mathcal{D}_2\wr \\ A \supset_c B & A\end{matrix}}{B}\ (\supset_c\mathrm{E})
$$

$$
\dfrac{\begin{matrix}[/A/]^{(u)} \\ /\mathcal{D}_1/ \\ B\end{matrix}}{A \supset_* B}\ (\supset_*\mathrm{I}),\ u
\quad
\dfrac{\begin{matrix}/\mathcal{D}_1/ & /\mathcal{D}_2/ \\ A \supset_* B & A\end{matrix}}{B}\ (\supset_*\mathrm{E})
$$

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Rules for conjunctions:

$$\dfrac{\begin{matrix}|\mathcal{D}_1| \\ A\end{matrix} \qquad \begin{matrix}|\mathcal{D}_2| \\ B\end{matrix}}{A \&_f B}\,(\&_f\mathrm{I}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \&_f B\end{matrix}}{A}\,(\&_f\mathrm{E1}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \&_f B\end{matrix}}{B}\,(\&_f\mathrm{E2})$$

$$\dfrac{\begin{matrix}\wr\mathcal{D}_1\wr \\ A\end{matrix} \qquad \begin{matrix}\wr\mathcal{D}_2\wr \\ B\end{matrix}}{A \&_c B}\,(\&_c\mathrm{I}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \&_c B\end{matrix}}{A}\,(\&_c\mathrm{E1}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \&_c B\end{matrix}}{B}\,(\&_c\mathrm{E2})$$

$$\dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A\end{matrix} \qquad \begin{matrix}/\mathcal{D}_2/ \\ B\end{matrix}}{A \&_* B}\,(\&_*\mathrm{I}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \&_* B\end{matrix}}{A}\,(\&_*\mathrm{E1}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \&_* B\end{matrix}}{B}\,(\&_*\mathrm{E2})$$

Rules for disjunctions:

$$\dfrac{\begin{matrix}|\mathcal{D}_1| \\ A\end{matrix}}{A \vee_f B}\,(\vee_f\mathrm{I1}) \qquad \dfrac{\begin{matrix}|\mathcal{D}_1| \\ B\end{matrix}}{A \vee_f B}\,(\vee_f\mathrm{I2}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \vee_f B\end{matrix} \qquad \begin{matrix}[/A/]^{(u)} \\ /\mathcal{D}_2/ \\ C\end{matrix} \qquad \begin{matrix}[/B/]^{(v)} \\ /\mathcal{D}_3/ \\ C\end{matrix}}{C}\,(\vee_f\mathrm{E}),\ u, v$$

$$\dfrac{\begin{matrix}\wr\mathcal{D}_1\wr \\ A\end{matrix}}{A \vee_c B}\,(\vee_c\mathrm{I1}) \qquad \dfrac{\begin{matrix}\wr\mathcal{D}_1\wr \\ B\end{matrix}}{A \vee_c B}\,(\vee_c\mathrm{I2}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \vee_c B\end{matrix} \qquad \begin{matrix}[/A/]^{(u)} \\ /\mathcal{D}_2/ \\ C\end{matrix} \qquad \begin{matrix}[/B/]^{(v)} \\ /\mathcal{D}_3/ \\ C\end{matrix}}{C}\,(\vee_c\mathrm{E}),\ u, v$$

$$\dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A\end{matrix}}{A \vee_* B}\,(\vee_*\mathrm{I1}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ B\end{matrix}}{A \vee_* B}\,(\vee_*\mathrm{I2}) \qquad \dfrac{\begin{matrix}/\mathcal{D}_1/ \\ A \vee_* B\end{matrix} \qquad \begin{matrix}[/A/]^{(u)} \\ /\mathcal{D}_2/ \\ C\end{matrix} \qquad \begin{matrix}[/B/]^{(v)} \\ /\mathcal{D}_3/ \\ C\end{matrix}}{C}\,(\vee_*\mathrm{E}),\ u, v$$

Rule for absurdity:

$$\dfrac{\begin{matrix}/\mathcal{D}_1/ \\ \bot\end{matrix}}{A}\,(\bot_*\mathrm{i})$$

Terminology: Call the conjunct A $[B]$ derived by an application of $\star^c$E1 $[\star^c$E2$]$ to $A \star^c B$ *selected* and the conjunct B $[A]$ *unselected*. Similarly, call the disjunct A $[B]$ in $A \star^d B$ derived by an application of $\star^d$I1 $[\star^d$I2$]$ to A $[B]$ *introducing* and the disjunct B $[A]$ in $A \star^d B$ *introduced*.

Side conditions:

SC1. $\supset_f$I:

(a) No empty discharge; and no empty discharge in $\mathcal{D}_1$.

(b) B does not depend on a $\star^c$-formula, other than itself, that contains an *unselected conjunct* in $\mathcal{D}_1$.

(c) A must not be an antecedent of an *introduced disjunct* in $\mathcal{D}_1$.

SC2. $\supset_c$I: Like SC1.

SC3. $\supset_c$E: In case A and B are distinct formulae, the conclusion of $\supset_c$E must not be the minor premiss of an application of another application of $\supset_c$E or of $\supset_*$E (*break formula*).

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SC4. $\&_\star$I: Both premisses have the same status.

SC5. $\vee_f$E: At least one of the premisses discharged by this rule is assumed in the factual mode.

SC6. $\vee_c$E: At least one of the premisses discharged by this rule is assumed in the counterfactual mode.

Assumption principles: The following principles are respected by any derivation $\mathcal{D}$ in M-systems:

AP1. No formula is assumed in more than one mode in $\mathcal{D}$.

AP2. The mode in which an antecedent A is assumed in $\star^i$I-applications in $\mathcal{D}$ determines the modal status of all antecedent A-nodes (i.e., minor premisses of $\star^i$E) in $\mathcal{D}$.

AP3. The mode in which a disjunct A is assumed in $\star^d$E-applications in $\mathcal{D}$ determines the modal status of all introducing disjunct A-nodes (i.e., premisses of $\star^d$I) in $\mathcal{D}$.

4 Preservation and normalization

The proof of normalization for M-systems proceeds largely along entirely familiar lines (cf. [25], [33]). However, as will be explained shortly, it is complicated by the requirements imposed on the rules of these systems.

Definition 4.1. A formula (of $L1$) which is the conclusion of an I-rule and at the same time the premiss of an E-rule is a *maximum formula* (or a *maximum*).

Definition 4.2. A *segment* σ_{M} of *length* n in a derivation $/\mathcal{D}/$ in an M-system is a sequence $A_1, ..., A_n$ of successive occurrences of a formula A in $/\mathcal{D}/$ such that:

1. for $1 < n$, $i < n$, A_i is a minor premiss of $\star^d$E in $/\mathcal{D}/$ with conclusion A_{i+1};
2. A_n is not a minor premiss of $\star^d$E;
3. A_1 is not the conclusion of $\star^d$E.

Definition 4.3. A segment σ_{M} is *maximal* in case A_n is the major premiss of an E-rule and either $n > 1$, or $n = 1$ and $A_1 \equiv A_n$ is the conclusion of an I-rule; $\equiv$ indicates literal identity (cf. [33]: 2). (A maximum, possibly an atomic sentence, is a special case of a maximal segment. A term assumption cannot be a maximum.)

Definition 4.4. *Rank and cut rank (M-systems)*:

1. The *rank* of a term assumption $\tau\Gamma$ is defined by: $r(\tau\Gamma) = 0$.
2. The *rank* of an atomic sentence $\varphi^n\alpha_1...\alpha_n$ is defined by: $r(\varphi^n\alpha_1...\alpha_n) = 1$.
3. Let A, B be formulae and let $\star \in \{\star^i, \star^c, \star^d\}$. The *rank* of $\bot$ $[A \star B]$ is defined by: $r(\bot) = 0$ $[r(A \star B) = max(r(A), r(B)) + 1]$.
4. The *cut rank* of an M-derivation $/\mathcal{D}/$ is defined by: $cr(/\mathcal{D}/) = \langle d, n\rangle$, where:
   (a) $d = max\{cr(\sigma_{\mathsf{M}})\colon \sigma_{\mathsf{M}}$ is a maximal segment in $/\mathcal{D}/\}$, where $cr(\sigma_{\mathsf{M}}) = |\ A\ |$ is the cut rank of a maximal segment σ_{M} with formula A. In case there is no maximal segment, $max\varnothing = 0$.

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(b) n = the sum of lengths of all critical cuts in $/\mathcal{D}/$, where a *critical cut* of an M-derivation $/\mathcal{D}/$ is a maximal segment of maximal cut rank from all maximal segments in $/\mathcal{D}/$.

Definition 4.5. Derivations in M-systems which do not contain critical cuts are said to be *normal* or in *normal form.*

Definition 4.6. *Conversions (M-systems).* Critical cuts are removed by means of the following conversions:

1. *Detour conversions (M-systems).*

(a) as_*-*Conversion*: $i \in \{0, ..., n\}$

$$\dfrac{\dfrac{\begin{array}{c}/\mathcal{D}_0/\\ \varphi_0^n\Gamma\end{array}\qquad \begin{array}{c}/\mathcal{D}_1/\\ \alpha_1\Gamma\end{array}\quad \dots \quad \begin{array}{c}/\mathcal{D}_n/\\ \alpha_n\Gamma\end{array}}{\varphi_0^n\alpha_1...\alpha_n}\,(as_*\mathrm{I})}{\tau_i\Gamma}\,(as_*\mathrm{E}_i) \quad \text{conv} \quad \begin{array}{c}/\mathcal{D}_i/\\ \tau_i\Gamma\end{array}$$

(b) $\star^i$-*Conversions*:

$$\dfrac{\dfrac{\begin{array}{c}[|A|]^{(u)}\\ /\llparenthesis\,\mathcal{D}_1\,\rrparenthesis/\\ B\end{array}}{A \supset_f B}\,(\supset_f\mathrm{I}),\, u \qquad \begin{array}{c}|\mathcal{D}_2|\\ A\end{array}}{B}\,(\supset_f\mathrm{E}) \quad \text{conv} \quad \begin{array}{c}|\mathcal{D}_2|\\ [A]\\ \llparenthesis\,\mathcal{D}_1\,\rrparenthesis\\ B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}[\wr A\wr]^{(u)}\\ /\wr\,\mathcal{D}_1\,\wr/\\ B\end{array}}{A \supset_c B}\,(\supset_c\mathrm{I}),\, u \qquad \begin{array}{c}\wr\mathcal{D}_2\wr\\ A\end{array}}{B}\,(\supset_c\mathrm{E}) \quad \text{conv} \quad \begin{array}{c}\wr\mathcal{D}_2\wr\\ [A]\\ \wr\mathcal{D}_1\wr\\ B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}[/A/]^{(u)}\\ /\mathcal{D}_1/\\ B\end{array}}{A \supset_* B}\,(\supset_*\mathrm{I}),\, u \qquad \begin{array}{c}/\mathcal{D}_2/\\ A\end{array}}{B}\,(\supset_*\mathrm{E}) \quad \text{conv} \quad \begin{array}{c}/\mathcal{D}_2/\\ [A]\\ /\mathcal{D}_1/\\ B\end{array}$$

(c) $\star^c$-*Conversions*: $i \in \{1, 2\}$

$$\dfrac{\dfrac{\begin{array}{c}|\mathcal{D}_1|\\ A_1\end{array}\quad \begin{array}{c}|\mathcal{D}_2|\\ A_2\end{array}}{A_1 \&_f A_2}\,(\&_f\mathrm{I})}{A_i}\,(\&_f\mathrm{E}i) \quad \text{conv} \quad \begin{array}{c}|\mathcal{D}_i|\\ A_i\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\wr\mathcal{D}_1\wr\\ A_1\end{array}\quad \begin{array}{c}\wr\mathcal{D}_2\wr\\ A_2\end{array}}{A_1 \&_c A_2}\,(\&_c\mathrm{I})}{A_i}\,(\&_c\mathrm{E}i) \quad \text{conv} \quad \begin{array}{c}\wr\mathcal{D}_i\wr\\ A_i\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}/\mathcal{D}_1/\\ A_1\end{array}\quad \begin{array}{c}/\mathcal{D}_2/\\ A_2\end{array}}{A_1 \&_* A_2}\,(\&_*\mathrm{I})}{A_i}\,(\&_*\mathrm{E}i) \quad \text{conv} \quad \begin{array}{c}/\mathcal{D}_i/\\ A_i\end{array}$$

(d) $\star^d$-*Conversions*: $i \in \{1, 2\}$

$$\dfrac{\dfrac{\begin{array}{c}|\mathcal{D}|\\ A_i\end{array}}{A_1 \vee_f A_2}\,(\vee_f\mathrm{I}) \qquad \begin{array}{c}[/A_1/]^{(u)}\\ /\mathcal{D}_1/\\ C\end{array} \qquad \begin{array}{c}[/A_2/]^{(v)}\\ /\mathcal{D}_2/\\ C\end{array}}{C}\,(\vee_f\mathrm{E}),\, u,\, v \quad \text{conv} \quad \begin{array}{c}|\mathcal{D}|\\ [A_i]\\ \llparenthesis\,\mathcal{D}_i\,\rrparenthesis\\ C\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\wr\mathcal{D}\wr\\ A_i\end{array}}{A_1 \vee_c A_2}\,(\vee_c\mathrm{I}) \qquad \begin{array}{c}[/A_1/]^{(u)}\\ /\mathcal{D}_1/\\ C\end{array} \qquad \begin{array}{c}[/A_2/]^{(v)}\\ /\mathcal{D}_2/\\ C\end{array}}{C}\,(\vee_c\mathrm{E}),\, u,\, v \quad \text{conv} \quad \begin{array}{c}\wr\mathcal{D}\wr\\ [A_i]\\ \wr\mathcal{D}_i\wr\\ C\end{array}$$

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$$\dfrac{\dfrac{\begin{matrix}/\mathcal{D}/\\ A_i\end{matrix}}{A_1 \vee_* A_2}(\vee_*\mathrm{I}) \qquad \begin{matrix}[/A_1/]^{(u)}\\ /\mathcal{D}_1/\\ C\end{matrix} \qquad \begin{matrix}[/A_2/]^{(v)}\\ /\mathcal{D}_2/\\ C\end{matrix}}{C}(\vee_*\mathrm{E}),\ u,\ v \qquad \text{conv} \qquad \begin{matrix}/\mathcal{D}/\\ [A_i]\\ /\mathcal{D}_i/\\ C\end{matrix}$$

2. *Permutation conversions (M-systems).* Cuts of length > 1 are removed after an upwards permutation of E-rules over minor premisses of $\star^d$E-applications (in case the E-rule is asE, C is an atomic sentence and D a term assumption).

$$\dfrac{\dfrac{\begin{matrix}/\mathcal{D}/\\ A \star^d B\end{matrix} \quad \begin{matrix}/\mathcal{D}_1/\\ C\end{matrix} \quad \begin{matrix}/\mathcal{D}_2/\\ C\end{matrix}}{C}(\star^d\mathrm{E}) \qquad /\mathcal{D}'/}{D}(\text{E-rule})$$

$$\text{perm}$$

$$\dfrac{\begin{matrix}/\mathcal{D}/\\ A \star^d B\end{matrix} \quad \dfrac{\begin{matrix}/\mathcal{D}_1/\\ C\end{matrix} \quad /\mathcal{D}'/}{D}(\text{E-rule}) \quad \dfrac{\begin{matrix}/\mathcal{D}_2/\\ C\end{matrix} \quad /\mathcal{D}'/}{D}(\text{E-rule})}{D}(\star^d\mathrm{E})$$

3. *Simplification conversions (M-systems).* An application of $\star^d$E to major premiss $A_1 \star^d A_2$ in which at least one of $[A_1]$, $[A_2]$ is empty in the derivations of its minor premisses C (possibly term assumptions) is redundant. Redundant $\star^d$E-applications are removed by simplification conversions for $\star^d$, where no assumptions are discharged by $\star^d$E in $\mathcal{D}_i$ ($i \in \{1, 2\}$).

$$\dfrac{\begin{matrix}/\mathcal{D}/\\ A_1 \star^d A_2\end{matrix} \quad \begin{matrix}/\mathcal{D}_1/\\ C\end{matrix} \quad \begin{matrix}/\mathcal{D}_2/\\ C\end{matrix}}{C}(\star^d\mathrm{E}) \qquad \text{simp} \qquad \begin{matrix}/\mathcal{D}_i/\\ C\end{matrix}$$

Due to the presence of various modes of assumption, very specific side conditions, and assumption principles, M-systems are considerably more complex than R-systems. For this reason, the question may arise whether the conversions of derivations in M-systems never transform derivations into non-derivations. An answer to this question is important, as a proper functioning of the conversions is vital to normalization. The answer is positive.

4.1 Preservation for detour conversions

We distinguish legitimate from illegitimate derivations.

Definition 4.7. A derivation is *legitimate* in case it is constructed according to Definition 3.11; otherwise it is *illegitimate*. (We also call the former 'derivations' and the latter 'non-derivations'.)

Theorem 4.1. The detour conversions of M-systems do not transform legitimate into illegitimate derivations.

Proof. A long proof by exhaustion.

- Part A: preservation for $\star^i$-conversions
- Part B: preservation for $\star^c$-conversions
- Part C: preservation for $\star^d$-conversions

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(Since no side conditions are imposed on the as_*-rules, preservation for as_*-conversions is guaranteed.) The formula for calculating the minimal number f of *cases* is $f = g^2 \times h$, where g is the number of I/E-rules for specific operators (only one for $\star^c$E, $\star^d$I) and h the number of detour conversions for specific operators. If the number of cases is larger than f, this is due to the fact that discharged assumptions may occur above both major and minor premisses of minor premiss rules.

Part A: Preservation for $\star^i$-detour conversions. Let $\begin{array}{c}/\mathcal{D}^a/\\ B\end{array}$, $\begin{array}{c}/\mathcal{D}^b/\\ A\end{array}$ be legitimate derivations. We show: If these derivations can be combined into a legitimate derivation $\mathcal{D}^*$, then the $\star^i$-conversions transform it into a legitimate derivation $\mathcal{D}^{**}$; otherwise, the combination $\mathcal{D}^*$ is illegitimate and an $\star^i$-conversion is precluded:

$$\mathcal{D}^* = \dfrac{\dfrac{\begin{array}{c}[/A/]^{(u)}\\ /\mathcal{D}^a/\\ B\end{array}}{A \star^i B}\,(\star^i\text{I}),\ u \qquad \begin{array}{c}/\mathcal{D}^b/\\ A\end{array}}{B}\,(\star^i\text{E}) \qquad \mathcal{D}^{**} = \begin{array}{c}/\mathcal{D}^b/\\ [A]\\ /\mathcal{D}^a/\\ B\end{array} \tag{3}$$

Top-down procedure. In order to establish the result we consider, for all specific instances of last rules applied in $\begin{array}{c}/\mathcal{D}^a/\\ B\end{array}$ and $\begin{array}{c}/\mathcal{D}^b/\\ A\end{array}$, the combination of these derivations into $\mathcal{D}^*$. In each case we attempt to derive the instance of the maximum formula $A \star^i B$ proceeding top-down starting from the assumptions in $\begin{array}{c}/\mathcal{D}^a/\\ B\end{array}$ according to the rules of M-systems until we arrive at the I-rule application which introduces $A \star^i B$. The procedure is top-down, since, in general, the modal status of assumptions determines the modal status of the formulae that appear below them in a derivation. Next, we attempt to derive the instance of the minor premiss A of the elimination of $A \star^i B$ proceeding in the aforementioned top-down manner with $\begin{array}{c}/\mathcal{D}^b/\\ A\end{array}$. This procedure determines whether $\mathcal{D}^*$ is legitimate and whether its conversion is successful or precluded.

Kinds of violation. Figure 1 lists for all violations which preclude conversion a *violation code* (*V-code*) and indicates the kind of operator affected by the violation (using the letter I for $\star^i$-, C for $\star^c$-, and D for $\star^d$-operators).

Preservation tables. Part A of the preservation proof is summarized in a condensed form in Tables A.1-9 below (pp. 20-22). Anticipating the observations on coincidence (Remark 4.2), Tables A.3, A.6, A.9 are not displayed for reasons of space. These tables list all the possible cases and the result of the top-down procedure for each case of an $\star^i$-conversion. For ease of orientation, each A-table is associated with a sequence of three letters (enclosed in brackets) which indicate the general kind of operator involved in the conversion:

A.1. (`III`)	A.4. (`ICI`)	A.7. (`IDI`)
A.2. (`IIC`)	A.5. (`ICC`)	A.8. (`IDC`)
A.3. (`IID`)	A.6. (`ICD`)	A.9. (`IDD`)

The first letter indicates the kind of main operator of the maximum formula, the second indicates the kind of operator of the last rule used in $\mathcal{D}^a$, and the third the kind of operator of the last rule used in $\mathcal{D}^b$. In each A-table, the entries below $\supset_f$-c [$\supset_c$-c, $\supset_*$-c] indicate the results for $\supset_f$ [$\supset_c$, $\supset_*$]-conversion, where $\bullet$ [$\circ$] indicates that the combination $\mathcal{D}^*$ is legitimate [illegitimate] and its the $\star^i$-conversion into $\mathcal{D}^{**}$ successful [precluded]. In case

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Code:	Violation:	Operator:
$V1$:	non-independent assumption of a non-$L0$-formula (respecting the convention in Definition 3.2)	I, D
$V2a$:	clash of assumption with AP2	I
$V2b$:	clash of assumption with AP3	D
$V3$:	minor premiss of $\supset_f$E has no factual status	I
$V4$:	minor premiss of $\supset_c$E has no counterfactual status	I
$V5a$:	presence of empty discharge (cf. SC1/2.a)	I
$V5b$:	presence of unselected conjunct (cf. SC1/2.b)	I, C
$V5c$:	presence of antecedent of introduced disjunct (cf. SC1/2.c)	I, D
$V6$:	break formula present (cf. SC3)	I
$V7$:	a premiss of $\&_f$I has no factual status	C
$V8$:	a premiss of $\&_c$I has no counterfactual status	C
$V9$:	premisses of $\&_\star$I differ in status	C
$V10$:	premiss of $\vee_f$I has no factual status	D
$V11$:	premiss of $\vee_c$I has no counterfactual status	D

Figure 1: Violations

of illegitimacy, the kind of violation is indicated. And for each kind of conversion, the number of illegitimate derivations (**I**-*number*) is listed. Likewise for B- and C-tables.

Tables A.1-3: In case $\mathcal{D}^a$ ends with $\star^i$E, there are two entries: The first [second] entry in the (c)- and (d)-cases indicates the result for the construction in which the $\star^i$-maximum is introduced discharging an assumption which is used to derive the major [minor] premiss of that $\star^i$E-application.

Tables A.4-6: All cases are single entry.

Tables A.7-9: In case $\mathcal{D}^a$ ends with $\star^d$E, there are two entries: The first [second] entry in the (c)- and (d)-cases indicates the result for the construction in which the $\star^i$-maximum is introduced discharging an assumption which is used to derive the major [a minor] premiss of that $\star^d$E-application.

Example 4.1. The examples given for the A-tables, as well as for the B- and C-tables further below, visualize some of the cases. They have been selected, since they belong to those in which implications, conjunctions, and disjunctions interact. In case there is no such interaction, the example has been chosen in order to illustrate a kind of violation that is not present in the interacting examples. Where applicable and convenient, a *cumulative notation* for applications of minor-premiss rules has been used:

$$\text{(a)}\ \dfrac{\overset{|\mathcal{D}_1}{A \star^i B} \qquad \overset{\mathcal{D}_2|}{A}}{B}\,(\star^i\text{E}) \qquad \text{(b)}\ \dfrac{\overset{\wr\mathcal{D}_1}{A \star^i B} \qquad \overset{\mathcal{D}_2\wr}{A}}{B}\,(\star^i\text{E}) \qquad \text{(c)}\ \dfrac{\overset{(\mathcal{D}_1}{A \star^i B} \qquad \overset{\mathcal{D}_2)}{A}}{B}\,(\star^i\text{E})$$

$$\text{(d)}\ \dfrac{\overset{|\mathcal{D}_1}{A \star^d B} \qquad \overset{\overset{[/A/]^{(u)}}{\mathcal{D}_2}}{C} \qquad \overset{\overset{[/B/]^{(v)}}{\mathcal{D}_3|}}{C}}{C}\,(\star^d\text{E}), u, v \qquad \text{(e)}\ \dfrac{\overset{\wr\mathcal{D}_1}{A \star^d B} \qquad \overset{\overset{[/A/]^{(u)}}{\mathcal{D}_2}}{C} \qquad \overset{\overset{[/B/]^{(v)}}{\mathcal{D}_3\wr}}{C}}{C}\,(\star^d\text{E}), u, v$$

$$\text{(f)}\ \dfrac{\overset{(\mathcal{D}_1}{A \star^d B} \qquad \overset{\overset{[/A/]^{(u)}}{\mathcal{D}_2}}{C} \qquad \overset{\overset{[/B/]^{(v)}}{\mathcal{D}_3)}}{C}}{C}\,(\star^d\text{E}), u, v$$

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A.1 (III)	$\mathcal{D}^a$:	$\mathcal{D}^b$:	α: $\supset_f$-c	β: $\supset_c$-c	γ: $\supset_*$-c
A.1.1.a	$\supset_f\mathrm{I}$	$\supset_f\mathrm{I}$	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.1.1.b	$\supset_f\mathrm{I}$	$\supset_f\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.1.1.c	$\supset_f\mathrm{E}$	$\supset_f\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1,3}$	$\bullet \mid \bullet$
A.1.1.d	$\supset_f\mathrm{E}$	$\supset_f\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.1.2.a	$\supset_f\mathrm{I}$	$\supset_c\mathrm{I}$	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.1.2.b	$\supset_f\mathrm{I}$	$\supset_c\mathrm{E}$	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.1.2.c	$\supset_f\mathrm{E}$	$\supset_c\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1,3}$	$\bullet \mid \bullet$
A.1.2.d	$\supset_f\mathrm{E}$	$\supset_c\mathrm{E}$	$\circ^{2a,3} \mid \circ^{2a,3}$	$\circ^{6} \mid \circ^{3,6}$	$\circ^{6} \mid \circ^{2a,6}$
A.1.3.a	$\supset_f\mathrm{I}$	$\supset_*\mathrm{I}$	$\bullet$	$\bullet$	$\bullet$
A.1.3.b	$\supset_f\mathrm{I}$	$\supset_*\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.1.3.c	$\supset_f\mathrm{E}$	$\supset_*\mathrm{I}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.1.3.d	$\supset_f\mathrm{E}$	$\supset_*\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.1.4.a	$\supset_c\mathrm{I}$	$\supset_f\mathrm{I}$	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.1.4.b	$\supset_c\mathrm{I}$	$\supset_f\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.1.4.c	$\supset_c\mathrm{E}$	$\supset_f\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.1.4.d	$\supset_c\mathrm{E}$	$\supset_f\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.1.5.a	$\supset_c\mathrm{I}$	$\supset_c\mathrm{I}$	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.1.5.b	$\supset_c\mathrm{I}$	$\supset_c\mathrm{E}$	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.1.5.c	$\supset_c\mathrm{E}$	$\supset_c\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.1.5.d	$\supset_c\mathrm{E}$	$\supset_c\mathrm{E}$	$\circ^{2a,3} \mid \circ^{2a,3}$	$\circ^{6} \mid \circ^{6}$	$\circ^{6} \mid \circ^{6}$
A.1.6.a	$\supset_c\mathrm{I}$	$\supset_*\mathrm{I}$	$\bullet$	$\bullet$	$\bullet$
A.1.6.b	$\supset_c\mathrm{I}$	$\supset_*\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.1.6.c	$\supset_c\mathrm{E}$	$\supset_*\mathrm{I}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.1.6.d	$\supset_c\mathrm{E}$	$\supset_*\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.1.7.a	$\supset_*\mathrm{I}$	$\supset_f\mathrm{I}$	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.1.7.b	$\supset_*\mathrm{I}$	$\supset_f\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.1.7.c	$\supset_*\mathrm{E}$	$\supset_f\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.1.7.d	$\supset_*\mathrm{E}$	$\supset_f\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.1.8.a	$\supset_*\mathrm{I}$	$\supset_c\mathrm{I}$	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.1.8.b	$\supset_*\mathrm{I}$	$\supset_c\mathrm{E}$	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.1.8.c	$\supset_*\mathrm{E}$	$\supset_c\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.1.8.d	$\supset_*\mathrm{E}$	$\supset_c\mathrm{E}$	$\circ^{2a,3} \mid \circ^{2a,3}$	$\circ^{6} \mid \circ^{6}$	$\circ^{6} \mid \circ^{6}$
A.1.9.a	$\supset_*\mathrm{I}$	$\supset_*\mathrm{I}$	$\bullet$	$\bullet$	$\bullet$
A.1.9.b	$\supset_*\mathrm{I}$	$\supset_*\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.1.9.c	$\supset_*\mathrm{E}$	$\supset_*\mathrm{I}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.1.9.d	$\supset_*\mathrm{E}$	$\supset_*\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I27	**I**30	**I**9

A.2 (IIC)	$\mathcal{D}^a$:	$\mathcal{D}^b$:	α: $\supset_f$-c	β: $\supset_c$-c	γ: $\supset_*$-c
A.2.1.a	$\supset_f\mathrm{I}$	$\&_f\mathrm{I}$	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.2.1.b	$\supset_f\mathrm{I}$	$\&_f\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.1.c	$\supset_f\mathrm{E}$	$\&_f\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2a,4} \mid \circ^{1,2a,3,4}$	$\circ^{2a} \mid \circ^{2a}$
A.2.1.d	$\supset_f\mathrm{E}$	$\&_f\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.2.2.a	$\supset_f\mathrm{I}$	$\&_c\mathrm{I}$	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.2.2.b	$\supset_f\mathrm{I}$	$\&_c\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.2.c	$\supset_f\mathrm{E}$	$\&_c\mathrm{I}$	$\circ^{1,2a,3} \mid \circ^{1,2a,3}$	$\circ^{1} \mid \circ^{1,3}$	$\circ^{2a} \mid \circ^{2a}$
A.2.2.d	$\supset_f\mathrm{E}$	$\&_c\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.2.3.a	$\supset_f\mathrm{I}$	$\&_*\mathrm{I}$	$\bullet$	$\bullet$	$\bullet$
A.2.3.b	$\supset_f\mathrm{I}$	$\&_*\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.3.c	$\supset_f\mathrm{E}$	$\&_*\mathrm{I}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.2.3.d	$\supset_f\mathrm{E}$	$\&_*\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
A.2.4.a	$\supset_c\mathrm{I}$	$\&_f\mathrm{I}$	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.2.4.b	$\supset_c\mathrm{I}$	$\&_f\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.4.c	$\supset_c\mathrm{E}$	$\&_f\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2a,4} \mid \circ^{1,2a,4}$	$\circ^{2a} \mid \circ^{2a}$
A.2.4.d	$\supset_c\mathrm{E}$	$\&_f\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.5.a	$\supset_c\mathrm{I}$	$\&_c\mathrm{I}$	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.2.5.b	$\supset_c\mathrm{I}$	$\&_c\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.5.c	$\supset_c\mathrm{E}$	$\&_c\mathrm{I}$	$\circ^{1,2a,3} \mid \circ^{1,2a,3}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2a} \mid \circ^{2a}$
A.2.5.d	$\supset_c\mathrm{E}$	$\&_c\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.6.a	$\supset_c\mathrm{I}$	$\&_*\mathrm{I}$	$\bullet$	$\bullet$	$\bullet$
A.2.6.b	$\supset_c\mathrm{I}$	$\&_*\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.6.c	$\supset_c\mathrm{E}$	$\&_*\mathrm{I}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.6.d	$\supset_c\mathrm{E}$	$\&_*\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.7.a	$\supset_*\mathrm{I}$	$\&_f\mathrm{I}$	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.2.7.b	$\supset_*\mathrm{I}$	$\&_f\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.7.c	$\supset_*\mathrm{E}$	$\&_f\mathrm{I}$	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2a,4} \mid \circ^{1,2a,4}$	$\circ^{2a} \mid \circ^{2a}$
A.2.7.d	$\supset_*\mathrm{E}$	$\&_f\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.8.a	$\supset_*\mathrm{I}$	$\&_c\mathrm{I}$	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.2.8.b	$\supset_*\mathrm{I}$	$\&_c\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.8.c	$\supset_*\mathrm{E}$	$\&_c\mathrm{I}$	$\circ^{1,2a,3} \mid \circ^{1,2a,3}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2a} \mid \circ^{2a}$
A.2.8.d	$\supset_*\mathrm{E}$	$\&_c\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.9.a	$\supset_*\mathrm{I}$	$\&_*\mathrm{I}$	$\bullet$	$\bullet$	$\bullet$
A.2.9.b	$\supset_*\mathrm{I}$	$\&_*\mathrm{E}$	$\bullet$	$\bullet$	$\bullet$
A.2.9.c	$\supset_*\mathrm{E}$	$\&_*\mathrm{I}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.2.9.d	$\supset_*\mathrm{E}$	$\&_*\mathrm{E}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I18	**I**22	**I**18

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A.4 (ICI)	$\mathcal{D}^a$:	$\mathcal{D}^b$:	α: $\supset_f$-c	β: $\supset_c$-c	γ: $\supset_*$-c
A.4.1.a	$\&_f$I	$\supset_f$I	$\circ^{1}$	$\circ^{1,7}$	$\bullet$
A.4.1.b	$\&_f$I	$\supset_f$E	$\bullet$	$\circ^{7}$	$\bullet$
A.4.1.c	$\&_f$E	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.1.d	$\&_f$E	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.4.2.a	$\&_f$I	$\supset_c$I	$\circ^{1}$	$\circ^{1,7}$	$\bullet$
A.4.2.b	$\&_f$I	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6,7}$	$\circ^{2a,6}$
A.4.2.c	$\&_f$E	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.2.d	$\&_f$E	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.4.3.a	$\&_f$I	$\supset_*$I	$\bullet$	$\circ^{7}$	$\bullet$
A.4.3.b	$\&_f$I	$\supset_*$E	$\bullet$	$\circ^{7}$	$\bullet$
A.4.3.c	$\&_f$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.4.3.d	$\&_f$E	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
A.4.4.a	$\&_c$I	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.4.b	$\&_c$I	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.4.4.c	$\&_c$E	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.4.d	$\&_c$E	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.4.5.a	$\&_c$I	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.5.b	$\&_c$I	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.4.5.c	$\&_c$E	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.5.d	$\&_c$E	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.4.6.a	$\&_c$I	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.4.6.b	$\&_c$I	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
A.4.6.c	$\&_c$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.4.6.d	$\&_c$E	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
A.4.7.a	$\&_*$I	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.7.b	$\&_*$I	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.4.7.c	$\&_*$E	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.7.d	$\&_*$E	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.4.8.a	$\&_*$I	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.8.b	$\&_*$I	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.4.8.c	$\&_*$E	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.4.8.d	$\&_*$E	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.4.9.a	$\&_*$I	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.4.9.b	$\&_*$I	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
A.4.9.c	$\&_*$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.4.9.d	$\&_*$E	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
			I18	**I**21	**I**6

A.5 (ICC)	$\mathcal{D}^a$:	$\mathcal{D}^b$:	α: $\supset_f$-c	β: $\supset_c$-c	γ: $\supset_*$-c
A.5.1.a	$\&_f$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4,7}$	$\circ^{2a}$
A.5.1.b	$\&_f$I	$\&_f$E	$\bullet$	$\circ^{7}$	$\bullet$
A.5.1.c	$\&_f$E	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.5.1.d	$\&_f$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.5.2.a	$\&_f$I	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1,7}$	$\circ^{2a}$
A.5.2.b	$\&_f$I	$\&_c$E	$\bullet$	$\circ^{7}$	$\bullet$
A.5.2.c	$\&_f$E	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.5.2.d	$\&_f$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.5.3.a	$\&_f$I	$\&_*$I	$\bullet$	$\circ^{7}$	$\bullet$
A.5.3.b	$\&_f$I	$\&_*$E	$\bullet$	$\circ^{7}$	$\bullet$
A.5.3.c	$\&_f$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.5.3.d	$\&_f$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
A.5.4.a	$\&_c$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.5.4.b	$\&_c$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.5.4.c	$\&_c$E	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.5.4.d	$\&_c$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.5.5.a	$\&_c$I	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.5.5.b	$\&_c$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.5.5.c	$\&_c$E	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.5.5.d	$\&_c$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.5.6.a	$\&_c$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.5.6.b	$\&_c$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
A.5.6.c	$\&_c$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.5.6.d	$\&_c$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
A.5.7.a	$\&_*$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.5.7.b	$\&_*$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.5.7.c	$\&_*$E	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.5.7.d	$\&_*$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.5.8.a	$\&_*$I	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.5.8.b	$\&_*$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.5.8.c	$\&_*$E	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.5.8.d	$\&_*$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.5.9.a	$\&_*$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.5.9.b	$\&_*$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
A.5.9.c	$\&_*$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.5.9.d	$\&_*$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
			I12	**I**16	**I**12

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A.7 (IDI)	$\mathcal{D}^a$:	$\mathcal{D}^b$:	α: $\supset_f$-c	β: $\supset_c$-c	γ: $\supset_*$-c
A.7.1.a	$\vee_f$I	$\supset_f$I	$\circ^{1}$	$\circ^{1,10}$	$\bullet$
A.7.1.b	$\vee_f$I	$\supset_f$E	$\bullet$	$\circ^{10}$	$\bullet$
A.7.1.c	$\vee_f$E	$\supset_f$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.7.1.d	$\vee_f$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.2.a	$\vee_f$I	$\supset_c$I	$\circ^{1}$	$\circ^{1,10}$	$\bullet$
A.7.2.b	$\vee_f$I	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6,10}$	$\circ^{2a,6}$
A.7.2.c	$\vee_f$E	$\supset_c$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.7.2.d	$\vee_f$E	$\supset_c$E	$\circ^{2a,3} \mid \circ^{2a,3}$	$\circ^{6} \mid \circ^{6}$	$\circ^{6} \mid \circ^{6}$
A.7.3.a	$\vee_f$I	$\supset_*$I	$\bullet$	$\circ^{10}$	$\bullet$
A.7.3.b	$\vee_f$I	$\supset_*$E	$\bullet$	$\circ^{10}$	$\bullet$
A.7.3.c	$\vee_f$E	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.3.d	$\vee_f$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.4.a	$\vee_c$I	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.7.4.b	$\vee_c$I	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.7.4.c	$\vee_c$E	$\supset_f$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.7.4.d	$\vee_c$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.5.a	$\vee_c$I	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.7.5.b	$\vee_c$I	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.7.5.c	$\vee_c$E	$\supset_c$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.7.5.d	$\vee_c$E	$\supset_c$E	$\circ^{2a,3} \mid \circ^{2a,3}$	$\circ^{6} \mid \circ^{6}$	$\circ^{6} \mid \circ^{6}$
A.7.6.a	$\vee_c$I	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.7.6.b	$\vee_c$I	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
A.7.6.c	$\vee_c$E	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.6.d	$\vee_c$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.7.a	$\vee_*$I	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.7.7.b	$\vee_*$I	$\supset_f$E	$\bullet$	$\bullet$	$\bullet$
A.7.7.c	$\vee_*$E	$\supset_f$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.7.7.d	$\vee_*$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.8.a	$\vee_*$I	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
A.7.8.b	$\vee_*$I	$\supset_c$E	$\circ^{2a,3}$	$\circ^{6}$	$\circ^{6}$
A.7.8.c	$\vee_*$E	$\supset_c$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
A.7.8.d	$\vee_*$E	$\supset_c$E	$\circ^{2a,3} \mid \circ^{2a,3}$	$\circ^{6} \mid \circ^{6}$	$\circ^{6} \mid \circ^{6}$
A.7.9.a	$\vee_*$I	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
A.7.9.b	$\vee_*$I	$\supset_*$E	$\bullet$	$\bullet$	$\bullet$
A.7.9.c	$\vee_*$E	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.7.9.d	$\vee_*$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I27	**I**30	**I**9

A.8 (IDC)	$\mathcal{D}^a$:	$\mathcal{D}^b$:	α: $\supset_f$-c	β: $\supset_c$-c	γ: $\supset_*$-c
A.8.1.a	$\vee_f$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4,10}$	$\circ^{2a}$
A.8.1.b	$\vee_f$I	$\&_f$E	$\bullet$	$\circ^{10}$	$\bullet$
A.8.1.c	$\vee_f$E	$\&_f$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2a,4} \mid \circ^{1,2a,4}$	$\circ^{2a} \mid \circ^{2a}$
A.8.1.d	$\vee_f$E	$\&_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.2.a	$\vee_f$I	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1,10}$	$\circ^{2a}$
A.8.2.b	$\vee_f$I	$\&_c$E	$\bullet$	$\circ^{10}$	$\bullet$
A.8.2.c	$\vee_f$E	$\&_c$I	$\circ^{1,2a,3} \mid \circ^{1,2a,3}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2a} \mid \circ^{2a}$
A.8.2.d	$\vee_f$E	$\&_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.3.a	$\vee_f$I	$\&_*$I	$\bullet$	$\circ^{10}$	$\bullet$
A.8.3.b	$\vee_f$I	$\&_*$E	$\bullet$	$\circ^{10}$	$\bullet$
A.8.3.c	$\vee_f$E	$\&_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.3.d	$\vee_f$E	$\&_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.4.a	$\vee_c$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.8.4.b	$\vee_c$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.8.4.c	$\vee_c$E	$\&_f$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2a,4} \mid \circ^{1,2a,4}$	$\circ^{2a} \mid \circ^{2a}$
A.8.4.d	$\vee_c$E	$\&_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.5.a	$\vee_c$I	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.8.5.b	$\vee_c$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.8.5.c	$\vee_c$E	$\&_c$I	$\circ^{1,2a,3} \mid \circ^{1,2a,3}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2a} \mid \circ^{2a}$
A.8.5.d	$\vee_c$E	$\&_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.6.a	$\vee_c$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.8.6.b	$\vee_c$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
A.8.6.c	$\vee_c$E	$\&_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.6.d	$\vee_c$E	$\&_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.7.a	$\vee_*$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2a,4}$	$\circ^{2a}$
A.8.7.b	$\vee_*$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
A.8.7.c	$\vee_*$E	$\&_f$I	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2a,4} \mid \circ^{1,2a,4}$	$\circ^{2a} \mid \circ^{2a}$
A.8.7.d	$\vee_*$E	$\&_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.8.a	$\vee_*$I	$\&_c$I	$\circ^{1,2a,3}$	$\circ^{1}$	$\circ^{2a}$
A.8.8.b	$\vee_*$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
A.8.8.c	$\vee_*$E	$\&_c$I	$\circ^{1,2a,3} \mid \circ^{1,2a,3}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2a} \mid \circ^{2a}$
A.8.8.d	$\vee_*$E	$\&_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.9.a	$\vee_*$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
A.8.9.b	$\vee_*$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
A.8.9.c	$\vee_*$E	$\&_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
A.8.9.d	$\vee_*$E	$\&_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I18	**I**22	**I**18

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For example, $|\mathcal{D}_1, \mathcal{D}_2|$ in (a) means that the premisses taken together have factual status; as a consequence, one of them may have independent status. Similarly in the remaining cases. Likewise with negated status markers. The cumulative notation can be combined with the discharge of assumptions.

1. *Case A.4.2.b.β*: $\mathcal{D}^a$ ends with $\&_f$I, $\mathcal{D}^b$ ends with $\supset_c$E: $\circ^{6,7}$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\wr A \wr]^{(u)}\\ /\wr\mathcal{D}_1\wr/\\ B\end{matrix}\qquad \begin{matrix}|\mathcal{D}_2|\\ C\end{matrix}}{B\&_f C}(\&_f\mathrm{I})}{A\supset_c(B\&_f C)}(\supset_c\mathrm{I}),\ u \qquad \dfrac{\begin{matrix}/\mathcal{D}_3/\\ D\supset_c A\end{matrix}\qquad \begin{matrix}\wr\mathcal{D}_4\wr\\ D\end{matrix}}{A}(\supset_c\mathrm{E})}{B\&_f C}(\supset_c\mathrm{E}) \tag{4}$$

2. *Case A.6.6.a.α*: $\mathcal{D}^a$ ends with $\&_c$I, $\mathcal{D}^b$ ends with $\vee_*$I: $\bullet$

$$\dfrac{\dfrac{\begin{matrix}[|A\vee_* B|]^{(u)}\\ \\ \end{matrix}\dfrac{\begin{matrix}\wr\mathcal{D}_1\wr\\ C\end{matrix}\qquad\begin{matrix}\wr\mathcal{D}_2\wr\\ D\end{matrix}}{C\&_c D}(\&_c\mathrm{I})}{(A\vee_* B)\supset_f(C\&_c D)}(\supset_f\mathrm{I}),\ u\qquad \dfrac{\begin{matrix}|\mathcal{D}_3|\\ A\end{matrix}}{A\vee_* B}(\vee_*\mathrm{I1})}{C\&_c D}(\supset_f\mathrm{E}) \quad \mathit{conv} \quad \begin{matrix}|\mathcal{D}_3|\\ \dfrac{A}{A\vee_* B}(\vee_*\mathrm{I1})\\ \dfrac{\begin{matrix}\wr\mathcal{D}_1\wr\\ C\end{matrix}\qquad\begin{matrix}\wr\mathcal{D}_2\wr\\ D\end{matrix}}{C\&_c D}(\&_c\mathrm{I})\end{matrix} \tag{5}$$

3. *Case A.6.6.d.α*: $\mathcal{D}^a$ ends with $\&_c$E, $\mathcal{D}^b$ ends with $\vee_*$E: $\bullet$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[|A|]^{(u)}\\ /\{\!|\,\mathcal{D}_1\,|\!\}/\\ B\&_c C\end{matrix}}{B}(\&_c\mathrm{E1})}{A\supset_f B}(\supset_f\mathrm{I}),\ u \qquad \dfrac{\begin{matrix}|\mathcal{D}_2\\ D\vee_* E\end{matrix}\qquad\begin{matrix}[/D/]^{(v)}\\ \mathcal{D}_3\\ A\end{matrix}\qquad\begin{matrix}[/E/]^{(w)}\\ \mathcal{D}_4|\\ A\end{matrix}}{A}(\vee_*\mathrm{E}),\ v,\ w}{B}(\supset_f\mathrm{E}) \quad \text{conv}$$

$$\begin{matrix}\dfrac{\begin{matrix}|\mathcal{D}_2\\ D\vee_* E\end{matrix}\qquad\begin{matrix}[/D/]^{(v)}\\ \mathcal{D}_3\\ A\end{matrix}\qquad\begin{matrix}[/E/]^{(w)}\\ \mathcal{D}_4|\\ A\end{matrix}}{A}(\vee_*\mathrm{E}),\ v,\ w\\ \{\!|\,\mathcal{D}_1\,|\!\}\\ \dfrac{B\&_c C}{B}(\&_c\mathrm{E1})\end{matrix} \tag{6}$$

4. *Case A.8.7.c.β*: $\mathcal{D}^a$ ends with $\vee_*$E, $\mathcal{D}^b$ ends with $\&_f$I: (c1): $\circ^{1,2a,4}$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\wr A\&_f B\wr]^{(u)}\\ /\wr\mathcal{D}_1\wr/\\ C\vee_* D\end{matrix}\qquad\begin{matrix}[/C/]^{(v)}\\ /\mathcal{D}_2/\\ E\end{matrix}\qquad\begin{matrix}[/D/]^{(w)}\\ /\mathcal{D}_3/\\ E\end{matrix}}{E}(\vee_*\mathrm{E}),\ v,\ w}{(A\&_f B)\supset_c E}(\supset_c\mathrm{I}),\ u\qquad \dfrac{\begin{matrix}|\mathcal{D}_4|\\ A\end{matrix}\qquad\begin{matrix}|\mathcal{D}_5|\\ B\end{matrix}}{A\&_f B}(\&_f\mathrm{I})}{E}(\supset_c\mathrm{E}) \tag{7}$$

(c2): $\circ^{1,2a,4}$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}/\mathcal{D}_1/\\ A\vee_* B\end{matrix}\qquad\begin{matrix}[/A/]^{(v)}\quad[\wr C\&_f D\wr]^{(u)}\\ /\wr\mathcal{D}_2\wr/\\ E\end{matrix}\qquad\begin{matrix}[/B/]^{(w)}\\ /\mathcal{D}_3/\\ E\end{matrix}}{E}(\vee_*\mathrm{E}),\ v,\ w}{(C\&_f D)\supset_c E}(\supset_c\mathrm{I}),\ u\qquad \dfrac{\begin{matrix}|\mathcal{D}_4|\\ C\end{matrix}\qquad\begin{matrix}|\mathcal{D}_5|\\ D\end{matrix}}{C\&_f D}(\&_f\mathrm{I})}{E}(\supset_c\mathrm{E}) \tag{8}$$

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Special cases (Part A). Let $\begin{matrix}/\mathcal{D}^a/\\ B\end{matrix}$ be $/B/$. Then:

$$\mathcal{D}^* = \cfrac{\cfrac{/B/}{A \star^i B}(\star^i\text{I}) \qquad \begin{matrix}/\mathcal{D}^b/\\ A\end{matrix}}{B}(\star^i\text{E}) \qquad \mathcal{D}^{**} = /B/ \tag{9}$$

This works only for $\star^i = \supset_*$. For $\star^i = \supset_f$ and $\star^i = \supset_c$, we get $V5a$; thus, unlike in the cases covered by the A-tables, we cannot presuppose that the $\star^i$I-rule is applied legitimately in these two special cases of $\mathcal{D}^*$. This presupposition can be made, since we deal with derivation schemes in the tables. In case the derivations are fully explicit, like in such special cases, this presupposition cannot be made.

Proof (Part B): Preservation for $\star^c$-detour conversions. Let $\begin{matrix}/\mathcal{D}^{a_1}/\\ A_1\end{matrix}$, $\begin{matrix}/\mathcal{D}^{a_2}/\\ A_2\end{matrix}$ be legitimate derivations. We show: If these derivations can be combined into a legitimate derivation $\mathcal{D}^*$, then the $\star^c$-conversions transform it into a legitimate derivation $\mathcal{D}^{**}$, where $i \in \{1, 2\}$; otherwise, the combination $\mathcal{D}^*$ is illegitimate and a $\star^c$-conversion precluded:

$$\mathcal{D}^* = \cfrac{\cfrac{\cfrac{\begin{matrix}/\mathcal{D}^{a_1}/\\ A_1\end{matrix} \quad \begin{matrix}/\mathcal{D}^{a_2}/\\ A_2\end{matrix}}{A_1 \star^c A_2}(\star^c\text{I})}{A_i}(\star^c\text{E}i)}{B}(\text{I/E-rule}) \qquad \mathcal{D}^{**} = \cfrac{\begin{matrix}/\mathcal{D}^{a_i}/\\ A_i\end{matrix}}{B}(\text{I/E-rule}) \tag{10}$$

We consider for all specific instances of last rules applied in $\begin{matrix}/\mathcal{D}^{a_1}/\\ A_1\end{matrix}$, $\begin{matrix}/\mathcal{D}^{a_2}/\\ A_2\end{matrix}$ their combination into $\mathcal{D}^*$. In each case we attempt to derive the instance of the maximum $A_1 \star^c A_2$ in the aforementioned top-down manner. For reasons of economy, we shall use $\star^c$E1 in eliminating the $\star^c$-maximum whenever possible. (A preference for $\star^c$E2 is possible and leads to the same results.) The results of the top-down procedure for the $\star^c$-conversions are collected in Tables B.1-9 which summarize Part B of the preservation proof (pp. 26-29). Anticipating Remark 4.2, B.8 and B.9 are omitted to save space.

Table B.1: In case both $\mathcal{D}^{a_i}$ and $\mathcal{D}^*$ end with $\star^i$I, there are two entries: The first entry in the (a)-cases indicates the result for the construction in which a conjunct of the $\star^c$-maximum is introduced by $\star^i$I and depends on an assumption which is discharged by the $\star^i$I-application which concludes $\mathcal{D}^*$. The second entry indicates the result for the construction in which a conjunct of the $\star^c$-maximum depends on an assumption which is discharged by the $\star^i$I-application which concludes $\mathcal{D}^*$. (Following the $\star^c$E1-convention, in the (a1)-cases, the $\star^c$-maximum is eliminated by means of $\star^c$E1 and by $\star^c$E2 in the (a2)-cases.) In case $\mathcal{D}^*$ ends with $\star^i$E, there are also two entries: The first [second] entry in the (b)- and (d)-cases indicates the result for the construction in which the $\star^c$-maximum is eliminated deriving the major [minor] premiss of that $\star^i$E-application. (Following the convention, in the (b1)-cases, the $\star^c$-maximum is eliminated by means of either $\star^c$E1 or $\star^c$E2, we therefore distinguish between (b1.1)-cases which apply $\star^c$E1 to the maximum and (b1.2)-cases which apply $\star^c$E2 to it. In the (b2)- and (d)-cases the maximum is eliminated by $\star^c$E1.) In case $\mathcal{D}^{a_i}$ ends with $\star^i$E and $\mathcal{D}^*$ with $\star^i$I, there are three entries: The first [second] entry in the (c)-cases indicates the result for the construction in which $\star^i$I discharges an assumption used to derive the major [minor] premiss of that $\star^i$E-application. The third entry indicates the result for the construction in which $\star^i$I discharges an assumption used to derive the second premiss of the $\star^c$I-application which introduces the $\star^c$-maximum. (In (c1) and (c2) $\star^c$E1 is used, in (c3) $\star^c$E2.)

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Tables B.2-3, B.5-6, B.8-9: All cases are single entry. B.2-3: In the (b)-cases, the $\star^c$-maximum is eliminated by means of $\star^c$E2; otherwise, we consider only its eliminations by $\star^c$E1. B.5: For the (b)-cases we have to distinguish (b.1)-cases which apply $\star^c$E1 to the $\star^c$-maximum from (b.2)-cases which apply $\star^c$E2 to it; otherwise, we consider only its eliminations by $\star^c$E1. The results of successful conversions in (b.1)-cases can be reduced further by $\star^c$-conversions. B.6 and B.8: Like B.2-3. B.9: Like B.5; however, the results of successful conversions in (b.1)-cases can be reduced further by $\star^d$-conversions.

Table B.4: In case both $\mathcal{D}^{a_i}$ and $\mathcal{D}^*$ end with $\star^i$I, there are two entries: The first entry in the (a)-cases indicates the result for the construction in which a conjunct of the $\star^c$-maximum is introduced by $\star^c$I and depends on an assumption which is discharged by the $\star^i$I-application which concludes $\mathcal{D}^*$. The second entry indicates the result for the construction in which a conjunct of the $\star^c$-maximum depends on an assumption which is discharged by the $\star^i$I-application which concludes $\mathcal{D}^*$. (Following the $\star^c$E1-convention, in the (a1)-cases, the $\star^c$-maximum is eliminated by means of $\star^c$E1 and by $\star^c$E2 in the (a2)-cases.) Thus, the situation is similar to the (a)-cases in B.1. In case $\mathcal{D}^*$ ends with $\star^i$E, there are also two entries: The first [second] entry in the (b)- and (d)-cases indicates the result for the construction in which the $\star^c$-maximum is eliminated deriving the major [minor] premiss of that $\star^i$E-application. (In the (b1)-cases, the maximum is eliminated by means of $\star^c$E2, in the (b2)- and (d)-cases by $\star^c$E1.) In case $\mathcal{D}^{a_i}$ ends with $\star^i$E and $\mathcal{D}^*$ with $\star^i$I, the situation is similar to the (a)-cases.

Table B.7: The situation here is largely analogous to B.4. In case both $\mathcal{D}^{a_i}$ and $\mathcal{D}^*$ end with $\star^i$I, there are two entries: The first entry in the (a)-cases indicates the result for the construction in which a conjunct of the $\star^c$-maximum is introduced by $\star^d$I and depends on an assumption which is discharged by the $\star^i$I-application which concludes $\mathcal{D}^*$. The second entry indicates the result for the construction in which a conjunct of the $\star^c$-maximum depends on an assumption which is discharged by the $\star^i$I-application which concludes $\mathcal{D}^*$. (In the (a1)-cases, the maximum is eliminated by means of $\star^c$E1 and by $\star^c$E2 in the (a2)-cases.) Thus, the situation here is similar to that of the (a)-cases in B.1 and B.4. In case $\mathcal{D}^*$ ends with $\star^i$E, there are also two entries: The first [second] entry in the (b)- and (d)-cases indicates the result for the construction in which the $\star^c$-maximum is eliminated deriving the major [minor] premiss of that $\star^i$E-application. (In the (b1)-cases, the maximum is eliminated by means of $\star^c$E2, in the (b2)- and (d)-cases by $\star^c$E1.) Unlike in B.4, there are three entries in (c)-cases. In case $\mathcal{D}^{a_i}$ ends with $\star^d$E and $\mathcal{D}^*$ with $\star^i$I, there are three entries, like in B.1: The first [second] entry in the (c)-cases indicates the result for the construction in which $\star^i$I discharges an assumption used to derive the major [minor] premiss of that $\star^d$E-application. The third entry indicates the result for the construction in which $\star^i$I discharges an assumption used to derive the second premiss of the $\star^c$I-application which introduces the maximum. (In (c1) and (c2) $\star^c$E1 is used, in (c3) $\star^c$E2.)

Example 4.2. 1. *Case B.3.4.a.γ*: $\mathcal{D}^{a_i}$ ends with $\supset_c$I, $\mathcal{D}^*$ ends with $\vee_f$I: •

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\wr A \wr]^{(u)}\\ |\wr \mathcal{D}_1 \wr|\\ B\end{matrix}}{A \supset_c B}\,(\supset_c\mathrm{I}),\, u \qquad \begin{matrix}|\mathcal{D}_2|\\ C\end{matrix}}{(A\supset_c B)\&_* C}\,(\&_*\mathrm{I})}{A\supset_c B}\,(\&_*\mathrm{E1})}{(A\supset_c B)\vee_f D}\,(\vee_f\mathrm{I1}) \quad conv \quad \dfrac{\dfrac{\begin{matrix}[\wr A \wr]^{(u)}\\ |\wr \mathcal{D}_1 \wr|\\ B\end{matrix}}{A \supset_c B}\,(\supset_c\mathrm{I}),\, u}{(A\supset_c B)\vee_f D}\,(\vee_f\mathrm{I1}) \tag{11}$$

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B.1 (CII)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.1.1.a	$\supset_f$I	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.1.b	$\supset_f$I	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.1.1.c	$\supset_f$E	$\supset_f$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.1.d	$\supset_f$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.1.2.a	$\supset_f$I	$\supset_c$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.2.b	$\supset_f$I	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.2.c	$\supset_f$E	$\supset_c$I	$\circ^7 \mid \circ^{3,7} \mid \circ^7$	$\bullet \mid \circ^3 \mid \bullet$	$\bullet \mid \circ^3 \mid \bullet$
B.1.2.d	$\supset_f$E	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.3.a	$\supset_f$I	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.3.b	$\supset_f$I	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.3.c	$\supset_f$E	$\supset_*$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.3.d	$\supset_f$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.4.a	$\supset_c$I	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.4.b	$\supset_c$I	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.1.4.c	$\supset_c$E	$\supset_f$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.4.d	$\supset_c$E	$\supset_f$E	$\circ^7 \mid \circ^{3,7}$	$\bullet \mid \circ^3$	$\bullet \mid \circ^3$
B.1.5.a	$\supset_c$I	$\supset_c$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.5.b	$\supset_c$I	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.5.c	$\supset_c$E	$\supset_c$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.5.d	$\supset_c$E	$\supset_c$E	$\circ^7 \mid \circ^{6,7}$	$\bullet \mid \circ^6$	$\bullet \mid \circ^6$
B.1.6.a	$\supset_c$I	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.6.b	$\supset_c$I	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.6.c	$\supset_c$E	$\supset_*$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.6.d	$\supset_c$E	$\supset_*$E	$\circ^7 \mid \circ^{6,7}$	$\bullet \mid \circ^6$	$\bullet \mid \circ^6$
B.1.7.a	$\supset_*$I	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.7.b	$\supset_*$I	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.1.7.c	$\supset_*$E	$\supset_f$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.7.d	$\supset_*$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.1.8.a	$\supset_*$I	$\supset_c$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.8.b	$\supset_*$I	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.8.c	$\supset_*$E	$\supset_c$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.8.d	$\supset_*$E	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.9.a	$\supset_*$I	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.9.b	$\supset_*$I	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.1.9.c	$\supset_*$E	$\supset_*$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.1.9.d	$\supset_*$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I32	**I**9	**I**4

B.2 (CIC)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.2.1.a	$\supset_f$I	$\&_f$I	$\bullet$	$\circ^7$	$\bullet$
B.2.1.b	$\supset_f$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.2.1.c	$\supset_f$E	$\&_f$I	$\bullet$	$\circ^7$	$\bullet$
B.2.1.d	$\supset_f$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.2.2.a	$\supset_f$I	$\&_c$I	$\circ^8$	$\bullet$	$\bullet$
B.2.2.b	$\supset_f$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.2.2.c	$\supset_f$E	$\&_c$I	$\circ^8$	$\bullet$	$\bullet$
B.2.2.d	$\supset_f$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.2.3.a	$\supset_f$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.2.3.b	$\supset_f$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.2.3.c	$\supset_f$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.2.3.d	$\supset_f$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.2.4.a	$\supset_c$I	$\&_f$I	$\bullet$	$\circ^7$	$\bullet$
B.2.4.b	$\supset_c$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.2.4.c	$\supset_c$E	$\&_f$I	$\circ^7$	$\circ^7$	$\circ^7$
B.2.4.d	$\supset_c$E	$\&_f$E	$\circ^7$	$\bullet$	$\bullet$
B.2.5.a	$\supset_c$I	$\&_c$I	$\circ^8$	$\bullet$	$\bullet$
B.2.5.b	$\supset_c$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.2.5.c	$\supset_c$E	$\&_c$I	$\circ^7$	$\bullet$	$\bullet$
B.2.5.d	$\supset_c$E	$\&_c$E	$\circ^7$	$\bullet$	$\bullet$
B.2.6.a	$\supset_c$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.2.6.b	$\supset_c$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.2.6.c	$\supset_c$E	$\&_*$I	$\circ^7$	$\bullet$	$\bullet$
B.2.6.d	$\supset_c$E	$\&_*$E	$\circ^7$	$\bullet$	$\bullet$
B.2.7.a	$\supset_*$I	$\&_f$I	$\bullet$	$\circ^7$	$\bullet$
B.2.7.b	$\supset_*$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.2.7.c	$\supset_*$E	$\&_f$I	$\bullet$	$\circ^7$	$\bullet$
B.2.7.d	$\supset_*$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.2.8.a	$\supset_*$I	$\&_c$I	$\circ^8$	$\bullet$	$\bullet$
B.2.8.b	$\supset_*$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.2.8.c	$\supset_*$E	$\&_c$I	$\circ^8$	$\bullet$	$\bullet$
B.2.8.d	$\supset_*$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.2.9.a	$\supset_*$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.2.9.b	$\supset_*$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.2.9.c	$\supset_*$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.2.9.d	$\supset_*$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
			I11	**I**6	**I**1

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B.3 (CID)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.3.1.a	$\supset_f$I	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.3.1.b	$\supset_f$I	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.3.1.c	$\supset_f$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.3.1.d	$\supset_f$E	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.3.2.a	$\supset_f$I	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.3.2.b	$\supset_f$I	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.3.2.c	$\supset_f$E	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.3.2.d	$\supset_f$E	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.3.3.a	$\supset_f$I	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.3.3.b	$\supset_f$I	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.3.3.c	$\supset_f$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.3.3.d	$\supset_f$E	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.3.4.a	$\supset_c$I	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.3.4.b	$\supset_c$I	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.3.4.c	$\supset_c$E	$\vee_f$I	$\circ^{7,10}$	$\circ^{10}$	$\circ^{10}$
B.3.4.d	$\supset_c$E	$\vee_f$E	$\circ^{7}$	$\bullet$	$\bullet$
B.3.5.a	$\supset_c$I	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.3.5.b	$\supset_c$I	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.3.5.c	$\supset_c$E	$\vee_c$I	$\circ^{7}$	$\bullet$	$\bullet$
B.3.5.d	$\supset_c$E	$\vee_c$E	$\circ^{7}$	$\bullet$	$\bullet$
B.3.6.a	$\supset_c$I	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.3.6.b	$\supset_c$I	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.3.6.c	$\supset_c$E	$\vee_*$I	$\circ^{7}$	$\bullet$	$\bullet$
B.3.6.d	$\supset_c$E	$\vee_*$E	$\circ^{7}$	$\bullet$	$\bullet$
B.3.7.a	$\supset_*$I	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.3.7.b	$\supset_*$I	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.3.7.c	$\supset_*$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.3.7.d	$\supset_*$E	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.3.8.a	$\supset_*$I	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.3.8.b	$\supset_*$I	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.3.8.c	$\supset_*$E	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.3.8.d	$\supset_*$E	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.3.9.a	$\supset_*$I	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.3.9.b	$\supset_*$I	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.3.9.c	$\supset_*$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.3.9.d	$\supset_*$E	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
			I11	**I**6	**I**1

B.4 (CCI)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.4.1.a	$\&_f$I	$\supset_f$I	$\bullet \mid \bullet$	$\circ^{8} \mid \circ^{8}$	$\bullet \mid \bullet$
B.4.1.b	$\&_f$I	$\supset_f$E	$\bullet \mid \bullet$	$\circ^{8} \mid \circ^{3,8}$	$\bullet \mid \bullet$
B.4.1.c	$\&_f$E	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.1.d	$\&_f$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
B.4.2.a	$\&_f$I	$\supset_c$I	$\circ^{7} \mid \circ^{7}$	$\circ^{7} \mid \circ^{8}$	$\circ^{7} \mid \circ^{9}$
B.4.2.b	$\&_f$I	$\supset_c$E	$\bullet \mid \circ^{4}$	$\circ^{8} \mid \circ^{8}$	$\bullet \mid \circ^{4}$
B.4.2.c	$\&_f$E	$\supset_c$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.2.d	$\&_f$E	$\supset_c$E	$\bullet \mid \circ^{4}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.3.a	$\&_f$I	$\supset_*$I	$\bullet \mid \bullet$	$\circ^{8} \mid \circ^{8}$	$\bullet \mid \bullet$
B.4.3.b	$\&_f$I	$\supset_*$E	$\bullet \mid \bullet$	$\circ^{8} \mid \circ^{8}$	$\bullet \mid \bullet$
B.4.3.c	$\&_f$E	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.3.d	$\&_f$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.4.a	$\&_c$I	$\supset_f$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.4.b	$\&_c$I	$\supset_f$E	$\circ^{7} \mid \circ^{3,7}$	$\bullet \mid \circ^{3}$	$\bullet \mid \circ^{3}$
B.4.4.c	$\&_c$E	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.4.d	$\&_c$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
B.4.5.a	$\&_c$I	$\supset_c$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.5.b	$\&_c$I	$\supset_c$E	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.5.c	$\&_c$E	$\supset_c$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.5.d	$\&_c$E	$\supset_c$E	$\bullet \mid \circ^{4}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.6.a	$\&_c$I	$\supset_*$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.6.b	$\&_c$I	$\supset_*$E	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.6.c	$\&_c$E	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.6.d	$\&_c$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.7.a	$\&_*$I	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.7.b	$\&_*$I	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
B.4.7.c	$\&_*$E	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.7.d	$\&_*$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
B.4.8.a	$\&_*$I	$\supset_c$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.8.b	$\&_*$I	$\supset_c$E	$\bullet \mid \circ^{4}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.8.c	$\&_*$E	$\supset_c$I	$\circ^{7} \mid \circ^{7}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.8.d	$\&_*$E	$\supset_c$E	$\bullet \mid \circ^{4}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.9.a	$\&_*$I	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.9.b	$\&_*$I	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.9.c	$\&_*$E	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.4.9.d	$\&_*$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I27	**I**17	**I**4

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B.5 (CCC)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.5.1.a	$\&_f$I	$\&_f$I	$\bullet$	$\circ^{7,8}$	$\bullet$
B.5.1.b	$\&_f$I	$\&_f$E	$\bullet$	$\circ^{8}$	$\bullet$
B.5.1.c	$\&_f$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
B.5.1.d	$\&_f$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.5.2.a	$\&_f$I	$\&_c$I	$\circ^{8}$	$\circ^{8}$	$\circ^{8}$
B.5.2.b	$\&_f$I	$\&_c$E	$\bullet$	$\circ^{8}$	$\bullet$
B.5.2.c	$\&_f$E	$\&_c$I	$\circ^{8}$	$\bullet$	$\bullet$
B.5.2.d	$\&_f$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.5.3.a	$\&_f$I	$\&_*$I	$\bullet$	$\circ^{8}$	$\bullet$
B.5.3.b	$\&_f$I	$\&_*$E	$\bullet$	$\circ^{8}$	$\bullet$
B.5.3.c	$\&_f$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.5.3.d	$\&_f$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.5.4.a	$\&_c$I	$\&_f$I	$\circ^{7}$	$\circ^{7}$	$\circ^{7}$
B.5.4.b	$\&_c$I	$\&_f$E	$\circ^{7}$	$\bullet$	$\bullet$
B.5.4.c	$\&_c$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
B.5.4.d	$\&_c$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.5.5.a	$\&_c$I	$\&_c$I	$\circ^{7}$	$\bullet$	$\bullet$
B.5.5.b	$\&_c$I	$\&_c$E	$\circ^{7}$	$\bullet$	$\bullet$
B.5.5.c	$\&_c$E	$\&_c$I	$\circ^{8}$	$\bullet$	$\bullet$
B.5.5.d	$\&_c$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.5.6.a	$\&_c$I	$\&_*$I	$\circ^{7}$	$\bullet$	$\bullet$
B.5.6.b	$\&_c$I	$\&_*$E	$\circ^{7}$	$\bullet$	$\bullet$
B.5.6.c	$\&_c$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.5.6.d	$\&_c$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.5.7.a	$\&_*$I	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
B.5.7.b	$\&_*$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.5.7.c	$\&_*$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
B.5.7.d	$\&_*$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
B.5.8.a	$\&_*$I	$\&_c$I	$\circ^{8}$	$\bullet$	$\bullet$
B.5.8.b	$\&_*$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.5.8.c	$\&_*$E	$\&_c$I	$\circ^{8}$	$\bullet$	$\bullet$
B.5.8.d	$\&_*$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
B.5.9.a	$\&_*$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.5.9.b	$\&_*$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
B.5.9.c	$\&_*$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
B.5.9.d	$\&_*$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
			I11	**I**11	**I**2

B.6 (CCD)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.6.1.a	$\&_f$I	$\vee_f$I	$\bullet$	$\circ^{8,10}$	$\bullet$
B.6.1.b	$\&_f$I	$\vee_f$E	$\bullet$	$\circ^{8}$	$\bullet$
B.6.1.c	$\&_f$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.6.1.d	$\&_f$E	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.6.2.a	$\&_f$I	$\vee_c$I	$\circ^{11}$	$\circ^{8}$	$\circ^{11}$
B.6.2.b	$\&_f$I	$\vee_c$E	$\bullet$	$\circ^{8}$	$\bullet$
B.6.2.c	$\&_f$E	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.6.2.d	$\&_f$E	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.6.3.a	$\&_f$I	$\vee_*$I	$\bullet$	$\circ^{8}$	$\bullet$
B.6.3.b	$\&_f$I	$\vee_*$E	$\bullet$	$\circ^{8}$	$\bullet$
B.6.3.c	$\&_f$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.6.3.d	$\&_f$E	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.6.4.a	$\&_c$I	$\vee_f$I	$\circ^{7,10}$	$\circ^{10}$	$\circ^{10}$
B.6.4.b	$\&_c$I	$\vee_f$E	$\circ^{7}$	$\bullet$	$\bullet$
B.6.4.c	$\&_c$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.6.4.d	$\&_c$E	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.6.5.a	$\&_c$I	$\vee_c$I	$\circ^{7}$	$\bullet$	$\bullet$
B.6.5.b	$\&_c$I	$\vee_c$E	$\circ^{7}$	$\bullet$	$\bullet$
B.6.5.c	$\&_c$E	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.6.5.d	$\&_c$E	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.6.6.a	$\&_c$I	$\vee_*$I	$\circ^{7}$	$\bullet$	$\bullet$
B.6.6.b	$\&_c$I	$\vee_*$E	$\circ^{7}$	$\bullet$	$\bullet$
B.6.6.c	$\&_c$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.6.6.d	$\&_c$E	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.6.7.a	$\&_*$I	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.6.7.b	$\&_*$I	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.6.7.c	$\&_*$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
B.6.7.d	$\&_*$E	$\vee_f$E	$\bullet$	$\bullet$	$\bullet$
B.6.8.a	$\&_*$I	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.6.8.b	$\&_*$I	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.6.8.c	$\&_*$E	$\vee_c$I	$\circ^{11}$	$\bullet$	$\bullet$
B.6.8.d	$\&_*$E	$\vee_c$E	$\bullet$	$\bullet$	$\bullet$
B.6.9.a	$\&_*$I	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.6.9.b	$\&_*$I	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
B.6.9.c	$\&_*$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
B.6.9.d	$\&_*$E	$\vee_*$E	$\bullet$	$\bullet$	$\bullet$
			I11	**I**11	**I**2

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B.7 (CDI)	$\mathcal{D}^{a_i}$:	$\mathcal{D}^*$:	α: $\&_f$-c	β: $\&_c$-c	γ: $\&_*$-c
B.7.1.a	$\vee_f$I	$\supset_f$I	$\bullet \mid \bullet$	$\circ^8 \mid \circ^8$	$\bullet \mid \bullet$
B.7.1.b	$\vee_f$I	$\supset_f$E	$\bullet \mid \bullet$	$\circ^8 \mid \circ^{3,8}$	$\bullet \mid \bullet$
B.7.1.c	$\vee_f$E	$\supset_f$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.1.d	$\vee_f$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.7.2.a	$\vee_f$I	$\supset_c$I	$\circ^{7,10} \mid \circ^7$	$\circ^{10} \mid \circ^8$	$\circ^{10} \mid \circ^9$
B.7.2.b	$\vee_f$I	$\supset_c$E	$\bullet \mid \circ^4$	$\circ^8 \mid \circ^8$	$\bullet \mid \circ^4$
B.7.2.c	$\vee_f$E	$\supset_c$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.2.d	$\vee_f$E	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.3.a	$\vee_f$I	$\supset_*$I	$\bullet \mid \bullet$	$\circ^8 \mid \circ^8$	$\bullet \mid \bullet$
B.7.3.b	$\vee_f$I	$\supset_*$E	$\bullet \mid \bullet$	$\circ^8 \mid \circ^8$	$\bullet \mid \bullet$
B.7.3.c	$\vee_f$E	$\supset_*$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.3.d	$\vee_f$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.4.a	$\vee_c$I	$\supset_f$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.4.b	$\vee_c$I	$\supset_f$E	$\circ^7 \mid \circ^{3,7}$	$\bullet \mid \circ^3$	$\bullet \mid \circ^3$
B.7.4.c	$\vee_c$E	$\supset_f$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.4.d	$\vee_c$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.7.5.a	$\vee_c$I	$\supset_c$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.5.b	$\vee_c$I	$\supset_c$E	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.5.c	$\vee_c$E	$\supset_c$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.5.d	$\vee_c$E	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.6.a	$\vee_c$I	$\supset_*$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.6.b	$\vee_c$I	$\supset_*$E	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.6.c	$\vee_c$E	$\supset_*$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.6.d	$\vee_c$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.7.a	$\vee_*$I	$\supset_f$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.7.b	$\vee_*$I	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.7.7.c	$\vee_*$E	$\supset_f$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.7.d	$\vee_*$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^3$	$\bullet \mid \bullet$
B.7.8.a	$\vee_*$I	$\supset_c$I	$\circ^7 \mid \circ^7$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.8.b	$\vee_*$I	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.8.c	$\vee_*$E	$\supset_c$I	$\circ^7 \mid \circ^7 \mid \circ^7$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.8.d	$\vee_*$E	$\supset_c$E	$\bullet \mid \circ^4$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.9.a	$\vee_*$I	$\supset_*$I	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.9.b	$\vee_*$I	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
B.7.9.c	$\vee_*$E	$\supset_*$I	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$	$\bullet \mid \bullet \mid \bullet$
B.7.9.d	$\vee_*$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I30	**I**17	**I**4

2. *Case B.3.4.c.γ*: $\mathcal{D}^{a_i}$ ends with $\supset_c$E, $\mathcal{D}^*$ ends with $\vee_f$I: $\circ^{10}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}/\mathcal{D}_1/\\ A \supset_c B\end{array} \quad \begin{array}{c}\wr\mathcal{D}_2\wr\\ A\end{array}}{B}\,(\supset_c\mathrm{E}) \quad \begin{array}{c}\wr\mathcal{D}_3\wr\\ C\end{array}}{B \&_* C}\,(\&_*\mathrm{I})}{B}\,(\&_*\mathrm{E}1)}{B \vee_f D}\,(\vee_f\mathrm{I}1) \tag{12}$$

3. *Case B.4.2.a2.γ*: $\mathcal{D}^{a_i}$ ends with $\&_f$I, $\mathcal{D}^*$ ends with $\supset_c$I: $\circ^9$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}|\mathcal{D}_1|\\ A\end{array} \quad \begin{array}{c}|\mathcal{D}_2|\\ B\end{array}}{A \&_f B}\,(\&_f\mathrm{I}) \quad \begin{array}{c}[\wr C\wr]^{(u)}\\ /\wr\mathcal{D}_3\wr/\\ D\end{array}}{(A \&_f B) \&_* D}\,(\&_*\mathrm{I})}{D}\,(\&_*\mathrm{E}2)}{C \supset_c D}\,(\supset_c\mathrm{I}),\ u \tag{13}$$

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4. *Case B.7.3.a.β*: $\mathcal{D}^{a_i}$ ends with $\vee_f$I, $\mathcal{D}^*$ ends with $\supset_*$I: (a1), (a2): $\circ^8$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\not\wr\, A \not\wr]^{(u)}\\ \not\wr\,|\mathcal{D}_1|\not\wr\\ B\end{array}}{B\vee_f C}(\vee_f\text{I1}) \qquad \begin{array}{c}\wr\mathcal{D}_2\wr\\ D\end{array}}{(B\vee_f C)\&_c D}(\&_c\text{I})}{B\vee_f C}(\&_c\text{E1})}{A\supset_*(B\vee_f C)}(\supset_*\text{I}),\ u \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}|\mathcal{D}_1|\\ A\end{array}}{A\vee_f B}(\vee_f\text{I1}) \qquad \begin{array}{c}[/C/]^{(u)}\\ /\wr\mathcal{D}_2\wr/\\ D\end{array}}{(A\vee_f B)\&_c D}(\&_c\text{I})}{D}(\&_c\text{E2})}{C\supset_* D}(\supset_*\text{I}),\ u \tag{14}$$

Special cases (Part B). Let $\begin{array}{c}/\mathcal{D}^{a_1}/\\ A_1\end{array}$ be $/A_1/$ and let $\begin{array}{c}/\mathcal{D}^{a_2}/\\ A_2\end{array}$ be $/A_2/$ in $\mathcal{D}^*$. Then:

$$\mathcal{D}^* = \dfrac{\dfrac{\dfrac{/A_1/ \qquad /A_2/}{A_1 \star^c A_2}(\star^c\text{I})}{A_i}(\star^c\text{E}i)}{B}(\text{I/E-rule}) \tag{15}$$

If the I/E-rule in $\mathcal{D}^*$ is $\supset_*$I, $\mathcal{D}^*$ is legitimate and eligible for $\star^c$-conversion. If the I/E-rule in $\mathcal{D}^*$ is $\supset_f$I [$\supset_c$I], $\mathcal{D}^*$ is illegitimate, due to a violation of SC1.b [SC2.b] ($V5b$), and not eligible for $\star^c$-conversion. Again, we cannot presuppose, as we did in the cases covered by the B-tables, that the $\star^i$I-rule is applied legitimately in these two special cases.

Proof (Part C): Preservation for $\star^d$-detour conversions. Let $\begin{array}{c}/\mathcal{D}^a/\\ A_i\end{array}$, $\begin{array}{c}/\mathcal{D}^{c_1}/\\ C\end{array}$, $\begin{array}{c}/\mathcal{D}^{c_2}/\\ C\end{array}$ be legitimate derivations. We show: If these derivations can be combined into a legitimate derivation $\mathcal{D}^*$, then the $\star^d$-conversions transform it into a legitimate derivation $\mathcal{D}^{**}$, where $i \in \{1,2\}$; otherwise, the combination is illegitimate and a $\star^d$-conversion precluded:

$$\mathcal{D}^* = \dfrac{\dfrac{\begin{array}{c}/\mathcal{D}^a/\\ A_i\end{array}}{A_1 \star^d A_2}(\star^d\text{I}) \qquad \begin{array}{c}[/A_1/]^{(u)}\\ /\mathcal{D}^{c_1}/\\ C\end{array} \qquad \begin{array}{c}[/A_2/]^{(v)}\\ /\mathcal{D}^{c_2}/\\ C\end{array}}{C}(\star^d\text{E}),\ u,\ v \qquad \mathcal{D}^{**} = \begin{array}{c}/\mathcal{D}^a/\\ [A_i]\\ /\mathcal{D}^{c_i}/\\ C\end{array}, \tag{16}$$

We consider for all specific instances of last rules applied in $\begin{array}{c}/\mathcal{D}^a/\\ A_i\end{array}$, $\begin{array}{c}/\mathcal{D}^{c_1}/\\ C\end{array}$, $\begin{array}{c}/\mathcal{D}^{c_2}/\\ C\end{array}$ the combination of these derivations into $\mathcal{D}^*$. We attempt to derive the instance of the maximum $A_1 \star^d A_2$ and the minor premisses of $\star^d$E proceeding in the top-down manner. For reasons of simplicity, we assume that it is the first disjunct of the maximum which has the status required by the side condition of the $\star^d$E-rule in question. Moreover, we may confine attention to the derivation of one of the minor premisses, the first one, as the situation for the other one is exactly analogous. The results of the top-down procedure for the $\star^d$-conversions are collected in Tables C.1-9 which summarize Part C of the preservation proof (pp. 31-33). Anticipating the observations in Remark 4.2, Tables C.7-9 are not displayed for reasons of space.

Tables C.1, C.4, C.7: In case $\mathcal{D}^{c_i}$ ends with $\star^i$E, there are two entries: The first [second] entry in the (b)- and (d)-cases indicates the result for the construction in which the $\star^d$-maximum is eliminated by discharging an assumption (appealing to the first-disjunct convention) which is used to derive the major [minor] premiss of that $\star^i$E-application.

Tables C.2, C.5, C.8: All cases are single entry.

Tables C.3, C.6, C.9: The description is like that for tables C.1, C.4, C.7, except that '$\star^i$E' is replaced by '$\star^d$E'.

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C.1 (DII)	$\mathcal{D}^a$:	$\mathcal{D}^{c_i}$:	α: $\vee_f$-c	β: $\vee_c$-c	γ: $\vee_*$-c
C.1.1.a	$\supset_f$I	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.1.1.b	$\supset_f$I	$\supset_f$E	$\circ^{1}$ | $\circ^{1}$	$\circ^{1}$ | $\circ^{1,3}$	$\bullet$ | $\bullet$
C.1.1.c	$\supset_f$E	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.1.1.d	$\supset_f$E	$\supset_f$E	$\bullet$ | $\bullet$	$\bullet$ | $\circ^{3}$	$\bullet$ | $\bullet$
C.1.2.a	$\supset_f$I	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.1.2.b	$\supset_f$I	$\supset_c$E	$\circ^{1}$ | $\circ^{1}$	$\circ^{1}$ | $\circ^{1}$	$\bullet$ | $\bullet$
C.1.2.c	$\supset_f$E	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.1.2.d	$\supset_f$E	$\supset_c$E	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.3.a	$\supset_f$I	$\supset_*$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.1.3.b	$\supset_f$I	$\supset_*$E	$\circ^{1}$ | $\circ^{1}$	$\circ^{1}$ | $\circ^{1}$	$\bullet$ | $\bullet$
C.1.3.c	$\supset_f$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.1.3.d	$\supset_f$E	$\supset_*$E	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.4.a	$\supset_c$I	$\supset_f$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.1.4.b	$\supset_c$I	$\supset_f$E	$\circ^{1}$ | $\circ^{1}$	$\circ^{1}$ | $\circ^{1,3}$	$\bullet$ | $\bullet$
C.1.4.c	$\supset_c$E	$\supset_f$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.1.4.d	$\supset_c$E	$\supset_f$E	$\circ^{2b,10}$ | $\circ^{2b,10}$	$\bullet$ | $\circ^{3}$	$\bullet$ | $\circ^{3}$
C.1.5.a	$\supset_c$I	$\supset_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.1.5.b	$\supset_c$I	$\supset_c$E	$\circ^{1}$ | $\circ^{1}$	$\circ^{1}$ | $\circ^{1}$	$\bullet$ | $\bullet$
C.1.5.c	$\supset_c$E	$\supset_c$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.1.5.d	$\supset_c$E	$\supset_c$E	$\circ^{2b,10}$ | $\circ^{2b,10}$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.6.a	$\supset_c$I	$\supset_*$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.1.6.b	$\supset_c$I	$\supset_*$E	$\circ^{1}$ | $\circ^{1}$	$\circ^{1}$ | $\circ^{1}$	$\bullet$ | $\bullet$
C.1.6.c	$\supset_c$E	$\supset_*$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.1.6.d	$\supset_c$E	$\supset_*$E	$\circ^{2b,10}$ | $\circ^{2b,10}$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.7.a	$\supset_*$I	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.1.7.b	$\supset_*$I	$\supset_f$E	$\bullet$ | $\bullet$	$\bullet$ | $\circ^{3}$	$\bullet$ | $\bullet$
C.1.7.c	$\supset_*$E	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.1.7.d	$\supset_*$E	$\supset_f$E	$\bullet$ | $\bullet$	$\bullet$ | $\circ^{3}$	$\bullet$ | $\bullet$
C.1.8.a	$\supset_*$I	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.1.8.b	$\supset_*$I	$\supset_c$E	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.8.c	$\supset_*$E	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.1.8.d	$\supset_*$E	$\supset_c$E	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.9.a	$\supset_*$I	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.1.9.b	$\supset_*$I	$\supset_*$E	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
C.1.9.c	$\supset_*$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.1.9.d	$\supset_*$E	$\supset_*$E	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$	$\bullet$ | $\bullet$
			I27	**I**22	**I**1

C.2 (DIC)	$\mathcal{D}^a$:	$\mathcal{D}^{c_i}$:	α: $\vee_f$-c	β: $\vee_c$-c	γ: $\vee_*$-c
C.2.1.a	$\supset_f$I	$\&_f$I	$\circ^{1}$	$\circ^{1,7}$	$\bullet$
C.2.1.b	$\supset_f$I	$\&_f$E	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.1.c	$\supset_f$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.2.1.d	$\supset_f$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.2.2.a	$\supset_f$I	$\&_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.2.b	$\supset_f$I	$\&_c$E	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.2.c	$\supset_f$E	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.2.2.d	$\supset_f$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.2.3.a	$\supset_f$I	$\&_*$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.3.b	$\supset_f$I	$\&_*$E	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.3.c	$\supset_f$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.2.3.d	$\supset_f$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
C.2.4.a	$\supset_c$I	$\&_f$I	$\circ^{1}$	$\circ^{1,7}$	$\bullet$
C.2.4.b	$\supset_c$I	$\&_f$E	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.4.c	$\supset_c$E	$\&_f$I	$\circ^{2b,10}$	$\circ^{7}$	$\circ^{7}$
C.2.4.d	$\supset_c$E	$\&_f$E	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.2.5.a	$\supset_c$I	$\&_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.5.b	$\supset_c$I	$\&_c$E	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.5.c	$\supset_c$E	$\&_c$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.2.5.d	$\supset_c$E	$\&_c$E	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.2.6.a	$\supset_c$I	$\&_*$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.6.b	$\supset_c$I	$\&_*$E	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.2.6.c	$\supset_c$E	$\&_*$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.2.6.d	$\supset_c$E	$\&_*$E	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.2.7.a	$\supset_*$I	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.2.7.b	$\supset_*$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.2.7.c	$\supset_*$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.2.7.d	$\supset_*$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.2.8.a	$\supset_*$I	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.2.8.b	$\supset_*$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.2.8.c	$\supset_*$E	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.2.8.d	$\supset_*$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.2.9.a	$\supset_*$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.2.9.b	$\supset_*$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
C.2.9.c	$\supset_*$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.2.9.d	$\supset_*$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
			I18	**I**16	**I**1

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C.3 (DID)	$\mathcal{D}^a$:	$\mathcal{D}^{c_i}$:	α: $\vee_f$-c	β: $\vee_c$-c	γ: $\vee_*$-c
C.3.1.a	$\supset_f$I	$\vee_f$I	$\circ^{1}$	$\circ^{1,10}$	$\bullet$
C.3.1.b	$\supset_f$I	$\vee_f$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
C.3.1.c	$\supset_f$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.3.1.d	$\supset_f$E	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.2.a	$\supset_f$I	$\vee_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.3.2.b	$\supset_f$I	$\vee_c$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
C.3.2.c	$\supset_f$E	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.3.2.d	$\supset_f$E	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.3.a	$\supset_f$I	$\vee_*$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.3.3.b	$\supset_f$I	$\vee_*$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
C.3.3.c	$\supset_f$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.3.3.d	$\supset_f$E	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.4.a	$\supset_c$I	$\vee_f$I	$\circ^{1}$	$\circ^{1,10}$	$\bullet$
C.3.4.b	$\supset_c$I	$\vee_f$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
C.3.4.c	$\supset_c$E	$\vee_f$I	$\circ^{2b,10}$	$\circ^{10}$	$\circ^{10}$
C.3.4.d	$\supset_c$E	$\vee_f$E	$\circ^{2b,10} \mid \circ^{2b,10}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.5.a	$\supset_c$I	$\vee_c$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.3.5.b	$\supset_c$I	$\vee_c$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
C.3.5.c	$\supset_c$E	$\vee_c$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.3.5.d	$\supset_c$E	$\vee_c$E	$\circ^{2b,10} \mid \circ^{2b,10}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.6.a	$\supset_c$I	$\vee_*$I	$\circ^{1}$	$\circ^{1}$	$\bullet$
C.3.6.b	$\supset_c$I	$\vee_*$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1} \mid \circ^{1}$	$\bullet \mid \bullet$
C.3.6.c	$\supset_c$E	$\vee_*$I	$\circ^{2b,10}$	$\bullet$	$\bullet$
C.3.6.d	$\supset_c$E	$\vee_*$E	$\circ^{2b,10} \mid \circ^{2b,10}$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.7.a	$\supset_*$I	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.3.7.b	$\supset_*$I	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.7.c	$\supset_*$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.3.7.d	$\supset_*$E	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.8.a	$\supset_*$I	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.3.8.b	$\supset_*$I	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.8.c	$\supset_*$E	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.3.8.d	$\supset_*$E	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.9.a	$\supset_*$I	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.3.9.b	$\supset_*$I	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.3.9.c	$\supset_*$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.3.9.d	$\supset_*$E	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I27	**I**22	**I**1

C.4 (DCI)	$\mathcal{D}^a$:	$\mathcal{D}^{c_i}$:	α: $\vee_f$-c	β: $\vee_c$-c	γ: $\vee_*$-c
C.4.1.a	$\&_f$I	$\supset_f$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.4.1.b	$\&_f$I	$\supset_f$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2b,11} \mid \circ^{1,2b,3,11}$	$\circ^{2b} \mid \circ^{2b}$
C.4.1.c	$\&_f$E	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.4.1.d	$\&_f$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
C.4.2.a	$\&_f$I	$\supset_c$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.4.2.b	$\&_f$I	$\supset_c$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2b,11} \mid \circ^{1,2b,11}$	$\circ^{2b} \mid \circ^{2b}$
C.4.2.c	$\&_f$E	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.4.2.d	$\&_f$E	$\supset_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.3.a	$\&_f$I	$\supset_*$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.4.3.b	$\&_f$I	$\supset_*$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2b,11} \mid \circ^{1,2b,11}$	$\circ^{2b} \mid \circ^{2b}$
C.4.3.c	$\&_f$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.4.3.d	$\&_f$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.4.a	$\&_c$I	$\supset_f$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.4.4.b	$\&_c$I	$\supset_f$E	$\circ^{1,2b,10} \mid \circ^{1,2b,10}$	$\circ^{1} \mid \circ^{1,3}$	$\circ^{2b} \mid \circ^{2b}$
C.4.4.c	$\&_c$E	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.4.4.d	$\&_c$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
C.4.5.a	$\&_c$I	$\supset_c$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.4.5.b	$\&_c$I	$\supset_c$E	$\circ^{1,2b,10} \mid \circ^{1,2b,10}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2b} \mid \circ^{2b}$
C.4.5.c	$\&_c$E	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.4.5.d	$\&_c$E	$\supset_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.6.a	$\&_c$I	$\supset_*$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.4.6.b	$\&_c$I	$\supset_*$E	$\circ^{1,2b,10} \mid \circ^{1,2b,10}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2b} \mid \circ^{2b}$
C.4.6.c	$\&_c$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.4.6.d	$\&_c$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.7.a	$\&_*$I	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.4.7.b	$\&_*$I	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
C.4.7.c	$\&_*$E	$\supset_f$I	$\bullet$	$\bullet$	$\bullet$
C.4.7.d	$\&_*$E	$\supset_f$E	$\bullet \mid \bullet$	$\bullet \mid \circ^{3}$	$\bullet \mid \bullet$
C.4.8.a	$\&_*$I	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.4.8.b	$\&_*$I	$\supset_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.8.c	$\&_*$E	$\supset_c$I	$\bullet$	$\bullet$	$\bullet$
C.4.8.d	$\&_*$E	$\supset_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.9.a	$\&_*$I	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.4.9.b	$\&_*$I	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.4.9.c	$\&_*$E	$\supset_*$I	$\bullet$	$\bullet$	$\bullet$
C.4.9.d	$\&_*$E	$\supset_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I18	**I**22	**I**18

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C.5 (DCC)	$\mathcal{D}^{a}$:	$\mathcal{D}^{c_i}$:	α: $\vee_f$-c	β: $\vee_c$-c	γ: $\vee_*$-c
C.5.1.a	$\&_f$I	$\&_f$I	$\circ^{1}$	$\circ^{1,2b,7,11}$	$\circ^{2b}$
C.5.1.b	$\&_f$I	$\&_f$E	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.5.1.c	$\&_f$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.5.1.d	$\&_f$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.5.2.a	$\&_f$I	$\&_c$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.5.2.b	$\&_f$I	$\&_c$E	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.5.2.c	$\&_f$E	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.5.2.d	$\&_f$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.5.3.a	$\&_f$I	$\&_*$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.5.3.b	$\&_f$I	$\&_*$E	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.5.3.c	$\&_f$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.5.3.d	$\&_f$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
C.5.4.a	$\&_c$I	$\&_f$I	$\circ^{1,2b,10}$	$\circ^{1,7}$	$\circ^{2b}$
C.5.4.b	$\&_c$I	$\&_f$E	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.5.4.c	$\&_c$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.5.4.d	$\&_c$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.5.5.a	$\&_c$I	$\&_c$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.5.5.b	$\&_c$I	$\&_c$E	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.5.5.c	$\&_c$E	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.5.5.d	$\&_c$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.5.6.a	$\&_c$I	$\&_*$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.5.6.b	$\&_c$I	$\&_*$E	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.5.6.c	$\&_c$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.5.6.d	$\&_c$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
C.5.7.a	$\&_*$I	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.5.7.b	$\&_*$I	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.5.7.c	$\&_*$E	$\&_f$I	$\bullet$	$\circ^{7}$	$\bullet$
C.5.7.d	$\&_*$E	$\&_f$E	$\bullet$	$\bullet$	$\bullet$
C.5.8.a	$\&_*$I	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.5.8.b	$\&_*$I	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.5.8.c	$\&_*$E	$\&_c$I	$\bullet$	$\bullet$	$\bullet$
C.5.8.d	$\&_*$E	$\&_c$E	$\bullet$	$\bullet$	$\bullet$
C.5.9.a	$\&_*$I	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.5.9.b	$\&_*$I	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
C.5.9.c	$\&_*$E	$\&_*$I	$\bullet$	$\bullet$	$\bullet$
C.5.9.d	$\&_*$E	$\&_*$E	$\bullet$	$\bullet$	$\bullet$
			I12	**I**16	**I**12

C.6 (DCD)	$\mathcal{D}^{a}$:	$\mathcal{D}^{c_i}$:	α: $\vee_f$-c	β: $\vee_c$-c	γ: $\vee_*$-c
C.6.1.a	$\&_f$I	$\vee_f$I	$\circ^{1}$	$\circ^{1,2b,10,11}$	$\circ^{2b}$
C.6.1.b	$\&_f$I	$\vee_f$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2b,11} \mid \circ^{1,2b,11}$	$\circ^{2b} \mid \circ^{2b}$
C.6.1.c	$\&_f$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.6.1.d	$\&_f$E	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.2.a	$\&_f$I	$\vee_c$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.6.2.b	$\&_f$I	$\vee_c$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2b,11} \mid \circ^{1,2b,11}$	$\circ^{2b} \mid \circ^{2b}$
C.6.2.c	$\&_f$E	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.6.2.d	$\&_f$E	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.3.a	$\&_f$I	$\vee_*$I	$\circ^{1}$	$\circ^{1,2b,11}$	$\circ^{2b}$
C.6.3.b	$\&_f$I	$\vee_*$E	$\circ^{1} \mid \circ^{1}$	$\circ^{1,2b,11} \mid \circ^{1,2b,11}$	$\circ^{2b} \mid \circ^{2b}$
C.6.3.c	$\&_f$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.6.3.d	$\&_f$E	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.4.a	$\&_c$I	$\vee_f$I	$\circ^{1,2b,10}$	$\circ^{1,10}$	$\circ^{2b}$
C.6.4.b	$\&_c$I	$\vee_f$E	$\circ^{1,2b,10} \mid \circ^{1,2b,10}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2b} \mid \circ^{2b}$
C.6.4.c	$\&_c$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.6.4.d	$\&_c$E	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.5.a	$\&_c$I	$\vee_c$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.6.5.b	$\&_c$I	$\vee_c$E	$\circ^{1,2b,10} \mid \circ^{1,2b,10}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2b} \mid \circ^{2b}$
C.6.5.c	$\&_c$E	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.6.5.d	$\&_c$E	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.6.a	$\&_c$I	$\vee_*$I	$\circ^{1,2b,10}$	$\circ^{1}$	$\circ^{2b}$
C.6.6.b	$\&_c$I	$\vee_*$E	$\circ^{1,2b,10} \mid \circ^{1,2b,10}$	$\circ^{1} \mid \circ^{1}$	$\circ^{2b} \mid \circ^{2b}$
C.6.6.c	$\&_c$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.6.6.d	$\&_c$E	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.7.a	$\&_*$I	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.6.7.b	$\&_*$I	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.7.c	$\&_*$E	$\vee_f$I	$\bullet$	$\circ^{10}$	$\bullet$
C.6.7.d	$\&_*$E	$\vee_f$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.8.a	$\&_*$I	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.6.8.b	$\&_*$I	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.8.c	$\&_*$E	$\vee_c$I	$\bullet$	$\bullet$	$\bullet$
C.6.8.d	$\&_*$E	$\vee_c$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.9.a	$\&_*$I	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.6.9.b	$\&_*$I	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
C.6.9.c	$\&_*$E	$\vee_*$I	$\bullet$	$\bullet$	$\bullet$
C.6.9.d	$\&_*$E	$\vee_*$E	$\bullet \mid \bullet$	$\bullet \mid \bullet$	$\bullet \mid \bullet$
			I18	**I**22	**I**18

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Example 4.3. 1. *Case C.4.1.a.β*: $\mathcal{D}^a$ ends with $\&_f$I, $\mathcal{D}^{c_i}$ ends with $\supset_f$I: $\circ^{1,2b,11}$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}|\mathcal{D}_1| & |\mathcal{D}_2|\\ A & B\end{matrix}}{A\&_f B}(\&_f\mathrm{I})}{(A\&_f B)\vee_c C}(\vee_c\mathrm{I1}) \qquad \dfrac{\begin{matrix}[\wr A\&_f B\wr]^{(u)} \quad [|D|]^{(v)}\\ /\wr\mathcal{D}_3\wr/\\ E\end{matrix}}{D\supset_f E}(\supset_f\mathrm{I}),\ v \qquad \begin{matrix}[/C/]^{(w)}\\ /\mathcal{D}_4/\\ D\supset_f E\end{matrix}}{D\supset_f E}(\vee_c\mathrm{E}),\ u,\ w \tag{17}$$

3. *Case C.4.1.d.β*: $\mathcal{D}^a$ ends with $\&_f$E, $\mathcal{D}^{c_i}$ ends with $\supset_f$E. (d1): •

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\wr\mathcal{D}_1\wr\\ A\&_f B\end{matrix}}{A}(\&_f\mathrm{E1})}{A\vee_c C}(\vee_c\mathrm{I1}) \qquad \dfrac{\begin{matrix}[\wr A\wr]^{(u)} & \\ /\wr\mathcal{D}_2\wr/ & |\mathcal{D}_3|\\ D\supset_f E & D\end{matrix}}{E}(\supset_f\mathrm{E}) \qquad \begin{matrix}[/C/]^{(v)}\\ /\mathcal{D}_4/\\ E\end{matrix}}{E}(\vee_c\mathrm{E}),\ u,\ v \qquad \text{conv}$$

$$\dfrac{\begin{matrix}\dfrac{\begin{matrix}\wr\mathcal{D}_1\wr\\ A\&_f B\end{matrix}}{A}(\&_f\mathrm{E1})\\ \wr\mathcal{D}_2\wr\\ D\supset_f E\end{matrix} \qquad \begin{matrix}|\mathcal{D}_3|\\ D\end{matrix}}{E}(\supset_f\mathrm{E}) \tag{18}$$

(d2): $\circ^3$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\wr\mathcal{D}_1\wr\\ A\&_f B\end{matrix}}{A}(\&_f\mathrm{E1})}{A\vee_c C}(\vee_c\mathrm{I1}) \qquad \dfrac{\begin{matrix} & [\wr A\wr]^{(u)}\\ /\mathcal{D}_2/ & /\wr\mathcal{D}_3\wr/\\ D\supset_f E & D\end{matrix}}{E}(\supset_f\mathrm{E}) \qquad \begin{matrix}[/C/]^{(v)}\\ /\mathcal{D}_4/\\ E\end{matrix}}{E}(\vee_c\mathrm{E}),\ u,\ v \tag{19}$$

Special cases (Part C). Consider the following special case of $\mathcal{D}^*$:

$$\dfrac{\dfrac{/A/}{A\star^d(B\star^i A)}(\star^d\mathrm{I}) \qquad [/A/]^{(u)} \qquad \dfrac{[/B\star^i A/]^{(v)} \quad /B/}{A}(\star^i\mathrm{E})}{A}(\star^d\mathrm{E}),\ u,\ v \tag{20}$$

In case $\star^i = \supset_*$, $\mathcal{D}^*$ can be continued by $\supset_*$I discharging $/B/$. The resulting derivation is eligible for $\star^d$-conversion. In case $\star^i = \supset_f$ [$\star^i = \supset_c$], a continuation of $\mathcal{D}^*$ by $\supset_f$I [$\supset_c$I] discharging $/B/$ will give us $V5c$; for these two special cases a remark analogous to those at the end of Part A and B applies.

□

Remark 4.1. The preservation tables do not only present the preservation proof in a condensed form, they also facilitate the comparison of the preservation behaviour for each kind of conversion. (For example, there are no violations $V8$, $V11$ in Part A, no $V1$, $V2a/b$ in Part B, and no $V4$, $V6$, $V8$ in Part C; the $\supset_*/\vee_*$-rules are never violated; and $V5a/b/c$ never show up in the preservation tables.) Moreover, the tables allow us to study the effects of dropping side conditions, or to locate changes to which additional restrictions on the rules for the operators might give rise. In this way, these tables may serve as a useful tool for the introduction of various kinds of modal natural deduction system.

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Type:	Coincidences:
Ty.1:	A.1 $=_n$ A.7
Ty.2:	A.2 $=_s$ A.3, A.8 $=_s$ A.9, C.4 $=_s$ C.7, C.6 $=_s$ C.9
Ty.3:	A.4
Ty.4:	A.5 $=_s$ A.6, C.5 $=_s$ C.8
Ty.5:	B.1
Ty.6:	B.2 $=_w$ B.3
Ty.7:	B.4
Ty.8:	B.5 $=_s$ B.8, B.6 $=_s$ B.9
Ty.9:	B.7
Ty.10:	C.1 $=_n$ C.3
Ty.11:	C.2

Figure 2: Types of preservation table

Remark 4.2. There are exactly eleven types of preservation table: see Figure 2. We say (i) that two tables *coincide numerically* ($=_n$), in case they display the same sequence of **I**-numbers, (ii) that they *coincide weakly* ($=_w$), in case they coincide numerically and display the same $\bullet$-pattern, (iii) that they *coincide strongly* ($=_s$), in case they coincide weakly and display the same V-codes, and (iv) that they *coincide exactly* ($=_e$), in case they coincide strongly and display the same rule entries in the $\mathcal{D}$-columns and the same $\alpha\beta\gamma$-entries. Since there is no exact coincidence, the number of cases cannot be, strictly speaking, reduced. However, on the basis of the other kinds of coincidence the number can be "reduced by analogy". If we drop the (c2)-cases from B.7, we get B.4 $=_w$ B.7 and analogy-reduce the number of types to ten. In general, we may reduce the number of cases further in this way by considering only one entry for each multiple-entry case, if the multiple entries do not differ in any respect.

4.2 Preservation for permutation and simplification conversions

Theorem 4.2. Permutation and simplification conversions of M-systems do not transform legitimate into illegitimate derivations.

Proof. The proof is by exhaustion. *Permutation*: Consider

$$\mathcal{D}^* = \dfrac{\dfrac{\overset{/\mathcal{D}/}{A \star^d B} \quad \overset{/\mathcal{D}_1/}{C} \quad \overset{/\mathcal{D}_2/}{C}}{C}\,(\star^d\text{E}) \qquad /\mathcal{D}'/}{D}\,(\text{E-rule}) \qquad \text{perm}$$

$$\mathcal{D}^{**} = \dfrac{\overset{/\mathcal{D}/}{A \star^d B} \quad \overset{/\mathcal{D}_1/}{\dfrac{C \quad /\mathcal{D}'/}{D}}\,(\text{E-rule}) \quad \overset{/\mathcal{D}_2/}{\dfrac{C \quad /\mathcal{D}'/}{D}}\,(\text{E-rule})}{D}\,(\star^d\text{E}) \tag{21}$$

We assume that $\mathcal{D}^*$ is legitimate and inspect, in each possible case, whether $\mathcal{D}^{**}$ remains legitimate after permutation. The P-table below lists the results for all the possible cases.

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$\vee_f$-perm	E-rule		$\vee_c$-perm	E-rule		$\vee_*$-perm	E-rule	
P.α.1	as_*E	•	P.β.1	as_*E	•	P.γ.1	as_*E	•
P.α.2	$\supset_f$E	•	P.β.2	$\supset_f$E	•	P.γ.2	$\supset_f$E	•
P.α.3	$\supset_c$E	•	P.β.3	$\supset_c$E	•	P.γ.3	$\supset_c$E	•
P.α.4	$\supset_*$E	•	P.β.4	$\supset_*$E	•	P.γ.4	$\supset_*$E	•
P.α.5	$\&_f$E	•	P.β.5	$\&_f$E	•	P.γ.5	$\&_f$E	•
P.α.6	$\&_c$E	•	P.β.6	$\&_c$E	•	P.γ.6	$\&_c$E	•
P.α.7	$\&_*$E	•	P.β.7	$\&_*$E	•	P.γ.7	$\&_*$E	•
P.α.8	$\vee_f$E	•	P.β.8	$\vee_f$E	•	P.γ.8	$\vee_f$E	•
P.α.9	$\vee_c$E	•	P.β.9	$\vee_c$E	•	P.γ.9	$\vee_c$E	•
P.α.10	$\vee_*$E	•	P.β.10	$\vee_*$E	•	P.γ.10	$\vee_*$E	•

We pick two cases. *Case P.α.1*: Let $\varphi_0(\alpha_i) =_{def} \varphi_0\alpha_1 ... \alpha_n$, where $i \in \{1, ..., n\}$, and let $j \in \{0, ..., n\}$:

$$\dfrac{\dfrac{\begin{array}{c}/\mathcal{D}/\\ A \vee_f B\end{array} \quad \begin{array}{c}/\mathcal{D}_1/\\ \varphi_0(\alpha_i)\end{array} \quad \begin{array}{c}/\mathcal{D}_2/\\ \varphi_0(\alpha_i)\end{array}}{\varphi_0(\alpha_i)}\,(\vee_f\mathrm{E})}{\tau_j\Gamma}\,(as_*\mathrm{E}j) \quad perm \quad \dfrac{\begin{array}{c}/\mathcal{D}/\\ A \vee_f B\end{array} \quad \dfrac{\begin{array}{c}/\mathcal{D}_1/\\ \varphi_0(\alpha_i)\end{array}}{\tau_j\Gamma}\,(as_*\mathrm{E}j) \quad \dfrac{\begin{array}{c}/\mathcal{D}_2/\\ \varphi_0(\alpha_i)\end{array}}{\tau_j\Gamma}\,(as_*\mathrm{E}j)}{\tau_j\Gamma}\,(\vee_f\mathrm{E}) \tag{22}$$

Case P.β.8:

$$\dfrac{\dfrac{\begin{array}{c}/\mathcal{D}/\\ A \vee_c B\end{array} \quad \begin{array}{c}/\mathcal{D}_1/\\ C \vee_f D\end{array} \quad \begin{array}{c}/\mathcal{D}_2/\\ C \vee_f D\end{array}}{C \vee_f D}\,(\vee_c\mathrm{E}) \quad \begin{array}{c}/\mathcal{D}_3/\\ E\end{array} \quad \begin{array}{c}/\mathcal{D}_4/\\ E\end{array}}{E}\,(\vee_f\mathrm{E})$$

perm

$$\dfrac{\begin{array}{c}/\mathcal{D}/\\ A \vee_c B\end{array} \quad \dfrac{\begin{array}{c}/\mathcal{D}_1/\\ C \vee_f D\end{array} \quad \begin{array}{c}/\mathcal{D}_3/\\ E\end{array} \quad \begin{array}{c}/\mathcal{D}_4/\\ E\end{array}}{E}\,(\vee_f\mathrm{E}) \quad \dfrac{\begin{array}{c}/\mathcal{D}_2/\\ C \vee_f D\end{array} \quad \begin{array}{c}/\mathcal{D}_3/\\ E\end{array} \quad \begin{array}{c}/\mathcal{D}_4/\\ E\end{array}}{E}\,(\vee_f\mathrm{E})}{E}\,(\vee_c\mathrm{E}) \tag{23}$$

Simplification: We consider

$$\mathcal{D}^* = \dfrac{\begin{array}{c}/\mathcal{D}/\\ A \vee_f B\end{array} \quad \begin{array}{c}/\mathcal{D}_1/\\ C\end{array} \quad \begin{array}{c}/\mathcal{D}_2/\\ C\end{array}}{C}\,(\vee_f\mathrm{E}) \quad simp \quad \mathcal{D}^{**} = \begin{array}{c}/\mathcal{D}_i/\\ C\end{array} \tag{24}$$

and proceed in an analogous way obtaining the results recorded in the S-table below.

simp	E-rule	
S.α	$\vee_f$E	•
S.β	$\vee_c$E	•
S.γ	$\vee_*$E	•

□

4.3 Normalization

Due to the preservation theorems we know that the conversions of M-systems never lead us from legitimate derivations to illegitimate ones.

Example 4.4. The following derivation is illegitimate, because of $V5a$. It cannot be reduced:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr A \wr]^{(u)}}{B \supset_* A}\,(\supset_*\mathrm{I}) \quad [\wr B \wr]^{(v)}}{A}\,(\supset_*\mathrm{E})}{B \supset_c A}\,(\supset_c\mathrm{I}),\ v\ V5a}{A \supset_c (B \supset_c A)}\,(\supset_c\mathrm{I}),\ u\ V5a \tag{25}$$

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Example 4.5. Consider the following derivation which contains the segments (a),(b) and (a′),(b) which are both of length 2:

$$\dfrac{\dfrac{\dfrac{/\mathcal{D}_1/}{Fa \vee_f Gba} \qquad \dfrac{\dfrac{/\mathcal{D}_2/}{H\Gamma} \quad \dfrac{[|Fa|]^{(1)}}{a\Gamma}\,(as_*\mathrm{E}_1)}{(a)Ha}\,(as_*\mathrm{I}) \qquad \dfrac{\dfrac{/\mathcal{D}_3/}{H\Gamma} \quad \dfrac{[/Gba/]^{(2)}}{a\Gamma}\,(as_*\mathrm{E}_2)}{(a')Ha}\,(as_*\mathrm{I})}{(b)Ha}\,(\vee_f\mathrm{E}),\ 1,\ 2}{H\Gamma}\,(as_*\mathrm{E}_0) \tag{26}$$

By permutation:

$$\dfrac{\dfrac{/\mathcal{D}_1/}{Fa \vee_f Gba} \qquad \dfrac{\dfrac{\dfrac{/\mathcal{D}_2/}{H\Gamma} \quad \dfrac{[|Fa|]^{(1)}}{a\Gamma}\,(as_*\mathrm{E}_1)}{Ha}\,(as_*\mathrm{I})}{H\Gamma}\,(as_*\mathrm{E}_0) \qquad \dfrac{\dfrac{\dfrac{/\mathcal{D}_3/}{H\Gamma} \quad \dfrac{[/Gba/]^{(2)}}{a\Gamma}\,(as_*\mathrm{E}_2)}{Ha}\,(as_*\mathrm{I})}{H\Gamma}\,(as_*\mathrm{E}_0)}{H\Gamma}\,(\vee_f\mathrm{E}),\ 1,\ 2 \tag{27}$$

By detour conversion for as:

$$\dfrac{\dfrac{/\mathcal{D}_1/}{Fa \vee_f Gba} \qquad \dfrac{/\mathcal{D}_2/}{H\Gamma} \qquad \dfrac{/\mathcal{D}_3/}{H\Gamma}}{H\Gamma}\,(\vee_f\mathrm{E}) \tag{28}$$

By simplification conversion ($i \in \{2, 3\}$):

$$\dfrac{/\mathcal{D}_i/}{H\Gamma} \tag{29}$$

Example 4.6. Consider:

$$\dfrac{\dfrac{(A \vee_c A) \qquad \dfrac{[\wr A \wr]^{(1)}}{A \vee_c B}\,(\vee_c\mathrm{I1}) \qquad \dfrac{[\wr A \wr]^{(2)}}{A \vee_c B}\,(\vee_c\mathrm{I1})}{A \vee_c B}\,(\vee_c\mathrm{E}),\ 1,\ 2 \qquad \dfrac{(A \supset_c C) \qquad [\wr A \wr]^{(3)}}{(a) \quad C}\,(\supset_c\mathrm{E}) \qquad \dfrac{(B \supset_f C) \qquad [|B|]^{(4)}}{(b) \quad C}\,(\supset_f\mathrm{E})}{(c) \quad C}\,(\vee_c\mathrm{E}),\ 3,\ 4 \tag{30}$$

By permutation:

$$\dfrac{(A \vee_c A) \qquad \dfrac{\dfrac{[\wr A \wr]^{(1)}}{A \vee_c B}\,(\vee_c\mathrm{I1}) \qquad \dfrac{(A \supset_c C) \qquad [\wr A \wr]^{(3)}}{(a) \quad C}\,(\supset_c\mathrm{E}) \qquad \dfrac{(B \supset_f C) \qquad [|B|]^{(4)}}{(b) \quad C}\,(\supset_f\mathrm{E})}{(d) \quad C}\,(\vee_c\mathrm{E}),\ 3,\ 4 \qquad /\mathcal{D}/}{(c) \quad C}\,(\vee_c\mathrm{E}),\ 1,\ 2 \tag{31}$$

where

$$/\mathcal{D}/ = \dfrac{\dfrac{[\wr A \wr]^{(2)}}{A \vee_c B}\,(\vee_c\mathrm{I1}) \qquad \dfrac{(A \supset_c C) \qquad [\wr A \wr]^{(3)}}{(a') \quad C}\,(\supset_c\mathrm{E}) \qquad \dfrac{(B \supset_f C) \qquad [|B|]^{(4)}}{(b') \quad C}\,(\supset_f\mathrm{E})}{(d') \quad C}\,(\vee_c\mathrm{E}),\ 3,\ 4$$

By conversion:

$$\dfrac{(A \vee_c A) \qquad \dfrac{(A \supset_c C) \qquad [\wr A \wr]^{(1)}}{(a) \quad C}\,(\supset_c\mathrm{E}) \qquad \dfrac{(A \supset_c C) \qquad [\wr A \wr]^{(2)}}{(a') \quad C}\,(\supset_c\mathrm{E})}{(c) \quad C}\,(\vee_c\mathrm{E}),\ 1,\ 2 \tag{32}$$

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Note that distinct single occurrences of the same formula in a derivation may differ with respect to modal status. Here, e.g., the C-nodes: (a), (a′) counterfactual, (b), (b′) factual, (c) independent, (d), (d′) counterfactual.

Theorem 4.3. *Normalization (M-systems).* Any derivation $/\mathcal{D}/$ in an M-system can be transformed into a normal M-derivation.

Proof. In view of Theorems 4.1-2, the conversions of M-systems are safe. Proceeding top-down in $/\mathcal{D}/$, we combine, in the familiar way (cf. [33]: 182), a main induction on d with a subinduction on n to obtain $cr(/\mathcal{D}/) = \langle 0, 0\rangle$ by applying the conversions. □

Remark 4.3. Due to the normalization theorem for M-systems, we may conclude that, ultimately, only normal derivations matter. Obviously, in such derivations assumption principles AP2 and AP3 can be ignored.

5 The structure of modal derivations

5.1 The subexpression property

Normal derivations in M-systems enjoy the subexpression property (and the subformula property as a special case of it). We now repeat the relevant definitions for M-systems and establish these results.

Definition 5.1. Let $/\mathcal{D}/$ be a derivation in an M-system.

1. A *unit* in $/\mathcal{D}/$ is either an occurrence of (i) a term assumption $\tau\Gamma$, or (ii) a segment σ_{M} in $/\mathcal{D}/$. (Any $L1$-formula is a special case of a segment). We use U, U' (possibly subscripted) for units of M-systems.

2. In case U is a term assumption $\tau\Gamma$ in $/\mathcal{D}/$, τ is *the expression in* U.

Definition 5.2. A *track* of a derivation $/\mathcal{D}/$ in an M-system is a sequence of occurrences of units $U_0, ..., U_n$ such that:

1. U_0 is either
   (a) a top formula occurrence $/A_0/$ in $/\mathcal{D}/$ not discharged by an application $\boldsymbol{a}$ of $\star^d$E, or
   (b) a top occurrence of a term assumption $\tau\Gamma_0$;
2. U_i for $i < n$ is either
   (a) a formula occurrence A_i which is not the minor premiss of an instance of $\star^i$E, and either
      i. A_i is not the major premiss of an instance $\boldsymbol{a}$ of $\star^d$E and A_{i+1} is directly below A_i, or
      ii. A_i is the major premiss of an instance $\boldsymbol{a}$ of $\star^d$E and $/A_{i+1}/$ is an assumption discharged by $\boldsymbol{a}$; or
   (b) U_i is an occurrence of a term assumption $\tau\Gamma_i$;
3. U_n is either
   (a) a formula occurrence A_n which is either
      i. the minor premiss of an instance of $\star^i$E, or

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ii. the conclusion of $/\mathcal{D}/$, or

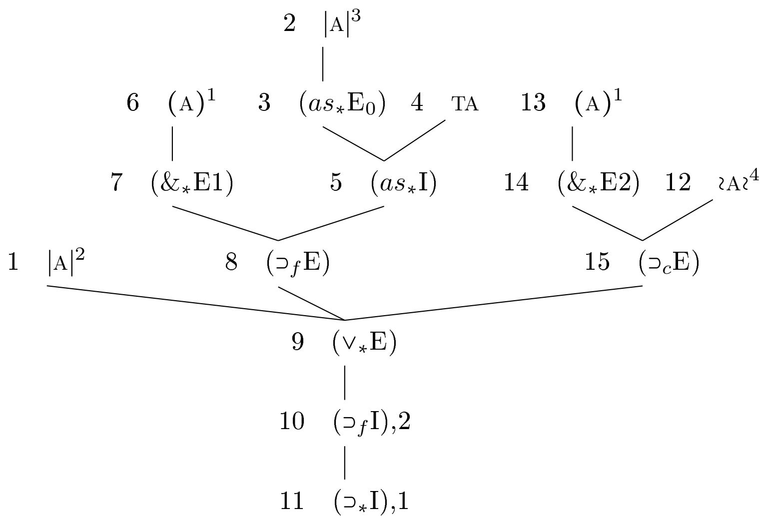
iii. the major premiss of an instance $\boldsymbol{a}$ of $\star^d$E, in case there is no assumption discharged by $\boldsymbol{a}$; or

(b) U_n is an occurrence of a term assumption $\tau\Gamma_n$.

Example 5.1. Consider:

$$\cfrac{\cfrac{[|Fa \vee_* Gbc|]^{(2)} \qquad \cfrac{\cfrac{[(\neg_f Fa \&_* \neg_c Gbc)]^{(1)}}{\neg_f Fa} \quad \cfrac{\cfrac{[|Fa|]^{(3)}}{F\Gamma} \quad a\Gamma}{Fa}}{\bot} \qquad \cfrac{\cfrac{[(\neg_f Fa \&_* \neg_c Gbc)]^{(1)}}{\neg_c Gbc} \quad [\wr Gbc \wr]^{(4)}}{\bot}}{\cfrac{\bot}{\neg_f (Fa \vee_* Gbc)}\,2}\,3,4}{(\neg_f Fa \&_* \neg_c Gbc) \supset_* \neg_f (Fa \vee_* Gbc)}\,1 \tag{33}$$

Structure tree:



(34)

Tracks: 1-3, 5 (Track 1); 1, 12 (Track 2); 4-5 (Track 3); 6-11 (Track 4); 13-15, 9-11 (Track 5).

Theorem 5.1. Let $/\mathcal{D}/$ be a normal derivation in an M-system, and let π be a track $U_0, ..., U_n$ in $/\mathcal{D}/$. There is a single unit U_i in π, the *minimum part* of π, which separates the possibly empty parts of π, called the *elimination (or E-)part* and the *introduction (or I-)part* of π, such that:

1. for each U_j in the E-part we have $j < i$, U_j is a major premiss of an E-rule, and U_{j+1} is a subexpression of U_j, and so each U_j is a subexpression of U_0;

2. for each U_j in the I-part we have $i < j$, and if $j < n$, then U_j is a premiss of an I-rule and a subexpression of U_{j+1}, thus, each U_j is a subexpression of U_n;

3. in case $i \neq n$, U_i is a premiss of an I-rule or of $\bot_*$i (so $U_i = \bot$) and a subexpression of U_0.

Proof. We use the fact that $/\mathcal{D}/$ is normal and inspect the rules of M-systems to verify that clauses 1-3 are satisfied.

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In case U_i is the first occurrence of a unit in π which is a premiss of an E-rule, all unit occurrences in π which are major premisses of E-rules precede all unit occurrences in π which are premisses of I-rules or $\bot$i. Otherwise, $/\mathcal{D}/$ would not be a normal derivation.

Next, let U_i be the first occurrence of a unit in π which is a premiss of an I-rule or of $\bot$i. Put $U_i = U_n$, in case there is no such unit. In these cases, U_i belongs to the minimum part of π.

Since, given these observations, U_i satisfies clauses 1 and 3, every unit occurrence U_j $(i < j < n)$ is a premiss of an I-rule or of $\bot$i. However, the case of $\bot$i is excluded, as the premiss of this rule is $\bot$, a formula that can only be derived by an E-rule. Hence, clause 2 is satisfied as well. □

Hence, all expressions in π are subexpressions of U_0 or U_n. An order can be imposed on tracks.

Definition 5.3. Let $/\mathcal{D}/$ be a normal derivation in an M-system.

1. A *track of order 0* (*main track*) in $/\mathcal{D}/$ is a track ending in a conclusion of $/\mathcal{D}/$.

2. A *track of order* $n+1$ in $/\mathcal{D}/$ is a track ending in the minor premiss of $\star^i$E, with the major premiss belonging to a track of order n.

A *main branch* of a derivation in an M-system is a branch π which passes only through premisses of I-rules and major premisses of E-rules beginning at a top-unit and ending in the conclusion of the derivation.

Remark 5.1. Applications of $as_{\star}$I and $\star^c$I undermine the uniqueness of main branches.

Example 5.2. Tracks 4 and 5 in Example 5.1 are main tracks and main branches, Tracks 1, 2 and 3 are tracks of order 1.

Theorem 5.2. In a normal derivation in an M-system each occurrence of a unit belongs to a track.

Proof. Proof omitted. □

Theorem 5.3. *Subexpression property (M-systems)*: If $/\mathcal{D}/$ is a normal M-derivation of a unit U from a set of units Γ, then each unit in $/\mathcal{D}/$ is a subexpression of an expression in $\Gamma \cup \{U\}$.

Proof. Let $/\mathcal{D}/$ be a normal derivation of U from Γ. The proof relies on Theorem 5.2 and proceeds by induction on the order of tracks n using Theorem 5.1.

Assume that the result holds for unit occurrences in tracks of order $< n$, let $\pi = U_0, ..., U_n$, and let U_i belong to the minimum part in π. There are two cases to consider.

Case 1. For U_n either $U_n = U$ or U_n is a minor premiss of $\star^i$E with the major premiss of the form $U_n \star^i B$ which appears in a track of order $n-1$. Hence, the result follows for all U_j $(i < j < n)$ by Theorem 5.1.

Case 2. For U_0 either $U_0 \in \Gamma$, or U_0 is discharged by an application $\boldsymbol{a}$ of $\star^i$I such that the conclusion of $\boldsymbol{a}$ has the form $U_0 \star^i B$, is contained in the I-part of π, or in some track of order $< n$, and U_0 is a subformula of the conditional. Hence, the result follows for all U_j $(j \leq i)$ by Theorem 5.1. □

Corollary 5.1. *Subformula property (M-systems)*: If $/\mathcal{D}/$ is a normal M-derivation of formula A from a set of formulae Γ, then each formula in $/\mathcal{D}/$ is a subformula of a formula in $\Gamma \cup \{A\}$.

Remark 5.2. In view of this result, we know that M-systems are consistent and that all theorems of M-systems can be established by means of normal canonical proofs.

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Definition 5.4. *Canonical derivation, canonical proof, thesis, and theorem (M-systems)*: The definitions of these notions for M-systems are analogous to those for R-systems (Definition 2.13).

5.2 The method of counter-derivations

Due to the subformula property, we may formulate a simple decision procedure for theoremhood.

Definition 5.5. *Method of counter-derivations* (cf. [41], [43]). Construct a candidate for a normal canonical M-proof of formula A by proceeding bottom-up using the rules for the operators ignoring the side conditions on them. In case (i) the construction has been successful, check whether the candidate violates a side condition. If this is the case, (ia) we obtain a *counter-derivation* for A, otherwise (ib) we obtain a normal M-proof of A. In case (ii) the construction of a candidate has not been successful, we may conclude that A cannot be derived as a theorem. Consequently, we get a decision concerning the M-derivability of A as a theorem. It is derivable as a theorem in case (ib), and underivable in cases (ia) and (ii).

Remark 5.3. The method can be used for the analysis of *counterfactual fallacies*. It essentially carves out counter-derivations from intuitionistic natural deduction proofs. Transitivity fails: $V6$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[((A \supset_c B)\&_*(B \supset_c C))]^{(1)}}{B \supset_c C}\,(\&_*\mathrm{E2}) \qquad \dfrac{\dfrac{[((A \supset_c B)\&_*(B \supset_c C))]^{(1)}}{A \supset_c B}\,(\&_*\mathrm{E1}) \qquad [\wr A \wr]^{(2)}}{B}\,(\supset_c\mathrm{E})\ V6}{C}\,(\supset_c\mathrm{E})}{A \supset_c C}\,(\supset_c\mathrm{I}),\ 2}{((A \supset_c B)\&_*(B \supset_c C)) \supset_* (A \supset_c C)}\,(\supset_*\mathrm{I}),\ 1 \tag{35}$$

Contraposition fails: $V6$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr \neg_* B \wr]^{(2)} \qquad \dfrac{[(A \supset_c B)]^{(1)} \qquad [\wr A \wr]^{(3)}}{B}\,(\supset_c\mathrm{E})\ V6}{\bot}\,(\supset_*\mathrm{E})}{\neg_* A}\,(\supset_*\mathrm{I}),\ 3}{(\neg_* B) \supset_c (\neg_* A)}\,(\supset_c\mathrm{I}),\ 2}{(A \supset_c B) \supset_* (\neg_* B \supset_c \neg_* A)}\,(\supset_*\mathrm{I}),\ 1 \tag{36}$$

Also the modus tolens version of contraposition $((A \supset_c B)\&_*\neg_* B) \supset_* \neg_* A$ is not a theorem due to $V6$. The converse of contraposition fails for M-systems, since case (ii) of the method applies. Also the following does not work for M-systems, given the fact that they are intuitionistic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\neg_* B \supset_c \neg_* A)]^{(1)} \qquad [\wr \neg_* B \wr]^{(3)}}{\neg_* A}\,(\supset_c\mathrm{E}) \qquad [\wr A \wr]^{(2)}}{\bot}\,(\supset_*\mathrm{E})}{B}\,(\bot_*\mathrm{c}),\ 3\ \textit{illeg.}}{A \supset_c B}\,(\supset_*\mathrm{I}),\ 2}{(\neg_* B \supset_c \neg_* A) \supset_* (A \supset_c B)}\,(\supset_*\mathrm{I}),\ 1 \tag{37}$$

Strengthening of the antecedent fails: $V5b$.

$$\dfrac{\dfrac{\dfrac{[(A \supset_c B)]^{(1)} \qquad \dfrac{[\wr A \&_* C \wr]^{(2)}}{A}\,(\&_*\mathrm{E1})}{B}\,(\supset_c\mathrm{E})}{(A \&_* C) \supset_c B}\,(\supset_c\mathrm{I}),\ 2\ V5b}{(A \supset_c B) \supset_* ((A \&_* C) \supset_c B)}\,(\supset_*\mathrm{I}),\ 1 \tag{38}$$

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Monotonicity fails for $\supset_c$: $V5a$.

$$\dfrac{\dfrac{[\wr A\wr]^{(1)}}{B \supset_c A}\,(\supset_c\text{I})\ V5a}{A \supset_c (B \supset_c A)}\,(\supset_c\text{I}),\ 1\ V5a \tag{39}$$

Monotonicity would be blocked also for derivations with detours, as Example 4.4 and what follows suggest:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr A\wr]^{(1)} \qquad [\wr B\wr]^{(2)}}{A \&_c B}\,(\&_c\text{I})}{A}\,(\&_c\text{E1})}{B \supset_c A}\,(\supset_c\text{I}),\ 2\ V5b}{A \supset_c (B \supset_c A)}\,(\supset_c\text{I}),\ 1\ V5b \tag{40}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr A\wr]^{(1)}}{A \vee_c (B \supset_c A)}\,(\vee_c\text{I1}) \qquad [\wr A\wr]^{(2)} \qquad \dfrac{[(B \supset_c A)]^{(3)} \qquad [\wr B\wr]^{(4)}}{A}\,(\supset_c\text{E})}{A}\,(\vee_c\text{E}),\ 2,\ 3}{B \supset_c A}\,(\supset_c\text{I}),\ 4\ V5c}{A \supset_c (B \supset_c A)}\,(\supset_c\text{I}),\ 1\ V5c \tag{41}$$

Remark 5.4. M-systems do justice to the fallacy of *transitivity* for $\supset_c$ by imposing SC3 on $\supset_c$E. This way of blocking transitivity is strong. It blocks the derivation of the transitivity formula $((A \supset_c B)\&_*(B \supset_c C)) \supset_* (A \supset_c C)$ as a theorem by blocking the transitive reasoning which gives rise to $V6$. A weaker way of blocking the derivation of the transitivity formula would be to impose a side condition on $\supset_c$I which requires that the premiss to which $\supset_c$I is applied must not have been derived by means of a break formula. We also mention that M-systems allow for subatomic transitive reasoning from counterfactual assumptions. This possibility may be interesting for a fine-grained assessment of specific cases.

$$\text{(a)}\ \dfrac{H\Gamma \qquad \dfrac{\dfrac{G\Gamma \qquad \dfrac{\wr Fa\wr}{a\Gamma}\,(as_*\text{E}_1)}{Ga}\,(as_*\text{I})}{a\Gamma}\,(as_*\text{E}_1)}{Ha}\,(as_*\text{I}) \qquad \text{(b)}\ \dfrac{\dfrac{\dfrac{H\Gamma \qquad \dfrac{\dfrac{[(Fa \supset_c Ga)]^{(1)} \qquad [\wr Fa\wr]^{(2)}}{Ga}\,(\supset_c\text{E})}{a\Gamma}\,(as_*\text{E}_1)}{Ha}\,(as_*\text{I})}{Fa \supset_c Ha}\,(\supset_c\text{I}),\ 2}{(Fa \supset_c Ga) \supset_* (Fa \supset_c Ha)}\,(\supset_c\text{I}),\ 1 \tag{42}$$

These derivations derive theses which are not theorems. In neither of them Ga is a break formula; note that it is a maximum formula in the first derivation.

Remark 5.5. Another application of the method is to the *assessment of some axioms* of model-theoretically defined conditional logics (e.g., Lewisian similarity-based conditional logics) from our proof-centered perspective on counterfactual inference (Figure 3; M-Th abbreviates 'M-theorem'). We consider CEM and CDA. CEM: Let $X = \neg_*((A \supset_c B) \vee_* (A \supset_c \neg_* B))$.

$$\dfrac{[(X)]^{(1)} \qquad \dfrac{\dfrac{\dfrac{[(\neg_*(A \supset_c B))]^{(2)} \qquad \dfrac{\dfrac{\dfrac{[(X)]^{(1)} \qquad \dfrac{\dfrac{[/\neg_* B/]^{(3)}}{A \supset_c \neg_* B}\,(\supset_c\text{I})\ V5a}{(A \supset_c B) \vee_* (A \supset_c \neg_* B)}\,(\vee_*\text{I2})}{\bot}\,(\supset_*\text{E})}{B}\,(\bot_*\text{c}),\ 3\ \textit{illeg.}}{A \supset_c B}\,(\supset_c\text{I})\ V5a}{\bot}\,(\supset_*\text{E})}{A \supset_c B}\,(\bot_*\text{c}),\ 2\ \textit{illeg.}}{(A \supset_c B) \vee_* (A \supset_c \neg_* B)}\,(\vee_*\text{I1})}{\dfrac{\bot}{(A \supset_c B) \vee_* (A \supset_c \neg_* B)}\,(\bot_*\text{c}),\ 1}\,(\supset_*\text{E}) \tag{43}$$

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	Name	Axiom	Fragm.	M-Th	Violation
1.	MOD	$(\neg_* A \supset_c A) \supset_* (B \supset_c A)$	I	no	$V5a$, $V6$, $[\bot_*\mathrm{c}]$
2.	MP	$(A \supset_c B) \supset_* (A \supset_* B)$	I	yes	none
3.	Imp	$((A \&_* B) \supset_c C)) \supset_* (A \supset_c (B \supset_c C))$	I, C	no	$V5b$
4.	Exp	$(A \supset_c (B \supset_c C)) \supset_* ((A \&_* B) \supset_c C)$	I, C	yes	none
5.	CC	$((A \supset_c B) \&_* (A \supset_c C)) \supset_* (A \supset_c (B \&_c C))$	I, C	yes	none
6.	CM	$((A \supset_c (B \&_* C)) \supset_* ((A \supset_c B) \&_* (A \supset_c C))$	I, C	no	$V5b$
7.	CS	$(A \&_* B) \supset_* (A \supset_c B)$	I, C	no	$V5a$, $V5b$
8.	CSO	$((A \supset_c B) \&_* (B \supset_c A)) \supset_* ((A \supset_c C) \leftrightarrow_* (B \supset_c C))$	I, C	no	$V6$
9.	CV	$((A \supset_c B) \&_* \neg (A \supset_c \neg C)) \supset_* ((A \&_* C) \supset_c B)$	I, C	no	$V5b$
10.	RT	$((A \&_* B) \supset_c C) \supset_* ((A \supset_c B) \supset_* (A \supset_c C))$	I, C	no	$V5b$
11.	SNCA	$(\neg_* (A \&_* B) \supset_c C) \supset_* ((\neg_* A \supset_c C) \&_* (\neg_* B \supset_c C))$	I, C	no	$V5b$
12.	CEM	$(A \supset_c B) \vee_* (A \supset_c \neg B)$	I, D	no	$V5a$, $[\bot_*\mathrm{c}]$
13.	CDA	$(A \supset_c (B \vee_* C)) \supset_* (A \supset_c (\neg_* B \supset_c C))$	I, D	yes	none
14.	CA	$((A \supset_c C) \&_* (B \supset_c C)) \supset_* ((A \vee_* B) \supset_c C)$	I, C, D	yes	none
15.	SDA	$((A \vee_* B) \supset_c C) \supset_* ((A \supset_c C) \&_* (B \supset_c C))$	I, C, D	yes	none

Figure 3: Assessment of some axioms

CDA (cf. [13]: 196):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(A \supset_c (B \vee_* C))]^{(1)} \quad [\wr A \wr]^{(2)}}{B \vee_* C}\,(\supset_c\mathrm{E}) \qquad \dfrac{\dfrac{[\wr \neg_* B \wr]^{(3)} \quad [\not\wr\, B \not\wr]^{(4)}}{\bot}\,(\supset_*\mathrm{E})}{C}\,(\bot_*\mathrm{i}) \qquad [/C/]^{(5)}}{C}\,(\vee_*\mathrm{E}),\ 4,\ 5}{\neg_* B \supset_c C}\,(\supset_c\mathrm{I}),\ 3}{A \supset_c (\neg_* B \supset_c C)}\,(\supset_c\mathrm{I}),\ 2}{(A \supset_c (B \vee_* C)) \supset_* (A \supset_c (\neg_* B \supset_c C))}\,(\supset_*\mathrm{I}),\ 1 \tag{44}$$

This derivation is not available for minimal M-systems, since it makes crucial use of the intuitionistic absurdity rule.

5.3 Internal completeness

M-systems can be classified as internally complete in the sense of Girard ([10]: 139-40), mentioned in the Introduction, where internal completeness is equated with the subformula property. We shall express this as follows:

Definition 5.6. A proof system is *internally complete* in case it enjoys the subformula property.

In order to adapt Girard's remarks that precede his equation to the present setting, we sharpen them as follows: we take "cut-free proof" to mean "normal canonical proof" and "calculus" to mean "M-system", thereby insisting on proofs in intuitionistic systems. Using the above definition, we obtain, by way of rephrasing Corollary 5.1:

Corollary 5.2. M-systems are internally complete.

Remark 5.6. 1. We may thus regard completeness as an internal property of M-systems. Their completeness does not need to be relativized to external structures—it is absolute in this sense.

2. Internal completeness in the sense of Definition 5.6 cannot be taken for granted, of course. In the present context, it is worth noting that labelled proof systems for counterfactual logics (e.g., [19], [24]) are not internally complete in this sense, since they do not possess the subformula property. For such systems a weaker form of internal completeness

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seems available, one that equates it with normalization (resp. cut-elimination) without insisting on the subformula property.

3. We mention that M-systems are also internally complete in the sense of [42], Theorem 3.47.

6 Proof-theoretic semantics

6.1 Meaning and truth

The semantics of $L1$ is defined proof-theoretically.

Definition 6.1. Let S be an M-system. The *meaning* of

1. a *non-logical constant* τ is given by the term assumptions $\tau\Gamma$ determined by the subatomic base of S for τ;
2. an $L1$-formula A is given by the set of canonical derivations of A in S (Definition 5.4).

Remark 6.1. 1. Unlike for $\supset_f$ and $\supset_c$, there seems to be no obvious natural language counterpart for $\&_f$, $\&_c$, $\vee_f$, and $\vee_c$. The rules for the latter can be seen as devices for sharpening our perception of the structure of inferences from factual and counterfactual assumptions. For the proof-theoretic modeling of most aspects of the meaning of natural language conjunction and disjunction the rules for $\&_*$ and $\vee_*$ are, presumably, sufficient.

2. It can be conjectured that an addition of factual, counterfactual, and independent existential [universal] quantifiers to M-systems will exploit the general analogy between the existential [universal] quantifier with disjunction [conjunction].

Definition 6.2. A formula A of $L1$ is a *truth* [*logical truth*], in case A is the conclusion of a canonical derivation [proof] in an M-system (cf. [43]: 417).

Remark 6.2. The explanations of meaning and truth given above are distinctively intuitionistic, since they rest on the notion of a canonical derivation. They would not be available for M-systems, if we were to replace $\bot_*$i by the *classical absurdity rule* $\bot_*$c. If we were to do this and to drop the violated side conditions, we would consider, e.g., MOD and CEM theorems. Note, however, that, by Definition 5.4, their derivations (e.g., that of CEM above) would not be canonical; and this lack of canonicity would preclude an intuitionistically acceptable proof-theoretic semantics. Obviously, also the subformula property, as given by Corollary 5.1, would be lost—and with it internal completeness as based on that corollary.

6.2 Expansions

Our definition of a proof-theoretic semantics for $L1$ in terms of canonical derivations is based entirely on the normalization result for intuitionistic M-systems. It does not insist on the availability of expansions (e.g., [7]) which, roughly, transform a derivation of A into one which first decomposes A by an E-rule in order to recompose it by the corresponding I-rule in the next step. Below, we formulate expansions and discuss their preservation behaviour.

Definition 6.3. Let A be a formula of $L1$. An *expansion* of a derivation $\genfrac{}{}{0pt}{}{/\mathcal{D}/}{A}$ in an M-system is a derivation $\genfrac{}{}{0pt}{}{/\mathcal{D}'/}{A}$ in that system in which A first occurs as major premiss of an application of an E-rule and immediately afterwards as a conclusion of an application an I-rule. The expansions of derivations in M-systems are:

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1. *as_*-Expansion*: Let A be $\varphi^n\alpha_1...\alpha_n$.

$$\begin{array}{c}/\mathcal{D}/\\ \varphi^n\alpha_1...\alpha_n\end{array} \quad \text{expn} \quad \dfrac{\dfrac{\begin{array}{c}/\mathcal{D}/\\ \varphi^n\alpha_1...\alpha_n\end{array}}{\varphi^n\Gamma}\,(as_*\mathrm{E}_0) \quad \dfrac{\begin{array}{c}/\mathcal{D}/\\ \varphi^n\alpha_1...\alpha_n\end{array}}{\alpha_1\Gamma}\,(as_*\mathrm{E}_1) \quad ... \quad \dfrac{\begin{array}{c}/\mathcal{D}/\\ \varphi^n\alpha_1...\alpha_n\end{array}}{\alpha_n\Gamma}\,(as_*\mathrm{E}_n)}{\varphi^n\alpha_1...\alpha_n}\,(as_*\mathrm{I})$$

2. *Expansions for $\star^i$ [$\star^c$, $\star^d$]*: We present only the expansions for the factual operators; those for the counterfactual [mode-sensitive] operators are similar.

$$\begin{array}{c}/\mathcal{D}/\\ A \supset_f B\end{array} \;\text{expn}\; \dfrac{\dfrac{\begin{array}{c}/\mathcal{D}/\\ A \supset_f B\end{array} \quad [|A|]^{(u)}}{B}\,(\supset_f\mathrm{E})}{A \supset_f B}\,(\supset_f\mathrm{I}),\, u \qquad \begin{array}{c}|\mathcal{D}|\\ A\&_f B\end{array} \;\text{expn}\; \dfrac{\dfrac{\begin{array}{c}|\mathcal{D}|\\ A\&_f B\end{array}}{A}\,(\&_f\mathrm{E1}) \quad \dfrac{\begin{array}{c}|\mathcal{D}|\\ A\&_f B\end{array}}{B}\,(\&_f\mathrm{E2})}{A\&_f B}\,(\&_f\mathrm{I})$$

$$\begin{array}{c}/\mathcal{D}/\\ A \vee_f B\end{array} \quad \text{expn} \quad \dfrac{\begin{array}{c}/\mathcal{D}/\\ A \vee_f B\end{array} \quad \dfrac{[|A|]^{(u)}}{A \vee_f B}\,(\vee_f\mathrm{I1}) \quad \dfrac{[|B|]^{(v)}}{A \vee_f B}\,(\vee_f\mathrm{I2})}{A \vee_f B}\,(\vee_f\mathrm{E}),\, u,\, v$$

Call the expansions for $\&_f$ [$\&_c$] *restricted*, as $/\mathcal{D}/$, $/\mathcal{D}'/$ are restricted to $|\mathcal{D}|$, $|\mathcal{D}'|$ [$\wr\mathcal{D}\wr$, $\wr\mathcal{D}'\wr$].

Theorem 6.1. The expansions do not transform legitimate into illegitimate derivations.

Proof. Assume, in each case, that the derivation to be expanded is legitimate. The result follows immediately. Note, however, that it fails for unrestricted expansions for $\&_f$ [$\&_c$]. □

Remark 6.3. The failure of preservation for the unrestricted expansions for $\&_f$ [$\&_c$] suggests that we may claim only a weak form of "local completeness" (cf. [7]: 93) for the rules for $\&_f$ [$\&_c$], since we cannot claim that every derivation of $A\&_f B$ [$A\&_c B$] can be expanded. It may be interesting to consider modified rules for these operators for which this claim can be made; e.g., ones which require factual [counterfactual] status for the premisses of $\&_f$E [$\&_c$E]. The preservation tables should be helpful here.

6.3 Refinements

In the literature, *might*-counterfactuals ($A >_m B$; cf. [2]: 189) are usually defined in terms of *would*-counterfactuals ($A > B$) on the basis of a model-theoretic semantics. Prominent definitions of this kind ($\Diamond_{[e]}$ means [epistemic] possibility) are:

(L) $A >_m B =_{def} \neg(A > \neg B)$ (cf. Lewis [17]: 21)

(S) $A >_m B =_{def} \Diamond_e(A > B)$ (cf. Stalnaker [29]: 101, [30])

(B) $A >_m B =_{def} A > \Diamond B$ (cf. Bennett [2]: 191)

According to (L), $>_m$ is a dual of $>$; and in (S) and (B) $\Diamond$ is a dual of $\Box$ (necessity). (Digression: Lewis has shown ([17]: 80) that an endorsement of (L) leads to a loss of the difference between $>_m$ and $>$, in case the logic governing the latter contains CEM which is validated in Stalnaker's preferred system (whose similarity truth conditions allow only for exactly one A-world closest to w). Due to this conflict with CEM, (L) is not an option for Stalnaker. It is essentially for this reason that he prefers (S), with an epistemic reading of $\Diamond$.) In each case, the duality rests on the interdefinability of the existential and the

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universal quantifiers (over worlds) in the context of classical logic. Since in an intuitionistic setting there is no such interdefinability (e.g., [40]: 468), we have to look for an alternative account of the *might*-counterfactual.

In a specific sense, such an account is implicit in our intuitionistic modal proof systems for counterfactual inference. It is, perhaps, noteworthy that this account does not require an introduction of possibility operators which draw on modes of assumption. Similarly, concerning the *would*-counterfactual, no use needs to be made of a necessity operator or the idea of the (variable) strictness of implication. Recall, the meaning of implications (cf. Definition 6.1(2)) has been explained in terms of canonical derivations. Their general form is:

$$\begin{array}{c} [/A/]^{(u)} \\ /\mathcal{D}_1/ \\ \dfrac{B}{A \star^i B}\,(\star^i\mathrm{I}),\, u \end{array} \tag{45}$$

These rules "listen", so to speak, only to the status of the assumed formula A, they do not listen to the status of the premiss B (i.e., the consequent-node) to which $\star^i$I is applied; in this sense, these rules are *"mono"*. What matters on the mono conception of $\star^i$I-rules, is the mere fact that B has been derived—irrespective of its status—from A that has been assumed in a given mode. For example, if the status of A is counterfactual, the formula introduced by $\star^i$I is a counterfactual, a *would*-counterfactual, irrespective of the status of B, provided that the side conditions have been respected.

In what follows, we shall define $\star^i$I-rules which may listen not only to the status of A, but also to that of B; these rules are *"stereo"*. This refinement enlarges the range of application of M-systems, as they may be used for the analysis of constructions other than (1.1) and (1.8). Consider the following constructions (with proposed readings):

Factuals:

(f1) Since A is the case, B is the case. (1.8)
(read: Since A is the case, B is factually the case)

(f2) Since A is the case, B might be the case. (1.11)
(read: Since A is the case, B is counterfactually the case)

(f3) Since A is the case, B.

Counterfactuals:

(c1) If A were the case, B would be the case. (1.1)
(read: If A were the case, B is factually the case)

(c2) If A were the case, B might be the case. (1.10)
(read: If A were the case, B is counterfactually the case)

(c3) If A were the case, B.

Mode-sensitives:

(m1) If A, B is factually the case.

(m2) If A, B is counterfactually the case.

(m3) If A, B.

The proposed readings indicate that we use 'would' in a factual sense, presupposing that B has factual status, and 'might' in a counterfactual sense, presupposing that B has counterfactual status.

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We refine $L1$ and the rules of M-systems so as to explain the meaning of (f1)-(m3) from a stereo perspective. To this end, we extend $L1$ with implication operators (with two subscripts) which indicate the status of the antecedent (first subscript) and the status of the consequent (second subscript).

Definition 6.4. *Stereo implications*:

f1:	$A \supset_{f,f} B$	f2:	$A \supset_{f,c} B$	f3:	$A \supset_{f,*} B$
c1:	$A \supset_{c,f} B$	c2:	$A \supset_{c,c} B$	c3:	$A \supset_{c,*} B$
m1:	$A \supset_{*,f} B$	m2:	$A \supset_{*,c} B$	m3:	$A \supset_{*,*} B$

The rules for these implications are special cases of the rules for the "mono" $\star^i$-operators of Definition 3.11.

Definition 6.5. *Rules for stereo implications. I-rules*:

$$\text{(f1.)}\ \dfrac{\begin{matrix}[|A|]^{(u)}\\ |\mathcal{D}_1|\\ B\end{matrix}}{A \supset_{f,f} B}\,(\supset_{f,f}\mathrm{I}),\ u \qquad \text{(f2.)}\ \dfrac{\begin{matrix}[|A|]^{(u)}\\ \wr\mathcal{D}_1\wr\\ B\end{matrix}}{A \supset_{f,c} B}\,(\supset_{f,c}\mathrm{I}),\ u \qquad \text{(f3.)}\ \dfrac{\begin{matrix}[|A|]^{(u)}\\ /\,\{\!\!\wr\, \mathcal{D}_1 \,\wr\!\!\}\,/\\ B\end{matrix}}{A \supset_{f,*} B}\,(\supset_{f,*}\mathrm{I}),\ u$$

$$\text{(c1.)}\ \dfrac{\begin{matrix}[\wr A\wr]^{(u)}\\ |\wr\mathcal{D}_1\wr|\\ B\end{matrix}}{A \supset_{c,f} B}\,(\supset_{c,f}\mathrm{I}),\ u \qquad \text{(c2.)}\ \dfrac{\begin{matrix}[\wr A\wr]^{(u)}\\ \wr\mathcal{D}_1\wr\\ B\end{matrix}}{A \supset_{c,c} B}\,(\supset_{c,c}\mathrm{I}),\ u \qquad \text{(c3.)}\ \dfrac{\begin{matrix}[\wr A\wr]^{(u)}\\ /\wr\mathcal{D}_1\wr/\\ B\end{matrix}}{A \supset_{c,*} B}\,(\supset_{c,*}\mathrm{I}),\ u$$

$$\text{(m1.)}\ \dfrac{\begin{matrix}[/A/]^{(u)}\\ |/\mathcal{D}_1/|\\ B\end{matrix}}{A \supset_{*,f} B}\,(\supset_{*,f}\mathrm{I}),\ u \qquad \text{(m2.)}\ \dfrac{\begin{matrix}[/A/]^{(u)}\\ \wr\mathcal{D}_1\wr\\ B\end{matrix}}{A \supset_{*,c} B}\,(\supset_{*,c}\mathrm{I}),\ u \qquad \text{(m3.)}\ \dfrac{\begin{matrix}[/A/]^{(u)}\\ /\mathcal{D}_1/\\ B\end{matrix}}{A \supset_{*,*} B}\,(\supset_{*,*}\mathrm{I}),\ u$$

E-rules: *Mutatis mutandis*, like those for mono implications. Side conditions: Like for mono implications.

Remark 6.4. 1. The I/E-rules for $\supset_{f,*}$ [$\supset_{c,*}$, $\supset_{*,*}$] are equivalent to those for $\supset_f$ [$\supset_c$, $\supset_*$]. The rules $\supset_{f,f}$I and $\supset_{f,c}$I are special cases of $\supset_{f,*}$I. Similarly, $\supset_{c,f}$I and $\supset_{c,c}$I are special cases of $\supset_{c,*}$I. And $\supset_{*,f}$I and $\supset_{*,c}$I are special cases of $\supset_{*,*}$I.

2. In the I-rules for stereo implications the status of B after the discharge matters. As a consequence, there can be only canonical derivations, but no canonical *proofs* (and so no proofs at all) of formulae of the forms $A \supset_{f,c} B$ and $A \supset_{c,f} B$ (including the case in which $B = A$).

3. Let the subscript 'i' stand for 'independent'. In $\supset_{f,*}$I: If $/\,\{\!\!\wr\, \mathcal{D}_1 \,\wr\!\!\}\,/ = (|\mathcal{D}_1|)$, then $\supset_{f,*}$ is $\supset_{f,i}$. In $\supset_{c,*}$: If $/\wr\mathcal{D}_1\wr/ = (\wr\mathcal{D}_1\wr)$, then $\supset_{c,*}$ is $\supset_{c,i}$. In $\supset_{*,*}$: If $/\mathcal{D}_1/ = (\mathcal{D}_1)$, then $\supset_{*,*}$ is $\supset_{*,i}$. If $[/A/]^{(u)} = [(A)]^{(u)}$, then $\supset_{*,f}$ is $\supset_{i,f}$ in $\supset_{*,f}$I, $\supset_{*,c}$ is $\supset_{i,c}$ in $\supset_{*,c}$I, and $\supset_{*,*}$ is $\supset_{i,*}$ in $\supset_{*,*}$I. The shape of canonical derivations of formulae of the form $A \supset_{i,i} B$ is, thus, obvious, as is the fact that the $\supset$-operator used in R-systems is a special case of the $\supset_{i,i}$-operator used in M-systems.

4. The shape of the I-rules for the $\star^c$- and the $\star^d$-operators precludes stereo-versions of them. It also seems less obvious than in the case of $\star^i$ that there might be a natural language rationale for considering such versions.

Definition 6.6. *Conversions for stereo implications.* The conversions for $\supset_{f,f}$ [$\supset_{f,c}$, $\supset_{f,*}$], $\supset_{c,f}$ [$\supset_{c,c}$, $\supset_{c,*}$], and $\supset_{*,f}$ [$\supset_{*,c}$, $\supset_{*,*}$] take the expected shapes.

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Remark 6.5. The results of conversions (i.e., the *conversa*; [33]: 12) for stereo factuals differ in shape, while there is only one form of conversum for stereo counterfactuals, and two for stereo mode-sensitives. Preservation for stereo implications is a special case of preservation for mono implications.

Remark 6.6. As the rules for stereo implications are set up, (f1) [(c1), (m1)] does not imply (f2) [(c2), (m2)], and vice versa. This is due to the factual understanding of the consequents of (f/c/m1) and to the counterfactual understanding of the consequents of (f/c/m2). The rules can be altered. For example, we may want (f1) [(c1), (m1)] to imply (f2) [(c2), (m2)], but not vice versa. To achieve this, we may espouse the following relaxed readings of (f/c/m2), on which the consequents can have either counterfactual or independent status:

(f2′) Since A is the case, B might be the case.
(read: Since A is the case, B is counterfactually/independently the case)

(c2′) If A were the case, B might be the case.
(read: If A were the case, B is counterfactually/independently the case)

(m2′) If A, B is counterfactually/independently the case.

The I-rules for these relaxed *might*-implications take the forms displayed below, whereas the E-rules are, *mutatis mutandis*, like those for the unrelaxed stereo implications:

$$\text{(f2}'.)\ \frac{\begin{array}{c}[|A|]^{(u)}\\ \Vert\!\wr\ \mathcal{D}_1\ \wr\!\Vert\\ B\end{array}}{A \supset_{f,c/i} B}\,(\supset_{f,c/i}\mathrm{I}),\ u \qquad \text{(c2}'.)\ \frac{\begin{array}{c}[\wr A\wr]^{(u)}\\ \Vert\ \wr\mathcal{D}_1\wr\ \Vert\\ B\end{array}}{A \supset_{c,c/i} B}\,(\supset_{c,c/i}\mathrm{I}),\ u \qquad \text{(m2}'.)\ \frac{\begin{array}{c}[/A/]^{(u)}\\ \Vert\ /\mathcal{D}_1/\ \Vert\\ B\end{array}}{A \supset_{*,c/i} B}\,(\supset_{*,c/i}\mathrm{I}),\ u$$

The conversions for $\supset_{f,c/i}$, $\supset_{c,c/i}$, $\supset_{*,c/i}$ should be obvious. On the relaxed reading, e.g., (f1) [(c1), (m1)] implies (f2′) [(c2′), (m2′)], but not conversely. We note that relaxed operators of the forms $\supset_{f,f/i}$, $\supset_{c,f/i}$, and $\supset_{*,f/i}$ can be introduced in a similar way.

Remark 6.7. In general, the exact shape of a modal proof system will depend upon decisions concerning, e.g., the choice of the reference proof system, the conception of an established fact, the modes in which formulae of a given form can be legitimately assumed, the shape of the rules, or the side conditions imposed on them (cf. [43]: 415).

Acknowledgements

Versions of this paper have been presented at the Carl Friedrich von Weizsäcker Center of the University of Tübingen (in April of 2023), at the University of Milan (Logic Colloquium 2023), at the Institute of History and Philosophy of Science and Technology (IHPST) of the University of Paris 1 Panthéon-Sorbonne (in June of 2023), at the Department of Mathematics of the University of Genoa (in October of 2023), at the Department of Philosophy of the University of Łódź (in November of 2023), and at the University of Zagreb (ZLC24). I wish to thank the audiences for very helpful feedback. I am particularly grateful to Francesca Poggiolesi, Sara Negri, and Andrzej Indrzejczak for substantial discussions during my visits, and to the anonymous referee for this journal for valuable comments and suggestions.

Funding

This research was supported by the Deutsche Forschungsgemeinschaft (grant number WI 3456/5-1).

https://doi.org/10.1017/bsl.2025.20 Published online by Cambridge University Press

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