Proceedings of the Edinburgh Mathematical Society (2006) **49**, 435–503 ©
DOI:10.1017/S0013091504000562 Printed in the United Kingdom

MODULI CURVES OF SUPERSINGULAR $K3$ SURFACES IN CHARACTERISTIC 2 WITH ARTIN INVARIANT 2

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(Received 11 June 2004)

Dedicated to Professor Igor V. Dolgachev for his 60th birthday

Abstract We construct moduli curves of polarized supersingular $K3$ surfaces in characteristic 2 with Artin invariant 2. As an application, we detect a 'jump' phenomenon in a family of automorphism groups of supersingular $K3$ surfaces with a constant Néron–Severi lattice.

Keywords: supersingular $K3$ surface; moduli; binary code; Cremona transformation

2000 *Mathematics subject classification:* Primary 14J28
Secondary 14J10; 14Q10

1. Introduction

A $K3$ surface is called *supersingular* if its numerical Néron–Severi lattice is of rank 22. Supersingular $K3$ surfaces exist only in positive characteristics. Artin showed in [**1**] that, in characteristic $p > 0$, the discriminant of the numerical Néron–Severi lattice of a supersingular $K3$ surface X is of the form $-p^{2\sigma(X)}$, where $\sigma(X)$ is a positive integer less than or equal to 10. This integer $\sigma(X)$ is called the *Artin invariant* of X.

We work over an algebraically closed field k of characteristic 2.

Definition 1.1. Let X be a supersingular $K3$ surface, and let $\mathcal{L}$ be a line bundle on X with $\mathcal{L}^2 = 2$. We say that $\mathcal{L}$ is a *polarization of type* ($\sharp$) if the following conditions are satisfied:

(i) the complete linear system $|\mathcal{L}|$ has no fixed components;

(ii) the set of curves contracted by the morphism $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ defined by $|\mathcal{L}|$ consists of 21 disjoint (-2)-curves.

In [**10**], we have shown that every supersingular $K3$ surface X in characteristic 2 has a polarization of type ($\sharp$), and that, if $\mathcal{L}$ is a polarization of type ($\sharp$) on X, then the morphism $\Phi_{|\mathcal{L}|}$ is purely inseparable. In [**11**], we have constructed a nine-dimensional moduli space $\mathfrak{M}$ of polarized supersingular $K3$ surfaces of type ($\sharp$). In this paper, we investigate the locus $\mathfrak{M}_2$ of $\mathfrak{M}$ corresponding to supersingular $K3$ surfaces with Artin invariant 2. We will show that the curve $\mathfrak{M}_2$ is a disjoint union of three affine lines punctured at the origin. We will also construct explicitly the universal family of polarized supersingular $K3$ surfaces over certain finite covers of these punctured affine lines. The construction involves investigations of configurations of lines and conics on the projective plane in characteristic 2. These configurations are encoded by certain binary codes. In order to construct the moduli curve, we have to determine the automorphism groups of these codes. The automorphism group of the polarized $K3$ surface is obtained from the automorphism group of the corresponding code.

Let us briefly review the construction of the moduli space $\mathfrak{M}$ in [**11**]. For a non-zero homogeneous polynomial $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ of degree 6, we denote by

$$\pi_G : Y_G \to \mathbb{P}^2$$

the purely inseparable double cover of $\mathbb{P}^2$ defined by $W^2 = G(X, Y, Z)$.

Definition 1.2. Let $\mathcal{U}$ denote the locus of all non-zero homogeneous polynomials $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ such that the surface Y_G has 21 ordinary nodes as its only singularities.

The locus $\mathcal{U}$ is Zariski open dense in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. Indeed, in characteristic 2, the differential $\mathrm{d}G$ of G can be defined as a global section of $\Omega^1_{\mathbb{P}^2}(6)$ for any homogeneous polynomial $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$, because, by the isomorphism $\mathcal{O}_{\mathbb{P}^2}(6) \cong \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$, we can assume that the transition functions of the line bundle corresponding to $\mathcal{O}_{\mathbb{P}^2}(6)$ are all squares. Since $c_2(\Omega^1_{\mathbb{P}^2}(6)) = 21$, the subscheme $Z(\mathrm{d}G)$ defined by $\mathrm{d}G = 0$ is reduced and of dimension 0 if and only if it consists of 21 points. The singular locus $\mathrm{Sing}(Y_G)$ of Y_G is equal to $\pi_G^{-1}(Z(\mathrm{d}G))$, and the singular point of Y_G lying over a reduced point of $Z(\mathrm{d}G)$ is an ordinary node. Hence, the condition that G be a point of $\mathcal{U}$ is equivalent to the open condition that $Z(\mathrm{d}G)$ be reduced and of dimension 0.

Let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type $(\sharp)$. There then exists a homogeneous polynomial $G \in \mathcal{U}$ such that the Stein factorization of $\Phi_{|\mathcal{L}|}$ may be written as

$$X \xrightarrow{\rho_G} Y_G \xrightarrow{\pi_G} \mathbb{P}^2.$$

Conversely, suppose that we are given $G \in \mathcal{U}$. Let $\rho_G : X_G \to Y_G$ be the minimal resolution of the surface Y_G. Then X_G is a supersingular $K3$ surface, and the invertible sheaf

$$\mathcal{L}_G := (\pi_G \circ \rho_G)^* \mathcal{O}_{\mathbb{P}^2}(1)$$

on X_G is a polarization of type $(\sharp)$.

We put

$$\mathcal{V} := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)).$$

Because we have $\mathrm{d}(G + H^2) = \mathrm{d}G$ for any $H \in \mathcal{V}$, the additive group $\mathcal{V}$ acts on the space $\mathcal{U}$ by

$$(G, H) \in \mathcal{U} \times \mathcal{V} \mapsto G + H^2 \in \mathcal{U}.$$

Proposition 1.3. *Let G and G' be homogeneous polynomials in $\mathcal{U}$. Then the following conditions are equivalent:*

(i) *Y_G and $Y_{G'}$ are isomorphic over $\mathbb{P}^2$,*

(ii) *$Z(\mathrm{d}G) = Z(\mathrm{d}G')$, and*

(iii) *there exist $c \in k^\times$ and $H \in \mathcal{V}$ such that $G' = cG + H^2$.*

See § 2 for the proof.

Therefore, the moduli space $\mathfrak{M}$ of polarized supersingular $K3$ surfaces of type (♯) is constructed by

$$\mathfrak{M} = \mathrm{PGL}(3, k) \setminus \mathbb{P}_*(\mathcal{U}/\mathcal{V}).$$

For $G \in \mathcal{U}$, let $[G]$ denote the point of $\mathfrak{M}$ corresponding to G, which corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_G, \mathcal{L}_G)$ of type (♯). By Proposition 1.3, the automorphism group $\mathrm{Aut}(X_G, \mathcal{L}_G)$ of the polarized supersingular $K3$ surface is canonically identified with

$$\{g \in \mathrm{PGL}(3, k) \mid g(Z(\mathrm{d}G)) = Z(\mathrm{d}G)\}.$$

The moduli space $\mathfrak{M}$ is stratified by the Artin invariant $\sigma(X_G)$ of X_G. We put

$$\mathfrak{M}_\sigma := \{[G] \in \mathfrak{M} \mid \sigma(X_G) = \sigma\} \quad \text{and} \quad \mathfrak{M}_{\leqslant\sigma} := \{[G] \in \mathfrak{M} \mid \sigma(X_G) \leqslant \sigma\}.$$

As was shown in [**11**], the locus $\mathfrak{M}_{\leqslant 1} = \mathfrak{M}_1$ consists of a single point $[G_{\mathrm{DK}}]$, where

$$G_{\mathrm{DK}} := XYZ(X^3 + Y^3 + Z^3)$$

is the homogeneous polynomial of Dolgachev and Kondo [**5**]. The points $Z(\mathrm{d}G_{\mathrm{DK}})$ coincide with the $\mathbb{F}_4$-rational points of $\mathbb{P}^2$ and, hence, the group $\mathrm{Aut}(X_{G_{\mathrm{DK}}}, \mathcal{L}_{G_{\mathrm{DK}}})$ is equal to $\mathrm{PGL}(3, \mathbb{F}_4)$. We call $[G_{\mathrm{DK}}]$ the *Dolgachev–Kondo point*.

Now we can state our main results.

Theorem 1.4. *The locus $\mathfrak{M}_{\leqslant 2}$ is a union of three irreducible curves $\overline{\mathfrak{M}}_A$, $\overline{\mathfrak{M}}_B$ and $\overline{\mathfrak{M}}_C$. In $\mathfrak{M}$, they are situated in such a way that, set-theoretically,*

$$\overline{\mathfrak{M}}_A \cap \overline{\mathfrak{M}}_B = \overline{\mathfrak{M}}_B \cap \overline{\mathfrak{M}}_C = \overline{\mathfrak{M}}_C \cap \overline{\mathfrak{M}}_A = \{[G_{\mathrm{DK}}]\}.$$

For $T = A$, B and C, we put

$$\mathfrak{M}_T := \overline{\mathfrak{M}}_T \setminus \{[G_{\mathrm{DK}}]\}.$$

Hence, $\mathfrak{M}_2$ is the disjoint union of $\mathfrak{M}_A$, $\mathfrak{M}_B$ and $\mathfrak{M}_C$.

Theorem 1.5. *For $T = A$, B and C, the curve $\mathfrak{M}_T$ is isomorphic to an affine line punctured at the origin.*

We will describe the curves $\mathfrak{M}_T$ more explicitly. Let $\omega \in \mathbb{F}_4$ be a primitive third root of unity, and let $\bar{\omega}$ be $\omega + 1 = \omega^2$.

We fix a finite set $\mathcal{P} := \{P_1, \dots, P_21\}$ consisting of 21 elements. A *marking* of a polarized supersingular $K3$ surface $(X_G, \mathcal{L}_G)$ of type (♯) is a bijective map $\gamma : \mathcal{P} \to Z(\mathrm{d}G)$.

Table 1. *Marking for* $\mathfrak{M}_A$

$\gamma_\lambda(P_1) = [1, \omega, 0]$	$\gamma_\lambda(P_8) = [1, 1+\lambda, 1]$	$\gamma_\lambda(P_{15}) = [1, \lambda, 1]$
$\gamma_\lambda(P_2) = [1, \bar{\omega}, 0]$	$\gamma_\lambda(P_9) = [1, 1, 1]$	$\gamma_\lambda(P_{16}) = [0, 1+\lambda, 1]$
$\gamma_\lambda(P_3) = [1+\lambda, 1+\lambda, 1]$	$\gamma_\lambda(P_{10}) = [0, \lambda, 1]$	$\gamma_\lambda(P_{17}) = [\lambda, 0, 1]$
$\gamma_\lambda(P_4) = [1+\lambda, \lambda, 1]$	$\gamma_\lambda(P_{11}) = [0, 1, 1]$	$\gamma_\lambda(P_{18}) = [1, 0, 0]$
$\gamma_\lambda(P_5) = [\lambda, \lambda, 1]$	$\gamma_\lambda(P_{12}) = [0, 1, 0]$	$\gamma_\lambda(P_{19}) = [1, 0, 1]$
$\gamma_\lambda(P_6) = [\lambda, 1+\lambda, 1]$	$\gamma_\lambda(P_{13}) = [1, 1, 0]$	$\gamma_\lambda(P_{20}) = [0, 0, 1]$
$\gamma_\lambda(P_7) = [1+\lambda, 1, 1]$	$\gamma_\lambda(P_{14}) = [\lambda, 1, 1]$	$\gamma_\lambda(P_{21}) = [1+\lambda, 0, 1]$

Theorem 1.6. *Let* Γ_A *be the group*

$$\left\{\lambda, \lambda+1, \frac{1}{\lambda}, \frac{1}{\lambda+1}, \frac{\lambda}{\lambda+1}, \frac{\lambda+1}{\lambda}\right\}$$

acting on the punctured λ*-line* $\mathbb{A}^1 \setminus \{0,1\} = \operatorname{Spec} k[\lambda, 1/\lambda(\lambda+1)]$. *We put*

$$J_A := \frac{(\lambda^2+\lambda+1)^3}{\lambda^2(\lambda+1)^2},$$

so that $k[\lambda, 1/\lambda(\lambda+1)]^{\Gamma_A} = k[J_A]$ *holds. We also put*

$$GA[\lambda] := XYZ(X+Y+Z)(X^2+Y^2+(\lambda^2+\lambda)Z^2+XY+YZ+ZX).$$

There then exists an isomorphism

$$\mathfrak{M}_A \cong \operatorname{Spec} k[J_A, 1/J_A]$$

such that the family $W^2 = GA[\lambda]$ *of sextic double planes over the finite Galois cover* $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} = \operatorname{Spec} k[\lambda, 1/(\lambda^4+\lambda)]$ *of the moduli curve* $\mathfrak{M}_A$ *yields the universal family of polarized supersingular* $K3$ *surfaces of type* $(\sharp)$ *with marking* $\gamma_\lambda : \mathcal{P} \to Z(\mathrm{d}GA[\lambda])$ *given in Table 1. The origin* $J_A = 0$ *corresponds to the Dolgachev–Kondo point.*

For $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, $\operatorname{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ *is equal to the group*

$$\left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & a \\ & & b \\ \hline 0 & 0 & 1 \end{array}\right] \in \mathrm{PGL}(3,k) \;\middle|\; \begin{array}{c} A \in \mathrm{GL}(2,\mathbb{F}_2), \\ a, b \in \{0,1,\alpha,\alpha+1\} \end{array} \right\} \tag{1.1}$$

of order 96.

Theorem 1.7. *We put*

$$Q_\lambda := (\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}Y^2 + \omega\lambda Z^2 + (\lambda+1)XY + (\bar{\omega}\lambda+\omega)YZ + (\lambda+1)ZX,$$

and

$$GB[\lambda] := XYZ(X+Y+Z)Q_\lambda.$$

https://doi.org/10.1017/S0013091504000562 Published online by Cambridge University Press

Table 2. *Marking for* $\mathfrak{M}_B$

P_i	$\gamma_\lambda(P_i)$
$P_1 = C(22)$	$[\lambda+1, \bar\omega\lambda+\omega, \bar\omega\lambda+\omega]$
$P_2 = C(20)$	$[1, \omega\lambda+\omega, \omega]$
$P_3 = C(21)$	$[\lambda+\bar\omega, 1, \lambda+1]$
$P_4 = T(20, 21, 22)$	$[1, \omega, \omega]$
$P_5 = C(02)$	$[\lambda, \bar\omega\lambda, \bar\omega\lambda+\bar\omega]$
$P_6 = C(12)$	$[\lambda+\bar\omega, \bar\omega\lambda+\bar\omega, \bar\omega\lambda]$
$P_7 = T(02, 12, 22)$	$[1, \bar\omega, \bar\omega]$
$P_8 = C(11)$	$[\lambda+1, 1, \lambda]$
$P_9 = T(02, 11, 20)$	$[1, \bar\omega, \omega]$
$P_{10} = C(10)$	$[1, \omega\lambda+1, 0]$
$P_{11} = T(02, 10, 21)$	$[1, \bar\omega, 0]$
$P_{12} = T(10, 11, 12)$	$[1, 1, 0]$
$P_{13} = C(01)$	$[\lambda, 0, \lambda+\bar\omega]$
$P_{14} = T(01, 12, 20)$	$[1, 0, \omega]$
$P_{15} = T(01, 11, 21)$	$[1, 0, 1]$
$P_{16} = T(01, 10, 22)$	$[1, 0, 0]$
$P_{17} = C(00)$	$[0, \lambda, 1]$
$P_{18} = T(00, 12, 21)$	$[0, 1, \omega]$
$P_{19} = T(00, 11, 22)$	$[0, 1, 1]$
$P_{20} = T(00, 10, 20)$	$[0, 1, 0]$
$P_{21} = T(00, 01, 02)$	$[0, 0, 1]$

Let Γ_B *be the group*

$$\left\{\lambda, \omega\lambda+1, \frac{1}{\lambda+1}, \frac{\lambda+\bar\omega}{\lambda+1}, \frac{\bar\omega\lambda+\omega}{\lambda}, \frac{\bar\omega}{\lambda}, \frac{\omega}{\lambda+\bar\omega}, \frac{\bar\omega(\lambda+1)}{\lambda+\bar\omega}, \frac{\bar\omega\lambda}{\lambda+1}, \frac{\lambda}{\lambda+\bar\omega}, \frac{\lambda+1}{\lambda}, \bar\omega(\lambda+1)\right\}$$

acting on the punctured λ*-line* $\mathbb{A}^1 \setminus \{0, 1, \bar\omega\} = \operatorname{Spec} k[\lambda, 1/\lambda(\lambda+1)(\lambda+\bar\omega)]$*. We put*

$$J_B := \frac{(\lambda+\omega)^{12}}{\lambda^3(\lambda+1)^3(\lambda+\bar\omega)^3}$$

so that $k[\lambda, 1/\lambda(\lambda+1)(\lambda+\bar\omega)]^{\Gamma_B} = k[J_B]$ *holds. There then exists an isomorphism*

$$\mathfrak{M}_B \cong \operatorname{Spec} k\left[J_B, \frac{1}{J_B}\right]$$

such that the family $W^2 = GB[\lambda]$ *of sextic double planes over the finite Galois cover* $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar\omega\} = \operatorname{Spec} k[\lambda, 1/(\lambda^4+\lambda)]$ *of the moduli curve* $\mathfrak{M}_B$ *yields the universal family of polarized supersingular K3 surfaces of type* $(\sharp)$ *with marking* $\gamma_\lambda : \mathcal{P} \to Z(\mathrm{d}GB[\lambda])$ *given in Table 2. The origin* $J_B = 0$ *corresponds to the Dolgachev–Kondo point.*

Table 3. *Marking for* $\mathfrak{M}_C$

$\gamma_\lambda(P_1) = [1,1,\lambda]$	$\gamma_\lambda(P_8) = [1,\bar\omega,\omega\lambda+\bar\omega]$	$\gamma_\lambda(P_{15}) = [1,0,\bar\omega]$
$\gamma_\lambda(P_2) = [1,1,\lambda+1]$	$\gamma_\lambda(P_9) = [1,\omega,\bar\omega\lambda+\bar\omega]$	$\gamma_\lambda(P_{16}) = [1,0,\omega]$
$\gamma_\lambda(P_3) = [1,1,\lambda+\omega]$	$\gamma_\lambda(P_{10}) = [1,\omega,\bar\omega\lambda]$	$\gamma_\lambda(P_{17}) = [0,1,1]$
$\gamma_\lambda(P_4) = [1,1,\lambda+\bar\omega]$	$\gamma_\lambda(P_{11}) = [1,\omega,\bar\omega\lambda+\omega]$	$\gamma_\lambda(P_{18}) = [0,1,0]$
$\gamma_\lambda(P_5) = [1,\bar\omega,\omega\lambda+\omega]$	$\gamma_\lambda(P_{12}) = [1,\omega,\bar\omega\lambda+1]$	$\gamma_\lambda(P_{19}) = [0,1,\bar\omega]$
$\gamma_\lambda(P_6) = [1,\bar\omega,\omega\lambda]$	$\gamma_\lambda(P_{13}) = [1,0,1]$	$\gamma_\lambda(P_{20}) = [0,1,\omega]$
$\gamma_\lambda(P_7) = [1,\bar\omega,\omega\lambda+1]$	$\gamma_\lambda(P_{14}) = [1,0,0]$	$\gamma_\lambda(P_{21}) = [0,0,1]$

For any $\alpha \in k \setminus \{0,1,\omega,\bar\omega\}$, $\mathrm{Aut}(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$ *is equal to the subgroup of* $\mathrm{PGL}(3,k)$ *generated by*

$$\begin{bmatrix} 0 & \omega & 0 \\ \bar\omega & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \bar\omega \\ 1 & 1 & 1 \\ \omega & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ \bar\omega & 1 & 0 \\ \omega & 0 & 1 \end{bmatrix}. \tag{1.2}$$

In particular, $\mathrm{Aut}(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$ *is isomorphic to the extended Heisenberg group of order* 18*.*

The meaning of C and T in Table 2 is explained in the proof of Theorem 1.7.

Theorem 1.8. *Let* Γ_C *be the group*

$$\{\alpha\lambda+\beta \mid \alpha \in \mathbb{F}_4^\times,\ \beta \in \mathbb{F}_4\}$$

of order 12 *acting on the* λ*-line* $\mathbb{A}^1 = k[\lambda]$*. We put*

$$J_C := (\lambda^4+\lambda)^3$$

so that $k[\lambda]^{\Gamma_C} = k[J_C]$ *holds. We also put*

$$GC[\lambda] := XYZ(X^3+Y^3+Z^3) + (\lambda^4+\lambda)X^3Y^3.$$

There then exists an isomorphism

$$\mathfrak{M}_C \cong \operatorname{Spec} k[J_C, 1/J_C]$$

such that the family $W^2 = GC[\lambda]$ *of sextic double planes over the finite Galois cover* $\mathbb{A}^1 \setminus \{0,1,\omega,\bar\omega\} = \operatorname{Spec} k[\lambda, 1/(\lambda^4+\lambda)]$ *of the moduli curve* $\mathfrak{M}_C$ *yields the universal family of polarized supersingular* $K3$ *surfaces of type* ($\sharp$) *with marking* $\gamma_\lambda : \mathcal{P} \to Z(\mathrm{d}GC[\lambda])$ *given in Table 3. The origin* $J_C = 0$ *corresponds to the Dolgachev–Kondo point.*

For $\alpha \in k \setminus \{0,1,\omega,\bar\omega\}$, $\mathrm{Aut}(X_{GC[\alpha]}, \mathcal{L}_{GC[\alpha]})$ *is equal to*

$$\left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ a^2c^2\alpha+e & b^2d^2\alpha+f & 1 \end{bmatrix} \in \mathrm{PGL}(3,k) \;\middle|\; \begin{array}{l} a,b,c,d,e,f \in \mathbb{F}_4, \\ ad+bc=1 \end{array} \right\} \tag{1.3}$$

of order 960*.*

Next we consider the isomorphism classes of *non-polarized* supersingular $K3$ surfaces with Artin invariant 2.

Definition 1.9. A reduced (possibly reducible) curve D in $\mathfrak{M}_T \times \mathfrak{M}_{T'}$ is called a *correspondence between* $\mathfrak{M}_T$ *and* $\mathfrak{M}_{T'}$. For a correspondence $D \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$, let D^{T} denote the correspondence in $\mathfrak{M}_{T'} \times \mathfrak{M}_T$ obtained from D by interchanging the first and the second factors. When D is a union of two curves D_1 and D_2 without common irreducible components, we write $D = D_1 + D_2$ and $D_2 = D - D_1$. Let $D_1 \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$ and $D_2 \subset \mathfrak{M}_{T'} \times \mathfrak{M}_{T''}$ be correspondences. The *composite* $D_1 * D_2 \subset \mathfrak{M}_T \times \mathfrak{M}_{T''}$ of D_1 and D_2 is defined as the image of

$$(D_1 \times \mathfrak{M}_{T''}) \cap (\mathfrak{M}_T \times D_2) \subset \mathfrak{M}_T \times \mathfrak{M}_{T'} \times \mathfrak{M}_{T''}$$

by the natural projection to $\mathfrak{M}_T \times \mathfrak{M}_{T''}$.

Definition 1.10. A correspondence D in $\mathfrak{M}_T \times \mathfrak{M}_{T'}$ is called an *isomorphism correspondence* if, for every point $([G], [G'])$ of D, the supersingular $K3$ surfaces X_G and $X_{G'}$ (without polarization) are isomorphic. An isomorphism correspondence $D \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$ is said to be *trivial* if T is equal to T' and D is the diagonal Δ_T of $\mathfrak{M}_T \times \mathfrak{M}_T$.

Using Cremona transformations by quintic curves, which played a central role in the study of $\mathrm{Aut}(X_{G_{\mathrm{DK}}})$ in [**5**], we have obtained examples of non-trivial isomorphism correspondences.

Definition 1.11. Let G be a homogeneous polynomial in $\mathcal{U}$. We say that a subset $\Sigma \subset Z(\mathrm{d}G)$ of cardinality 6 is a *centre of Cremona transformation for* $(X_G, \mathcal{L}_G)$ or *for* G if Σ satisfies the following conditions:

(i) no three points of Σ are collinear;

(ii) for each $p_i \in \Sigma$, there exists a conic curve $N_i' \subset \mathbb{P}^2$ such that $N_i' \cap Z(\mathrm{d}G) = \Sigma \setminus \{p_i\}$.

Note that the conic curve N_i' is necessarily non-singular.

Let $\Sigma = \{p_1, \dots, p_6\}$ be a centre of Cremona transformation for $(X_G, \mathcal{L}_G)$. Consider the linear system $|\mathcal{I}_\Sigma^2(5)| \subset |\mathcal{O}_{\mathbb{P}^2}(5)|$ of quintic curves that pass through all the points of Σ and are singular at each point of Σ. Then $|\mathcal{I}_\Sigma^2(5)|$ is of dimension 2, and defines a birational map

$$\mathrm{CT}_\Sigma : \mathbb{P}^2 \cdots \to \mathbb{P}^2.$$

The birational map CT_Σ is the composite of the blowing up $\beta : S \to \mathbb{P}^2$ of the points of Σ and the blowing down $\beta' : S \to \mathbb{P}^2$ of the strict transforms N_i of the conic curves N_i'. We denote by p_i' the image of N_i by β'. Note that, if $p \in \mathbb{P}^2 \setminus \Sigma$, then the point $\mathrm{CT}_\Sigma(p) \in \mathbb{P}^2$ is well defined.

Proposition 1.12 (Dolgachev–Kondo [5]). *We put*

$$Z' := \{\mathrm{CT}_\Sigma(p) \mid p \in Z(\mathrm{d}G) \setminus \Sigma\} \cup \{p_1', \dots, p_6'\}.$$

There then exists a homogeneous polynomial $G' \in \mathcal{U}$ *such that* $Z' = Z(\mathrm{d}G')$. *The birational map* CT_Σ *of* $\mathbb{P}^2$ *lifts to an isomorphism*

$$\widetilde{\mathrm{CT}}_\Sigma : X_G \xrightarrow{\sim} X_{G'}$$

of supersingular $K3$ *surfaces.*

Proposition 1.12 is proved in § 8. Note that the polynomial G' is not uniquely determined, but the point $[G'] \in \mathfrak{M}$ is uniquely determined by G and Σ. We call $\widetilde{\mathrm{CT}}_\Sigma$ the *Cremona transformation of* X_G *with centre* Σ.

Let T be A, B or C. As Tables 1–3 show, the family

$$\{(p, \lambda) \mid p \in Z(\mathrm{d}GT[\lambda])\} \subset \mathbb{P}^2 \times (\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\})$$

of the points $Z(\mathrm{d}GT[\lambda])$ consists of 21 connected components, each of which is étale of degree 1 over the punctured λ-line $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$. Therefore, it makes sense to talk about a family $\Sigma[\lambda]$ of subsets of $Z(\mathrm{d}GT[\lambda])$ that depends on λ continuously. It can be shown that, if $\Sigma[\alpha]$ is a centre of Cremona transformation for $GT[\alpha]$ at one $\alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}$, then so is $\Sigma[\alpha]$ at every $\alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}$. In this case, we say that $\Sigma[\lambda]$ is a centre of Cremona transformation for $GT[\lambda]$ or for $(X_{GT[\lambda]}, \mathcal{L}_{GT[\lambda]})$.

Suppose that $\Sigma[\lambda]$ is a centre of Cremona transformation for $GT[\lambda]$. There then exist a family $G'[\lambda]$ of homogeneous polynomials in $\mathcal{U}$ and a family of isomorphisms

$$\widetilde{\mathrm{CT}}_{\Sigma[\lambda]} : X_{GT[\lambda]} \xrightarrow{\sim} X_{G'[\lambda]}$$

depending on the parameter λ. The points $[G'[\lambda]]$ are of course contained in $\mathfrak{M}_2 = \mathfrak{M}_A \sqcup \mathfrak{M}_B \sqcup \mathfrak{M}_C$. Suppose that $[G'[\lambda]] \in \mathfrak{M}_{T'}$. Then the curve

$$\{([GT[\lambda]], [G'[\lambda]]) \in \mathfrak{M}_T \times \mathfrak{M}_{T'} \mid \lambda \in \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}\}$$

is an irreducible isomorphism correspondence between $\mathfrak{M}_T$ and $\mathfrak{M}_{T'}$.

Theorem 1.13.

(1) *There exist* 1644 *centres of Cremona transformation for the family* $(X_{GA[\lambda]}, \mathcal{L}_{GA[\lambda]})$. *They yield the following isomorphism correspondences:*

  (i) 156 *of them give the trivial correspondence* Δ_A*;*

  (ii) 144 *of them give the correspondence*

$$D_{A,A,1} : 1 + J_A J'_A + {J_A}^2 {J'_A}^2 + {J_A}^2 {J'_A}^3 + {J_A}^3 {J'_A}^2 = 0$$

  in $\mathfrak{M}_A \times \mathfrak{M}_A$*;*

  (iii) 720 *of them give the correspondence*

$$D_{A,A,2} := D_{A,A,1} * D_{A,A,1} - \Delta_A \subset \mathfrak{M}_A \times \mathfrak{M}_A;$$

Table 4. *Non-trivial irreducible isomorphism correspondences*

T	T'	name	equation
A	A	$D_{A,A,1}$	$J_A^3 {J'_A}^2 + J_A^2 {J'_A}^3 + J_A^2 {J'_A}^2 + J_A J'_A + 1$
A	A	$D_{A,A,2}$	$J_A^6 {J'_A}^2 + J_A^4 {J'_A}^4 + J_A^2 {J'_A}^6 + J_A^4 {J'_A}^3 + J_A^3 {J'_A}^4 + J_A^4 {J'_A}^2 + J_A^3 {J'_A}^3 + J_A^2 {J'_A}^4 + J_A^4 J'_A + J_A {J'_A}^4 + J_A^3 J'_A + J_A^2 {J'_A}^2 + J_A {J'_A}^3 + J_A^3 + J_A^2 J'_A + J_A {J'_A}^2 + {J'_A}^3$
B	B	$D_{B,B,1}$	${J_B}^4 J'_B + {J_B}^3 {J'_B}^2 + {J_B}^2 {J'_B}^3 + J_B {J'_B}^4 + {J_B}^3 J'_B + {J_B}^2 {J'_B}^2 + J_B {J'_B}^3 + 1$
C	C	$D_{C,C,1}$	${J_C}^4 {J'_C}^4 + {J_C}^3 J'_C + {J_C}^2 {J'_C}^2 + J_C {J'_C}^3 + {J_C}^3 + {J_C}^2 J'_C + J_C {J'_C}^2 + {J'_C}^3$
A	B	$D_{A,B,1}$	$J_A^4 + J_A^2 J_B + J_A {J_B}^2 + J_A J_B + J_B$
A	B	$D_{A,B,2}$	$J_A^6 J_B + J_A^5 J_B + J_A^4 {J_B}^2 + J_A^3 {J_B}^3 + J_A^4 J_B + J_A^2 {J_B}^2 + J_A^2 J_B + J_A J_B + 1$
B	C	$D_{B,C,1}$	$J_B J_C + 1$
B	C	$D_{B,C,2}$	$J_B^4 {J_C}^3 + {J_B}^3 {J_C}^3 + {J_B}^3 {J_C}^2 + {J_B}^2 {J_C}^2 + {J_C}^4 + {J_B}^2 J_C + J_B J_C + J_B$
C	A	$D_{C,A,1}$	$J_C^2 J_A^4 + J_C J_A^2 + J_C J_A + J_C + J_A$
C	A	$D_{C,A,2}$	$J_C^2 J_A^6 + {J_C}^2 J_A^5 + {J_C}^2 J_A^4 + J_C J_A^4 + {J_C}^2 J_A^2 + {J_C}^3 + {J_C}^2 J_A + J_C J_A^2 + J_A^3$

(iv) 576 *of them give the correspondence*

$$D_{A,B,1} : J_B + J_A J_B + J_A {J_B}^2 + {J_A}^2 J_B + {J_A}^4 = 0$$

in $\mathfrak{M}_A \times \mathfrak{M}_B$;

(v) 48 *of them give the correspondence*

$$D_{A,C,1} : J_C + J_A + J_A J_C + {J_A}^2 J_C + {J_A}^4 {J_C}^2 = 0$$

in $\mathfrak{M}_A \times \mathfrak{M}_C$.

(2) *There exist* 1374 *centres of Cremona transformation for* $(X_{GB[\lambda]}, \mathcal{L}_{GB[\lambda]})$. *They yield the following isomorphism correspondences:*

(i) 798 *of them give the trivial correspondence* Δ_B;

(ii) 216 *of them give the correspondence*

$$D_{B,A,1} := {D_{A,B,1}}^{\mathrm{T}} \subset \mathfrak{M}_B \times \mathfrak{M}_A;$$

(iii) 360 *of them give the correspondence*

$$D_{B,B,1} := D_{B,A,1} * D_{A,B,1} - \Delta_B \subset \mathfrak{M}_B \times \mathfrak{M}_B.$$

(3) *There exist* 2224 *centres of Cremona transformation for* $(X_{GC[\lambda]}, \mathcal{L}_{GC[\lambda]})$. *They yield the following isomorphism correspondences:*

(i) 1200 *of them give the trivial correspondence* Δ_C;

(ii) 960 *of them give the correspondence*

$$D_{C,A,1} := {D_{A,C,1}}^{\mathrm{T}} \subset \mathfrak{M}_C \times \mathfrak{M}_A;$$

(iii) 64 *of them give the correspondence*

$$D_{C,C,1} := D_{C,A,1} * D_{A,C,1} - \Delta_C \subset \mathfrak{M}_C \times \mathfrak{M}_C.$$

https://doi.org/10.1017/S0013091504000562 Published online by Cambridge University Press

Starting from the isomorphism correspondences by Cremona transformation above, making transposes and composites, and taking irreducible components, we obtain non-trivial irreducible isomorphism correspondences given in Table 4.

When $T \neq T'$, we denote by $D_{T',T,\nu}$ the correspondence ${D_{T,T',\nu}}^{\mathrm{T}}$ for $\nu = 1$ and 2. They have the relations in Appendix A.

Question 1.14. Are there any non-trivial irreducible isomorphism correspondences other than the ones in Table 4 and their transposes?

The Cremona transformations that yield the trivial isomorphism correspondence are also interesting, because they give automorphisms of the supersingular $K3$ surface X that may not be contained in $\mathrm{Aut}(X, \mathcal{L})$ (see Remark 7.12).

Observation 1.15. Consider a Cremona transformation $\widetilde{\mathrm{CT}}_\Sigma$ on $(X_{GA[\lambda]}, \mathcal{L}_{GA[\lambda]})$ that yields the non-trivial isomorphism correspondence $D_{A,A,1}$. The curve $D_{A,A,1}$ intersects the diagonal Δ_A at two points $(J_A, J'_A) = (\omega, \omega)$ and $(\bar{\omega}, \bar{\omega})$. Let η be an element of k such that the J_A-invariant of $(X_{GA[\eta]}, \mathcal{L}_{GA[\eta]})$ is ω or $\bar{\omega}$; that is, η is a root of

$$(\lambda^4 + \lambda^3 + 1)(\lambda^4 + \lambda + 1)(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1) = 0.$$

The Cremona transformation $\widetilde{\mathrm{CT}}_\Sigma$ gives rise to an automorphism of $X_{GA[\eta]}$, which cannot be deformed to any automorphisms of $X_{GA[\lambda]}$ for a generic λ. In other words, the automorphism group $\mathrm{Aut}(X_{GA[\lambda]})$ of the non-polarized supersingular $K3$ surface $X_{GA[\lambda]}$ *jumps* at $\lambda = \eta$, even though the numerical Néron–Severi lattice of $X_{GA[\lambda]}$ is constant around $\lambda = \eta$. Note that the automorphism group of a supersingular $K3$ surface is always embedded into the orthogonal group of its numerical Néron–Severi lattice [**8**, § 8, Proposition 3].

The plan of this paper is as follows. In § 2, we recall from [**11**] the definition of the binary code associated with a polarized supersingular $K3$ surface of type (♯). We stratify the moduli space $\mathfrak{M}$ according to the isomorphism classes $[\boldsymbol{C}]$ of the codes, and give a method to construct the stratum $\mathfrak{M}_{[\boldsymbol{C}]}$ from the code $\boldsymbol{C}$. In § 3, we present three isomorphism classes $[\boldsymbol{C}_A]$, $[\boldsymbol{C}_B]$ and $[\boldsymbol{C}_C]$ of codes that are associated with polarized supersingular $K3$ surfaces of type (♯) with Artin invariant 2. In §§ 4–6, we carry out the method of the construction of $\mathfrak{M}_{[\boldsymbol{C}]}$ for $\boldsymbol{C} = \boldsymbol{C}_A$, $\boldsymbol{C}_B$ and $\boldsymbol{C}_C$, and prove Theorems 1.6, 1.7 and 1.8, respectively. In § 7, we review from [**5**] the theory of Cremona transformations by quintic curves. In § 8, we explain the algorithm to calculate the isomorphism correspondences given by Cremona transformations, and prove Theorem 1.13.

The isomorphism classes of codes associated with polarized supersingular $K3$ surfaces of Artin invariant $\sigma \geqslant 3$ are also given in [**11**]. For $\sigma = 3$, there are 13 isomorphism classes, and for $\sigma = 4$, there are 41 isomorphism classes. It would be a challenging problem in computational algebraic geometry to construct explicitly the moduli spaces of dimension $\sigma - 1$ corresponding to these isomorphism classes of codes, and to investigate the relations between them.

In [**7**], Rudakov and Shafarevich gave families of supersingular $K3$ surfaces in characteristic 2 for Artin invariants $\sigma = 1, \ldots, 10$. The equation of the family for $\sigma = 2$

is

$$y^2 = x^3 + \mu t^6 x + t^5 (t+1)^4,$$

where μ is the 'modulus'. We would like to know the relation between μ and our moduli J_A, J_B and J_C.

The polarized supersingular $K3$ surface of type ($\sharp$) is an example of *Zariski surfaces*. A general theory of Zariski surfaces has been developed in [**2**].

1.1. Notation and terminology

(1) Let A be a commutative ring, and S a set. We denote by A^S the A-module of all maps from S to A.

(2) Let S be a finite set. The full symmetric group of S is denoted by $\mathfrak{S}(S)$, which acts on S from left. We denote by $\mathrm{Pow}(S)$ the power set of S. A canonical identification between $\mathrm{Pow}(S)$ and $\mathbb{F}_2^S$ is given by $f \in \mathbb{F}_2^S \mapsto f^{-1}(1) \subset S$. Hence, $\mathrm{Pow}(S)$ has a structure of the $\mathbb{F}_2$-vector space by the symmetric difference

$$T_1 + T_2 = (T_1 \cup T_2) \setminus (T_1 \cap T_2), \quad T_1, T_2 \subset S.$$

A linear subspace of $\mathbb{F}_2^S = \mathrm{Pow}(S)$ is called a *code*, and an element of a code is called a *word*. A word is expressed either as a vector of dimension $|S|$ with coefficients in $\mathbb{F}_2$, or as a subset of S. The cardinality $|A|$ of a word $A \subset S$ is called the *weight* of A. The automorphism group $\mathrm{Aut}(\boldsymbol{C})$ of a code $\boldsymbol{C} \subset \mathrm{Pow}(S)$ is the subgroup of $\mathfrak{S}(S)$ consisting of all permutations preserving $\boldsymbol{C}$. Two codes $\boldsymbol{C}$ and $\boldsymbol{C}'$ in $\mathrm{Pow}(S)$ are said to be *isomorphic* if there exists a permutation $\sigma \in \mathfrak{S}(S)$ such that $\sigma(\boldsymbol{C}) = \boldsymbol{C}'$. The isomorphism class of codes represented by a code $\boldsymbol{C}$ is denoted by $[\boldsymbol{C}]$.

(3) A *lattice* is a free $\mathbb{Z}$-module Λ of finite rank equipped with a non-degenerate symmetric bilinear form $\Lambda \times \Lambda \to \mathbb{Z}$. A lattice is called *even* if $v^2 \in 2\mathbb{Z}$ holds for every $v \in \Lambda$. A lattice is called *hyperbolic* if the signature of the symmetric bilinear form on $\Lambda \otimes \mathbb{R}$ is $(1, r-1)$, where r is the rank of Λ. The *dual lattice* $\Lambda^\vee$ of Λ is the $\mathbb{Z}$-module $\mathrm{Hom}(\Lambda, \mathbb{Z})$. There exists a canonical embedding $\Lambda \hookrightarrow \Lambda^\vee$ of finite cokernel. Hence, $\Lambda^\vee$ can be regarded as a submodule of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We have a natural $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ that extends the $\mathbb{Z}$-valued bilinear form on Λ. An *overlattice* of Λ is a submodule Λ' of $\Lambda^\vee$ containing Λ such that the canonical $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ takes values in $\mathbb{Z}$ on Λ'.

2. The codes associated with the supersingular $K3$ surfaces

First we give a proof of Proposition 1.3.

Proof of Proposition 1.3. The equivalence of (i) and (iii) follows from the structure of the graded ring $\bigoplus_{m \geqslant 0} H^0(X, \mathcal{L}^{\otimes m})$, where X is a $K3$ surface and $\mathcal{L}$ is a line bundle of degree 2 (see [**11**, § 7]). By [**11**, Theorem 2.1], $Z(\mathrm{d}G) = Z(\mathrm{d}G')$ holds if and only if $\mathrm{d}G = c\,\mathrm{d}G'$ for some $c \in k^\times$. Since the kernel of $G \mapsto \mathrm{d}G$ is equal to $\{H^2 \mid H \in \mathcal{V}\}$, the equivalence of (ii) and (iii) follows. $\square$

2.1. Definition of the code $\mathcal{C}(X, \mathcal{L}, \gamma)$

Let us fix a finite set

$$\mathcal{P} := \{P_1, \dots, P_{21}\}$$

consisting of 21 elements, on which the full symmetric group $\mathfrak{S}(\mathcal{P})$ acts from left.

Definition 2.1. We denote by $\mathcal{G}$ the space of all injective maps $\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$ such that there exists a homogeneous polynomial $G \in \mathcal{U}$ satisfying $\gamma(\mathcal{P}) = Z(\mathrm{d}G)$.

The space $\mathcal{G}$ are constructed as follows. For $G \in \mathcal{U}$, let $\langle G \rangle \in \mathbb{P}_*(\mathcal{U}/\mathcal{V})$ denote the point corresponding to G. We denote by

$$\mathcal{Z} := \{(p, \langle G \rangle) \in \mathbb{P}^2 \times \mathbb{P}_*(\mathcal{U}/\mathcal{V}) \mid p \in Z(\mathrm{d}G)\} \to \mathbb{P}_*(\mathcal{U}/\mathcal{V})$$

the family of $Z(\mathrm{d}G)$, which is finite and étale of degree 21 over $\mathbb{P}_*(\mathcal{U}/\mathcal{V})$. We prepare 21 copies of $\mathcal{Z}$ and make the fibre-product $\mathcal{Z}^{(21)}$ of them over $\mathbb{P}_*(\mathcal{U}/\mathcal{V})$. Then $\mathcal{G}$ is the union of irreducible components of $\mathcal{Z}^{(21)}$ that do not intersect the big diagonal.

Remark 2.2. We fix a base point $\langle G_0 \rangle \in \mathbb{P}_*(\mathcal{U}/\mathcal{V})$, and consider the monodromy action

$$\mu : \pi_1(\mathbb{P}_*(\mathcal{U}/\mathcal{V}), \langle G_0 \rangle) \to \mathfrak{S}(Z(\mathrm{d}G_0))$$

of the algebraic fundamental group of $\mathbb{P}_*(\mathcal{U}/\mathcal{V})$ on $Z(\mathrm{d}G_0)$. Then the number of irreducible components of $\mathcal{G}$ is equal to the index of the image of μ in $\mathfrak{S}(Z(\mathrm{d}G_0))$. It was shown in [**2**, Chapter 4, Appendix 2] that the monodromy group on the singular points of a generic Zariski surface in characteristic greater than or equal to 5 is equal to the full-symmetric group.

The group $\mathfrak{S}(\mathcal{P})$ acts on $\mathcal{G}$ from right, and $\mathrm{PGL}(3, k)$ acts on $\mathcal{G}$ from left. By Proposition 1.3, we have

$$\mathfrak{M} = \mathrm{PGL}(3, k) \setminus \mathcal{G} / \mathfrak{S}(\mathcal{P}).$$

Let

$$N_0 := \mathbb{Z}^{\mathcal{P}} \oplus \mathbb{Z}h = \bigoplus_{i=1}^{21} \mathbb{Z}e_i \oplus \mathbb{Z}h$$

be a free $\mathbb{Z}$-module of rank 22 generated by vectors $e_1, \dots, e_{21}$ corresponding to $P_1, \dots, P_{21} \in \mathcal{P}$ and a vector h. We equip N_0 with a structure of the even hyperbolic lattice by

$$e_i^2 = -2, \quad h^2 = 2, \quad e_i e_j = 0 \quad \text{if } i \neq j, \quad he_i = 0.$$

The dual lattice

$$N_0^\vee = \mathrm{Hom}(N_0, \mathbb{Z}) \subset N_0 \otimes_{\mathbb{Z}} \mathbb{Q}$$

is generated by $e_i/2$ $(i = 1, \dots, 21)$ and $h/2$. Thus, we have a canonical isomorphism

$$N_0^\vee / N_0 \cong \mathbb{F}_2^{\mathcal{P}} \oplus \mathbb{F}_2 = \mathrm{Pow}(\mathcal{P}) \oplus \mathbb{F}_2.$$

Hence, we can write an element of $N_0^\vee/N_0$ in the form (A,α), where A is a subset of $\mathcal{P}$ and $\alpha \in \mathbb{F}_2$. We denote by

$$\mathrm{pr} : N_0^\vee \to N_0^\vee/N_0 = \mathrm{Pow}(\mathcal{P}) \oplus \mathbb{F}_2$$

the natural projection. We also denote by

$$\rho : N_0^\vee/N_0 = \mathrm{Pow}(\mathcal{P}) \oplus \mathbb{F}_2 \to \mathrm{Pow}(\mathcal{P})$$

the natural projection onto the first factor. The following lemma is obvious.

Lemma 2.3. *Let $\tilde{\mathcal{C}}$ be a subspace of the $\mathbb{F}_2$-vector space $\mathrm{Pow}(\mathcal{P}) \oplus \mathbb{F}_2$. Then the submodule $\mathrm{pr}^{-1}(\tilde{\mathcal{C}})$ of $N_0^\vee$ is an even overlattice of N_0 if and only if*

$$|A| \equiv \begin{cases} 0 \bmod 4 & \text{if } \alpha = 0, \\ 1 \bmod 4 & \text{if } \alpha = 1 \end{cases}$$

holds for every $(A,\alpha) \in \tilde{\mathcal{C}}$.

Let $(X,\mathcal{L})$ be a polarized supersingular $K3$ surface of type ($\sharp$), and let $\mathrm{NS}(X)$ denote the numerical Néron–Severi lattice of X. There exists $G \in \mathcal{U}$ such that $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ factors through $\pi_G : Y_G \to \mathbb{P}^2$. We put

$$Z_{(X,\mathcal{L})} := Z(\mathrm{d}G) = \pi_G(\mathrm{Sing}\, Y_G).$$

There also exists a point $\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$ of $\mathcal{G}$, unique up to the action of $\mathfrak{S}(\mathcal{P})$, that induces a bijection from $\mathcal{P}$ to $Z_{(X,\mathcal{L})}$. We fix such a point $\gamma \in \mathcal{G}$. Let E_i be the (-2)-curve on X such that $\Phi_{|\mathcal{L}|}(E_i)$ is the point $\gamma(P_i) \in Z_{(X,\mathcal{L})}$. Then we obtain an embedding

$$\iota_\gamma : N_0 \hookrightarrow \mathrm{NS}(X)$$

of the lattice N_0 into $\mathrm{NS}(X)$ by $e_i \mapsto [E_i]$ and $h \mapsto [\mathcal{L}]$. By the embedding ι_γ, we can regard $\mathrm{NS}(X)$ as a submodule of $N_0^\vee$. We put

$$\tilde{\mathcal{C}}(X,\mathcal{L},\gamma) := \mathrm{NS}(X)/N_0 \subset \mathrm{Pow}(\mathcal{P}) \oplus \mathbb{F}_2$$

and

$$\mathcal{C}(X,\mathcal{L},\gamma) := \rho(\tilde{\mathcal{C}}(X,\mathcal{L},\gamma)) \subset \mathrm{Pow}(\mathcal{P}).$$

Since $\mathrm{NS}(X)$ is an even overlattice of N_0, the code $\tilde{\mathcal{C}}(X,\mathcal{L},\gamma)$ is uniquely recovered from $\mathcal{C}(X,\mathcal{L},\gamma)$ by Lemma 2.3, and hence the lattice $\mathrm{NS}(X)$ is also uniquely recovered from the code $\mathcal{C}(X,\mathcal{L},\gamma)$. In particular, the Artin invariant $\sigma(X)$ of X is given by

$$\sigma(X) = 11 - \dim_{\mathbb{F}_2} \mathcal{C}(X,\mathcal{L},\gamma).$$

Note that the isomorphism class of the code $\mathcal{C}(X,\mathcal{L},\gamma)$ does not depend on the choice of γ. The following is one of the main results of [**11**].

Theorem 2.4. *For an isomorphism class* $[\boldsymbol{C}]$ *of codes in* $\mathrm{Pow}(\mathcal{P})$*, the following two conditions are equivalent.*

(i) *There exists a polarized supersingular* $K3$ *surface* $(X, \mathcal{L})$ *of type* $(\sharp)$ *such that, for a (and hence any) bijection* γ *from* $\mathcal{P}$ *to* $Z_{(X,\mathcal{L})}$*, the code* $\mathcal{C}(X, \mathcal{L}, \gamma)$ *is in the isomorphism class* $[\boldsymbol{C}]$.

(ii) *A (and hence any) code* $\boldsymbol{C} \in [\boldsymbol{C}]$ *satisfies the following:*

(a) $\dim \boldsymbol{C} \leqslant 10$,

(b) *the word* $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$ *is contained in* $\boldsymbol{C}$*, and*

(c) $|A| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ *for every word* $A \in \boldsymbol{C}$.

2.2. Geometry of $Z_{(X,\mathcal{L})}$ and the code $\mathcal{C}(X, \mathcal{L}, \gamma)$

Let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type $(\sharp)$. We fix a bijection γ from $\mathcal{P}$ to $Z_{(X,\mathcal{L})}$. Let $G \in \mathcal{U}$ be a homogeneous polynomial such that $\Phi_{|\mathcal{L}|}$ factors through Y_G, or equivalently, such that $Z(\mathrm{d}G) = Z_{(X,\mathcal{L})}$ holds. For the proofs of the facts stated in this subsection, we refer the reader to [**11**, §§ 6 and 7].

Definition 2.5. Let $C \subset \mathbb{P}^2$ be a reduced irreducible curve. We say that C *splits in* $(X, \mathcal{L})$ if the proper transform of C by $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ is non-reduced. We say that a reduced (possibly reducible) curve C' *splits in* $(X, \mathcal{L})$ if every irreducible component of C' splits in $(X, \mathcal{L})$.

Since $\Phi_{|\mathcal{L}|}$ is purely inseparable of degree 2, the proper transform of a splitting curve C by $\Phi_{|\mathcal{L}|}$ is written as $2F_C$, where F_C is a reduced divisor of X. We denote by $w(C) \in \mathrm{Pow}(\mathcal{P})$ the image of the numerical equivalence class $[F_C] \in \mathrm{NS}(X)$ by

$$\mathrm{NS}(X) \to \mathrm{NS}(X)/N_0 \hookrightarrow N_0^{\vee}/N_0 \xrightarrow{\rho} \mathrm{Pow}(\mathcal{P}),$$

where $N_0 \hookrightarrow \mathrm{NS}(X)$ is obtained from the fixed bijection $\gamma : \mathcal{P} \xrightarrow{\sim} Z_{(X,\mathcal{L})}$. By definition, we have

$$w(C) \in \mathcal{C}(X, \mathcal{L}, \gamma).$$

It is easy to see that

$$w(C) = \{P_i \in \mathcal{P} \mid \text{the multiplicity of } C \text{ at } \gamma(P_i) \text{ is odd}\}.$$

If C is a non-singular curve splitting in $(X, \mathcal{L})$, then

$$w(C) = \gamma^{-1}(C \cap Z_{(X,\mathcal{L})}).$$

If C_1 and C_2 are two splitting curves without common irreducible components, then $w(C_1 \cup C_2) = w(C_1) + w(C_2)$ holds.

Proposition 2.6. *Let $\mathcal{I}_{Z(\mathrm{d}G)} \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf defining the subscheme $Z(\mathrm{d}G)$. The linear system $|\mathcal{I}_{Z(\mathrm{d}G)}(5)|$ of quintic curves passing through all the points of $Z(\mathrm{d}G)$ is of dimension 2, and spanned by the curves defined by*

$$\frac{\partial G}{\partial X} = 0, \quad \frac{\partial G}{\partial Y} = 0 \quad \text{and} \quad \frac{\partial G}{\partial Z} = 0.$$

A general member C of $|\mathcal{I}_{Z(\mathrm{d}G)}(5)|$ splits in $(X, \mathcal{L})$, and the word $w(C) \in \mathcal{C}(X, \mathcal{L}, \gamma)$ is equal to $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$.

Proposition 2.7. *Let C be a reduced curve splitting in $(X, \mathcal{L})$, and let p be a point of C.*

(1) *If p is an ordinary node of C, then $p \in Z_{(X,\mathcal{L})}$.*

(2) *If p is an ordinary tacnode of C, then $p \notin Z_{(X,\mathcal{L})}$.*

Proposition 2.8. *Let C be a reduced curve of degree 6 splitting in $(X, \mathcal{L})$, and let $G' = 0$ be a defining equation of C. If C has only ordinary nodes as its singularities, then the homogeneous polynomial G' is a point of $\mathcal{U}$, and the point $[G'] \in \mathfrak{M}$ corresponds to the isomorphism class of $(X, \mathcal{L})$.*

Proposition 2.9. *Let $L \subset \mathbb{P}^2$ be a line. The following conditions are equivalent:*

(i) *L splits in $(X, \mathcal{L})$;*

(ii) *$|L \cap Z_{(X,\mathcal{L})}| \geqslant 3$;*

(iii) *$|L \cap Z_{(X,\mathcal{L})}| = 5$.*

Proposition 2.10. *Let $Q \subset \mathbb{P}^2$ be a non-singular conic curve. The following conditions are equivalent:*

(i) *Q splits in $(X, \mathcal{L})$;*

(ii) *$|Q \cap Z_{(X,\mathcal{L})}| \geqslant 6$;*

(iii) *$|Q \cap Z_{(X,\mathcal{L})}| = 8$.*

Corollary 2.11. *The word $w(L) = \gamma^{-1}(L \cap Z_{(X,\mathcal{L})})$ of a splitting line L is of weight 5, and the word $w(Q) = \gamma^{-1}(Q \cap Z_{(X,\mathcal{L})})$ of a splitting non-singular conic curve Q is of weight 8.*

Definition 2.12. A pencil $\mathcal{E}$ of cubic curves in $\mathbb{P}^2$ is called a *regular pencil* if the following hold:

(i) the base locus $\mathrm{Bs}(\mathcal{E})$ of $\mathcal{E}$ consists of nine distinct points;

(ii) every singular member of $\mathcal{E}$ has only one ordinary node as its singularities.

We say that a regular pencil $\mathcal{E}$ *splits in* $(X, \mathcal{L})$ if every member of $\mathcal{E}$ splits in $(X, \mathcal{L})$.

Proposition 2.13. *Let $\mathcal{E}$ be a regular pencil of cubic curves spanned by E_0 and E_∞. Let $H_0 = 0$ and $H_\infty = 0$ be the defining equations of E_0 and E_∞, respectively. Then $\mathcal{E}$ splits in $(X, \mathcal{L})$ if and only if there exist $c \in k^\times$ and $H \in \mathcal{V}$ such that*

$$G = cH_0H_\infty + H^2 \tag{2.1}$$

holds. If $\mathcal{E}$ splits in $(X, \mathcal{L})$, then $\mathrm{Bs}(\mathcal{E})$ is contained in $Z_{(X,\mathcal{L})}$, and

$$w(E_t) = \gamma^{-1}(\mathrm{Bs}(\mathcal{E}))$$

holds for every member E_t of $\mathcal{E}$. In particular, the word $w(E_t)$ is of weight 9.

Remark 2.14. The condition (2.1) is equivalent to

$$Z(\mathrm{d}(H_0H_\infty)) = Z(\mathrm{d}G) = Z_{(X,\mathcal{L})}$$

by Proposition 1.3.

Remark 2.15. A regular pencil $\mathcal{E}$ has 12 singular members $E^{(1)}, \dots, E^{(12)}$. We denote by $N^{(i)}$ the ordinary node of $E^{(i)}$. Suppose that $\mathcal{E}$ splits in $(X, \mathcal{L})$. Then $Z_{(X,\mathcal{L})}$ is a disjoint union of $\mathrm{Bs}(\mathcal{E})$ and $\{N^{(1)}, \dots, N^{(12)}\}$.

Let L_1 and L_2 be distinct lines splitting in $(X, \mathcal{L})$. Then the intersection point of L_1 and L_2 is in $Z_{(X,\mathcal{L})}$ by Proposition 2.7, and hence

$$w(L_1 \cup L_2) = w(L_1) + w(L_2)$$

is a word of weight 8.

Let L_1, L_2 and L_3 be lines splitting in $(X, \mathcal{L})$ such that $L_1 \cap L_2 \cap L_3 = \emptyset$. Then the three ordinary nodes of $L_1 \cup L_2 \cup L_3$ are in $Z_{(X,\mathcal{L})}$ by Proposition 2.7, and hence

$$w(L_1 \cup L_2 \cup L_3) = w(L_1) + w(L_2) + w(L_3)$$

is a word of weight 9.

Let Q be a non-singular conic curve splitting in $(X, \mathcal{L})$, and let L be a line splitting in $(X, \mathcal{L})$. Using Proposition 2.7, we see that L intersects Q transversely if and only if $w(L \cup Q) = w(L) + w(Q)$ is of weight 9. We also see that L is tangent to Q if and only if $w(L) \cap w(Q) = \emptyset$.

Definition 2.16. Let $\boldsymbol{C} \subset \mathrm{Pow}(\mathcal{P})$ be a code satisfying the conditions in Theorem 2.4(ii), and let A be a word of $\boldsymbol{C}$ with $|A| \in \{5, 8, 9\}$.

(i) We say that A is a *linear word of* $\boldsymbol{C}$ if $|A| = 5$.

(ii) Suppose $|A| = 8$. If A is *not* a sum of two linear words of $\boldsymbol{C}$, then we say that A is a *quadratic word of* $\boldsymbol{C}$.

(iii) Suppose $|A| = 9$. If A is neither a sum of three linear words of $\boldsymbol{C}$ nor a sum of a linear and a quadratic word of $\boldsymbol{C}$, then we say that A is a *cubic word of* $\boldsymbol{C}$.

Proposition 2.17.

(1) *The correspondence $L \mapsto w(L)$ yields a bijection from the set of lines splitting in $(X, \mathcal{L})$ to the set of linear words in $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

(2) *The correspondence $Q \mapsto w(Q)$ yields a bijection from the set of non-singular conic curves splitting in $(X, \mathcal{L})$ to the set of quadratic words in $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

(3) *The correspondence $\mathcal{E} \mapsto \gamma^{-1}(\mathrm{Bs}(\mathcal{E}))$ yields a bijection from the set of regular pencils of cubic curves splitting in $(X, \mathcal{L})$ to the set of cubic words in $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

By Theorem 2.4, the code $\mathcal{C}(X, \mathcal{L}, \gamma)$ is generated by the word $\mathcal{P}$ and by the linear, quadratic and cubic words in $\mathcal{C}(X, \mathcal{L}, \gamma)$. Combining this fact with Proposition 2.17, we obtain the following corollary.

Corollary 2.18. *Let g be an element of the group*

$$\mathrm{Aut}(X, \mathcal{L}) = \{h \in \mathrm{PGL}(3, k) \mid h(Z_{(X,\mathcal{L})}) = Z_{(X,\mathcal{L})}\}.$$

We then have $\mathcal{C}(X, \mathcal{L}, \gamma) = \mathcal{C}(X, \mathcal{L}, g \circ \gamma)$. Hence, there exists a unique element $\sigma_g \in \mathrm{Aut}(\mathcal{C}(X, \mathcal{L}, \gamma))$ such that $g \circ \gamma = \gamma \circ \sigma_g$ holds. By $g \mapsto \sigma_g$, we can embed $\mathrm{Aut}(X, \mathcal{L})$ into $\mathrm{Aut}(\mathcal{C}(X, \mathcal{L}, \gamma))$.

2.3. Construction of $\mathfrak{M}_{[\boldsymbol{C}]}$ from $\boldsymbol{C}$

Let $[\boldsymbol{C}]$ be an isomorphism class of codes satisfying the conditions of Theorem 2.4(ii). We denote by

$$\mathfrak{M}_{[\boldsymbol{C}]} \subset \mathfrak{M}$$

the locus of all isomorphism classes of polarized supersingular $K3$ surfaces $(X, \mathcal{L})$ of type ($\sharp$) such that $\mathcal{C}(X, \mathcal{L}, \gamma)$ is contained in $[\boldsymbol{C}]$ for a (and hence any) bijection γ from $\mathcal{P}$ to $Z_{(X,\mathcal{L})}$. We also denote by $\mathcal{G}_{[\boldsymbol{C}]}$ the pull-back of $\mathfrak{M}_{[\boldsymbol{C}]}$ by the quotient map

$$\mathcal{G} \to \mathfrak{M} = \mathrm{PGL}(3, k) \setminus \mathcal{G}/\mathfrak{S}(\mathcal{P}).$$

We will describe the locus $\mathcal{G}_{[\boldsymbol{C}]}$.

Definition 2.19. For a point γ of $\mathcal{G}$, let $\mathcal{C}[\gamma]$ denote the code in $\mathrm{Pow}(\mathcal{P})$ generated by the following words:

(i) $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$;

(ii) words A of weight 5 such that the points $\gamma(A)$ are collinear;

(iii) words A of weight 8 such that there exists a non-singular conic curve containing $\gamma(A)$;

(iv) words A of weight 9 such that there exists a regular pencil $\mathcal{E}$ of cubic curves spanned by $E_0 = \{H_0 = 0\}$ and $E_\infty = \{H_\infty = 0\}$ such that $\mathrm{Bs}(\mathcal{E}) = \gamma(A)$ and $Z(\mathrm{d}(H_0 H_\infty)) = \gamma(\mathcal{P})$ hold.

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From the results above, we obtain the following.

Corollary 2.20. *Suppose that* $\gamma \in \mathcal{G}$, *and let* $(X, \mathcal{L})$ *be a polarized supersingular* $K3$ *surface of type* $(\sharp)$ *such that* $\gamma(\mathcal{P}) = Z_{(X,\mathcal{L})}$. *Then the code* $\mathcal{C}[\gamma]$ *coincides with the code* $\mathcal{C}(X, \mathcal{L}, \gamma)$.

By definition, we have

$$\mathcal{C}[\gamma \circ \sigma] = \sigma^{-1}(\mathcal{C}[\gamma]) \quad \text{for any } \sigma \in \mathfrak{S}(\mathcal{P}).$$

For each code $\boldsymbol{C} \in [\boldsymbol{C}]$, we put

$$\mathcal{G}_{\boldsymbol{C}} := \{\gamma \in \mathcal{G} \mid \mathcal{C}[\gamma] = \boldsymbol{C}\}.$$

Then we have

$$\mathcal{G}_{\boldsymbol{C}}^{\sigma} = \mathcal{G}_{\sigma^{-1}(\boldsymbol{C})},$$

where $\mathcal{G}_{\boldsymbol{C}}^{\sigma}$ denotes the image of $\mathcal{G}_{\boldsymbol{C}}$ by the action of $\sigma \in \mathfrak{S}(\mathcal{P})$. Therefore, we obtain

$$\mathcal{G}_{[\boldsymbol{C}]} = \bigsqcup_{\boldsymbol{C}' \in [\boldsymbol{C}]} \mathcal{G}_{\boldsymbol{C}'} = \bigsqcup_{\sigma} \mathcal{G}_{\boldsymbol{C}}^{\sigma} \quad \text{(disjoint union)},$$

where σ runs through the set of representatives for the right cosets in $\mathfrak{S}(\mathcal{P})$ with respect to the subgroup $\operatorname{Aut}(\boldsymbol{C}) \subset \mathfrak{S}(\mathcal{P})$. Hence, we have

$$\mathfrak{M}_{[\boldsymbol{C}]} = \mathrm{PGL}(3, k) \setminus \mathcal{G}_{\boldsymbol{C}} / \operatorname{Aut}(\boldsymbol{C}).$$

For $\gamma \in \mathcal{G}_{\boldsymbol{C}}$, let $[\gamma] \in \mathrm{PGL}(3, k) \setminus \mathcal{G}_{\boldsymbol{C}}$ denote the projective equivalence class of γ. From Corollary 2.18, we obtain the following.

Corollary 2.21. *Let* $(X, \mathcal{L})$ *be a polarized supersingular* $K3$ *surface of type* $(\sharp)$ *corresponding to the image of* $[\gamma] \in \mathrm{PGL}(3, k) \setminus \mathcal{G}_{\boldsymbol{C}}$ *by the quotient map*

$$\mathrm{PGL}(3, k) \setminus \mathcal{G}_{\boldsymbol{C}} \to \mathfrak{M}_{[\boldsymbol{C}]} = \mathrm{PGL}(3, k) \setminus \mathcal{G}_{\boldsymbol{C}} / \operatorname{Aut}(\boldsymbol{C}).$$

Via the natural embedding of $\operatorname{Aut}(X, \mathcal{L})$ *into* $\operatorname{Aut}(\mathcal{C}(X, \mathcal{L}, \gamma)) = \operatorname{Aut}(\boldsymbol{C})$, *the automorphism group* $\operatorname{Aut}(X, \mathcal{L})$ *is equal to the stabilizer subgroup of the point* $[\gamma]$.

3. The isomorphism classes of codes with Artin invariant 1 and 2

We have classified all isomorphism classes of codes satisfying the conditions of Theorem 2.4(ii). The list is given in [**11**, § 8]. Using the classification, we have obtained the following theorem [**11**, Corollary 1.11].

Theorem 3.1. *There exists exactly one isomorphism class* $[\boldsymbol{C}_0]$ *of codes of dimension* 10 *satisfying the conditions in Theorem 2.4(ii). The moduli space* $\mathfrak{M}_{[\boldsymbol{C}_0]}$ *consists of a single point corresponding to the Dolgachev–Kondo polynomial*

$$G_{\mathrm{DK}} := XYZ(X^3 + Y^3 + Z^3).$$

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Table 5. *Generators of the code $\boldsymbol{C}_A$*

[1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1]
[0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1]
[0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1]
[0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	1	0]
[0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	1	0	0]
[0	0	0	0	0	0	1	0	1	0	1	0	0	1	0	0	0	1	0	0	0]
[0	0	0	0	1	1	0	0	0	0	0	1	0	1	0	0	0	1	1	1	1]
[0	0	0	1	0	1	0	0	0	0	1	0	0	1	1	1	0	0	1	0	1]
[0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	1	0	1	1	0	0]

Table 6. *Generators of the code $\boldsymbol{C}_B$*

[1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1]
[0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1]
[0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1]
[0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	1	0]
[0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	1	0	0]
[0	0	0	0	0	1	1	0	0	0	0	1	0	1	0	0	0	1	0	0	0]
[0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1]
[0	0	1	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0]
[0	1	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0]

We call the point $[G_{\mathrm{DK}}]$ constituting $\mathfrak{M}_1 = \mathfrak{M}_{[\boldsymbol{C}_0]}$ the *Dolgachev–Kondo point*. We define the *Dolgachev–Kondo code*

$$\boldsymbol{C}_{\mathrm{DK}} \subset \mathrm{Pow}(\mathbb{P}^2(\mathbb{F}_4))$$

to be the code generated by the words $\Lambda(\mathbb{F}_4)$, where Λ are $\mathbb{F}_4$-rational lines in $\mathbb{P}^2$. The codes in the isomorphism class $[\boldsymbol{C}_0]$ are precisely the codes $\gamma^{-1}(\boldsymbol{C}_{\mathrm{DK}})$, where γ runs through the set of all bijections from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4) = Z(\mathrm{d}G_{\mathrm{DK}})$. The weight enumerator of any code in $[\boldsymbol{C}_0]$ is

$$1 + 21z^5 + 210z^8 + 280z^9 + 280z^{12} + 210z^{13} + 21z^{16} + z^{21}.$$

There are no quadratic or cubic words in $\boldsymbol{C}_0$.

From the list in [**11**, § 8], we obtain the following proposition.

Proposition 3.2. *There are exactly three isomorphism classes $[\boldsymbol{C}_A]$, $[\boldsymbol{C}_B]$, $[\boldsymbol{C}_C]$ of codes of dimension 9 satisfying the conditions in Theorem 2.4(ii).*

As representatives of these isomorphism classes, we can take codes $\boldsymbol{C}_A$, $\boldsymbol{C}_B$ and $\boldsymbol{C}_C$ generated by vectors in Tables 5, 6 and 7.

The numbers of linear, quadratic and cubic words in these codes are given in Table 8.

Table 7. *Generators of the code $\boldsymbol{C}_C$*

[1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1]
[0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1]
[0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1]
[0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1]
[0	0	0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0]
[0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0]
[0	0	0	0	1	0	0	1	0	1	1	0	0	1	1	0	0	1	1	0	1]
[0	0	1	1	0	0	0	0	0	0	1	1	0	1	0	1	0	1	1	0	0]
[0	1	0	1	0	0	0	0	0	1	0	1	0	1	1	0	0	0	1	1	0]

Table 8. *Numbers of linear, quadratic and cubic words in $\boldsymbol{C}_A$, $\boldsymbol{C}_B$ and $\boldsymbol{C}_C$*

	linear	quadratic	cubic
$\boldsymbol{C}_A$	13	28	0
$\boldsymbol{C}_B$	9	66	0
$\boldsymbol{C}_C$	5	120	0

The weight enumerators of these codes are as follows:

$$\boldsymbol{C}_A : 1 + 13z^5 + 106z^8 + 136z^9 + 136z^{12} + 106z^{13} + 13z^{16} + z^{21},$$
$$\boldsymbol{C}_B : 1 + 9z^5 + 102z^8 + 144z^9 + 144z^{12} + 102z^{13} + 9z^{16} + z^{21},$$
$$\boldsymbol{C}_C : 1 + 5z^5 + 130z^8 + 120z^9 + 120z^{12} + 130z^{13} + 5z^{16} + z^{21}.$$

Remark 3.3. The Dolgachev–Kondo code $\boldsymbol{C}_{\mathrm{DK}}$ is related to the binary Golay code $\boldsymbol{C}_{24}$ in the following way. Let $M := \{\mu_1, \dots, \mu_{24}\}$ be the set of positions of the Miracle Octad Generator as is indicated in [**9**, Table 6.1]. The definition of $\boldsymbol{C}_{24}$ as a subcode of $\mathrm{Pow}(M)$ is described in [**3**, Chapter 11]. We put $N := \{\mu_{22}, \mu_{23}, \mu_{24}\} \subset M$, and consider the ten-dimensional subcode

$$\boldsymbol{C}_{22} := \{w \in \boldsymbol{C}_{24} \mid w \supset N \text{ or } w \cap N = \emptyset\}$$

of $\boldsymbol{C}_{24}$. We then define a map

$$\mathbb{P}^2(\mathbb{F}_4) \to M$$

by [**9**, Table 6.2]. The pull-back of $\boldsymbol{C}_{22}$ by this map is just the Dolgachev–Kondo code $\boldsymbol{C}_{\mathrm{DK}}$.

Remark 3.4. The codes $\boldsymbol{C}_A$, $\boldsymbol{C}_B$ and $\boldsymbol{C}_C$ are isomorphic to linear subcodes of $\boldsymbol{C}_{\mathrm{DK}}$ defined as follows. Let $F = \{Q_1, Q_2, Q_3, Q_4\}$ be a set of four points of $\mathbb{P}^2(\mathbb{F}_4)$, and let $\boldsymbol{C}_F$ be the nine-dimensional linear subcode of $\boldsymbol{C}_{\mathrm{DK}}$ defined by

$$\boldsymbol{C}_F := \{w \in \boldsymbol{C}_{\mathrm{DK}} \mid |w \cap F| \text{ is even}\}.$$

If no three points of F are collinear, then $\boldsymbol{C}_F$ is isomorphic to $\boldsymbol{C}_A$. If exactly one triplet of the points of F is collinear, then $\boldsymbol{C}_F$ is isomorphic to $\boldsymbol{C}_B$, while if F is on a line, then $\boldsymbol{C}_F$ is isomorphic to $\boldsymbol{C}_C$.

For $T = A$, B and C, we will write $\mathfrak{M}_T$ instead of $\mathfrak{M}_{[\boldsymbol{C}_T]}$, and $\mathcal{G}_T$ instead of $\mathcal{G}_{\boldsymbol{C}_T}$. In the next three sections, we will construct explicitly the space

$$\mathfrak{M}_T = \mathrm{PGL}(3,k) \setminus \mathcal{G}_T / \operatorname{Aut}(\boldsymbol{C}_T)$$

for $T = A, B, C$, and prove Theorems 1.6, 1.7 and 1.8 stated in § 1. For this purpose, we must determine the group $\operatorname{Aut}(\boldsymbol{C}_T)$ and the space $\mathcal{G}_T$. Since $\boldsymbol{C}_T$ is generated by $\mathcal{P}$ and the set of linear and quadratic words, we obtain the following proposition.

Proposition 3.5. *Let $W_1(\boldsymbol{C}_T)$ and $W_2(\boldsymbol{C}_T)$ be the sets of linear and quadratic words in $\boldsymbol{C}_T$, respectively. An element σ of $\mathfrak{S}(\mathcal{P})$ is contained in $\operatorname{Aut}(\boldsymbol{C}_T)$ if and only if the following hold:*

$$\sigma(W_1(\boldsymbol{C}_T)) = W_1(\boldsymbol{C}_T) \quad \text{and} \quad \sigma(W_2(\boldsymbol{C}_T)) = W_2(\boldsymbol{C}_T).$$

Proposition 3.6. *Suppose that a map $\gamma : \mathcal{P} \to \mathbb{P}^2$ is given. Then γ is contained in $\mathcal{G}_T = \{\gamma \in \mathcal{G} \mid \mathcal{C}[\gamma] = \boldsymbol{C}_T\}$ if and only if the following hold:*

(i) *γ is injective;*

(ii) *there exists a homogeneous polynomial G of degree 6 such that $\gamma(\mathcal{P}) = Z(\mathrm{d}G)$;*

(iii) *for every linear word l of $\boldsymbol{C}_T$, there exists a line $L \subset \mathbb{P}^2$ containing $\gamma(l)$;*

(iv) *for every quadratic word q of $\boldsymbol{C}_T$, there exists a non-singular conic curve $Q \subset \mathbb{P}^2$ containing $\gamma(q)$.*

Proof. The 'only if' part is obvious from the definition of $\mathcal{G}_T$. Suppose that γ satisfies (i)–(iv). By (i) and (ii), we have $\gamma \in \mathcal{G}$. Since $\boldsymbol{C}_T$ is generated by the word $\mathcal{P}$ and linear and quadratic words, the properties (iii) and (iv) imply that $\boldsymbol{C}_T \subseteq \mathcal{C}[\gamma]$. If $\boldsymbol{C}_T \neq \mathcal{C}[\gamma]$, then, by Theorem 3.1, the code $\mathcal{C}[\gamma] \subset \operatorname{Pow}(\mathcal{P})$ is isomorphic to the Dolgachev–Kondo code $\boldsymbol{C}_{\mathrm{DK}} \subset \operatorname{Pow}(\mathbb{P}^2(\mathbb{F}_4))$ by some bijection from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4)$. Hence, there exists $g \in \mathrm{PGL}(3,k)$ such that

$$g(\gamma(\mathcal{P})) = Z(\mathrm{d}G_{\mathrm{DK}}) = \mathbb{P}^2(\mathbb{F}_4).$$

However, there are no eight points in $\mathbb{P}^2(\mathbb{F}_4)$ that are contained in a non-singular conic curve. □

In fact, we will prove the following assertions for $T = A$, B and C.

(1) The space $\mathrm{PGL}(3,k) \setminus \mathcal{G}_T$ has exactly two connected components, both of which are isomorphic to $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$. Let $N_T \subset \operatorname{Aut}(\boldsymbol{C}_T)$ be the subgroup consisting of the elements that do not interchange the two connected components, and let Γ_T be the image of N_T in $\operatorname{Aut}(\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\})$. Then N_T is of index 2 in $\operatorname{Aut}(\boldsymbol{C}_T)$. The moduli curve $\mathfrak{M}_T$ is the quotient of $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ by Γ_T.

Table 9. *Orders of the groups given in assertion 2.*

T	$\lvert\mathrm{Aut}(\boldsymbol{C}_T)\rvert$	$=$	2	$\times$	$\lvert\Gamma_T\rvert$	$\times$	$\lvert\mathrm{Aut}(X,\mathcal{L})\rvert$
A	1 152	$=$	2	$\times$	6	$\times$	96
B	432	$=$	2	$\times$	12	$\times$	18
C	23 040	$=$	2	$\times$	12	$\times$	960

(2) The action of Γ_T on the punctured affine line $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ is free. Hence, the order of the stabilizer subgroup $\mathrm{Stab}([\gamma]) \subset \mathrm{Aut}(\boldsymbol{C}_T)$ of a point $[\gamma] \in \mathrm{PGL}(3, k) \setminus \mathcal{G}_T$ is constant on $\mathrm{PGL}(3, k) \setminus \mathcal{G}_T$. By Corollary 2.21, $\mathrm{Stab}([\gamma])$ is equal to $\mathrm{Aut}(X, \mathcal{L})$, where $(X, \mathcal{L})$ corresponds to the image of $[\gamma]$ in $\mathfrak{M}_T$. Hence, we have an exact sequence

$$1 \to \mathrm{Aut}(X, \mathcal{L}) \to N_T \to \Gamma_T \to 1$$

for any polarized supersingular $K3$ surface $(X, \mathcal{L})$ corresponding to a point of $\mathfrak{M}_T$.

The orders of the groups above are given as in Table 9.

Remark 3.7. The following algorithm will be used frequently. Suppose that we are given eight points

$$p_i = [\xi_i, \eta_i, \zeta_i], \quad i = 1, \dots, 8,$$

on $\mathbb{P}^2$. In order for them to be on a (possibly singular) conic curve, it is necessary and sufficient that the 8×6 matrix

$$M := \begin{bmatrix} \xi_1^2, \eta_1^2, \zeta_1^2, \xi_1\eta_1, \eta_1\zeta_1, \zeta_1\xi_1 \\ \vdots \\ \xi_8^2, \eta_8^2, \zeta_8^2, \xi_8\eta_8, \eta_8\zeta_8, \zeta_8\xi_8 \end{bmatrix}$$

is of rank less than 6. When the rank of M is less than 6, a non-zero solution

$$[A, B, C, D, E, F]^{\mathrm{T}}$$

of the linear equation $M\boldsymbol{x} = 0$ gives us a defining equation

$$AX^2 + BY^2 + CZ^2 + DXY + EYZ + FZX = 0 \tag{3.1}$$

of a conic curve containing $p_1, \dots, p_8$.

The following are phenomena peculiar to projective geometry in characteristic 2.

Remark 3.8. The conic curve defined by equation (3.1) is singular if and only if

$$AE^2 + BF^2 + CD^2 + DEF = 0$$

holds.

Definition 3.9. Let $L \subset \mathbb{P}^2$ be a line, and let $Q \subset \mathbb{P}^2$ be a (possibly singular) conic curve. We say that L and Q are *tangent* if they fail to intersect at distinct two points.

Table 10. *Linear words in* $\boldsymbol{C}_A$

$m: \{1, 2, 12, 13, 18\}$
$l_{12,1}: \{10, 11, 12, 16, 20\}$
$l_{12,2}: \{8, 9, 12, 15, 19\}$
$l_{12,3}: \{5, 6, 12, 14, 17\}$
$l_{12,4}: \{3, 4, 7, 12, 21\}$
$l_{13,1}: \{13, 14, 15, 16, 21\}$
$l_{13,2}: \{7, 8, 10, 13, 17\}$
$l_{13,3}: \{4, 6, 11, 13, 19\}$
$l_{13,4}: \{3, 5, 9, 13, 20\}$
$l_{18,1}: \{17, 18, 19, 20, 21\}$
$l_{18,2}: \{7, 9, 11, 14, 18\}$
$l_{18,3}: \{4, 5, 10, 15, 18\}$
$l_{18,4}: \{3, 6, 8, 16, 18\}$

Remark 3.10. Let L be a line. Then the conic curves tangent to L form a *linear* system in $|\mathcal{O}_{\mathbb{P}^2}(2)|$. If three distinct lines L_1, L_2 and L_3 are concurrent, then every conic curve that is tangent to L_1 and L_2 is tangent to L_3.

Remark 3.11. Let $A, B, C, D \in \mathbb{P}^2$ be distinct points. Suppose that no three of them are collinear. Let O (respectively, P) (respectively, Q) be the intersection point of the lines $\overline{AB}$ and $\overline{CD}$ (respectively, $\overline{AC}$ and $\overline{BD}$) (respectively, $\overline{AD}$ and $\overline{BC}$). Then O, P and Q are collinear.

4. The moduli curve corresponding to the code $\boldsymbol{C}_A$

In this section, we prove Theorem 1.6.

The linear words of $\boldsymbol{C}_A$ are listed in Table 10.

From now on, we sometimes abbreviate, for example, the set $\{P_8, P_9, P_{12}, P_{15}, P_{19}\}$ to $\{8, 9, 12, 15, 19\}$. The linear word m stands out from the rest in that there are two points P_1 and P_2 in m through which no other linear words pass. We call m the *special linear word*. The other linear words are divided into three groups according to the intersection point with m. For $\nu = 12, 13, 18$ and $i = 1, 2, 3, 4$, the non-special linear word $l_{\nu,i}$ intersects m at the point P_ν. For each of P_1 and P_2, there exists only one linear word m containing it. For each of P_{12}, P_{13} and P_{18}, there exist exactly five linear words containing it. For each of the other 16 points, there exist exactly three linear words containing it. For each $\alpha, \beta = 1, \dots, 4$, there exists a unique $\gamma = \gamma(\alpha, \beta)$ such that the three linear words $l_{12,\alpha}$, $l_{13,\beta}$ and $l_{18,\gamma}$ have a point in common.

We call such a triple (α, β, γ) a *concurrent triple*. The list of concurrent triples is given in Table 11. For a concurrent triple (α, β, γ), we denote by $T_{\alpha\beta\gamma}$ the intersection point of $l_{12,\alpha}$, $l_{13,\beta}$ and $l_{18,\gamma}$.

The 28 quadratic words in $\boldsymbol{C}_A$ are divided into two groups. The quadratic words $q'_1, \dots, q'_{12}$ listed in Table 13 are disjoint from the special linear word m, and intersect each of the non-special linear words $l_{\nu,i}$ at distinct two points. On the other hand, for

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Table 11. *Function $\gamma(\alpha, \beta)$ of concurrent triples*

	α			
β	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

Table 12. *Points $T_{\alpha\beta\gamma}$*

$\alpha\beta\gamma$	$T_{\alpha\beta\gamma}$	$\alpha\beta\gamma$	$T_{\alpha\beta\gamma}$
114	P_{16}	312	P_{14}
123	P_{10}	321	P_{17}
132	P_{11}	334	P_{6}
141	P_{20}	343	P_{5}
213	P_{15}	411	P_{21}
224	P_{8}	422	P_{7}
231	P_{19}	433	P_{4}
242	P_{9}	444	P_{3}

Table 13. *Quadratic words q'_ν in $\boldsymbol{C}_A$*

$q'_1 : \{5, 6, 7, 9, 10, 16, 19, 21\}$
$q'_2 : \{5, 6, 7, 8, 11, 15, 20, 21\}$
$q'_3 : \{4, 6, 8, 9, 10, 14, 20, 21\}$
$q'_4 : \{4, 6, 7, 9, 15, 16, 17, 20\}$
$q'_5 : \{4, 5, 8, 9, 11, 16, 17, 21\}$
$q'_6 : \{4, 5, 7, 8, 14, 16, 19, 20\}$
$q'_7 : \{3, 6, 9, 10, 11, 15, 17, 21\}$
$q'_8 : \{3, 6, 7, 10, 14, 15, 19, 20\}$
$q'_9 : \{3, 5, 8, 10, 11, 14, 19, 21\}$
$q'_{10} : \{3, 5, 7, 11, 15, 16, 17, 19\}$
$q'_{11} : \{3, 4, 9, 10, 14, 16, 17, 19\}$
$q'_{12} : \{3, 4, 8, 11, 14, 15, 17, 20\}$

each concurrent triple (α, β, γ), there exists a unique quadratic word $q_{\alpha\beta\gamma}$ that is disjoint from the three linear words $l_{12,\alpha}$, $l_{13,\beta}$, $l_{18,\gamma}$, and intersects other ten linear words at distinct two points. The list of these quadratic words $q_{\alpha\beta\gamma}$ is given in Table 14.

In order to study $\mathrm{Aut}(\boldsymbol{C}_A)$, we embed $\boldsymbol{C}_A$ into the Dolgachev–Kondo code $\boldsymbol{C}_{\mathrm{DK}} \subset \mathrm{Pow}(\mathbb{P}^2(\mathbb{F}_4))$ by the bijection $\phi : \mathcal{P} \xrightarrow{\sim} \mathbb{P}^2(\mathbb{F}_4)$ given in Table 15.

Table 14. *Quadratic words $q_{\alpha\beta\gamma}$ in $\boldsymbol{C}_A$*

$q_{114}: \{1,2,4,5,7,9,17,19\}$
$q_{123}: \{1,2,3,6,9,14,19,21\}$
$q_{132}: \{1,2,3,5,8,15,17,21\}$
$q_{141}: \{1,2,4,6,7,8,14,15\}$
$q_{213}: \{1,2,3,6,7,11,17,20\}$
$q_{224}: \{1,2,4,5,11,14,20,21\}$
$q_{231}: \{1,2,3,5,7,10,14,16\}$
$q_{242}: \{1,2,4,6,10,16,17,21\}$
$q_{312}: \{1,2,3,4,8,10,19,20\}$
$q_{321}: \{1,2,3,4,9,11,15,16\}$
$q_{334}: \{1,2,7,9,10,15,20,21\}$
$q_{343}: \{1,2,7,8,11,16,19,21\}$
$q_{411}: \{1,2,5,6,8,9,10,11\}$
$q_{422}: \{1,2,5,6,15,16,19,20\}$
$q_{433}: \{1,2,8,9,14,16,17,20\}$
$q_{444}: \{1,2,10,11,14,15,17,19\}$

Table 15. *Bijection ϕ from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4)$*

$\phi(P_1) = [1,\omega,0]$	$\phi(P_8) = [1,\bar{\omega},1]$	$\phi(P_{15}) = [1,\omega,1]$
$\phi(P_2) = [1,\bar{\omega},0]$	$\phi(P_9) = [1,1,1]$	$\phi(P_{16}) = [0,1,\omega]$
$\phi(P_3) = [1,1,\omega]$	$\phi(P_{10}) = [0,1,\bar{\omega}]$	$\phi(P_{17}) = [1,0,\bar{\omega}]$
$\phi(P_4) = [1,\bar{\omega},\omega]$	$\phi(P_{11}) = [0,1,1]$	$\phi(P_{18}) = [1,0,0]$
$\phi(P_5) = [1,1,\bar{\omega}]$	$\phi(P_{12}) = [0,1,0]$	$\phi(P_{19}) = [1,0,1]$
$\phi(P_6) = [1,\omega,\bar{\omega}]$	$\phi(P_{13}) = [1,1,0]$	$\phi(P_{20}) = [0,0,1]$
$\phi(P_7) = [1,\omega,\omega]$	$\phi(P_{14}) = [1,\bar{\omega},\bar{\omega}]$	$\phi(P_{21}) = [1,0,\omega]$

The following can be checked easily.

(1) If l is a linear word of $\boldsymbol{C}_A$, then the points in $\phi(l)$ are collinear. The linear words of $\boldsymbol{C}_A$ coincide with $\phi^{-1}(\Lambda(\mathbb{F}_4))$, where Λ are $\mathbb{F}_4$-rational lines containing at least one of $\phi(P_{12})$, $\phi(P_{13})$, $\phi(P_{18})$.

(2) The words $q'_1, \dots, q'_{12}$ coincide with the words written as

$$\phi^{-1}(\Lambda_1(\mathbb{F}_4) + \Lambda_2(\mathbb{F}_4)),$$

where Λ_1 and Λ_2 are distinct $\mathbb{F}_4$-rational lines such that both $\Lambda_1(\mathbb{F}_4)$ and $\Lambda_2(\mathbb{F}_4)$ are disjoint from $\{\phi(P_{12}), \phi(P_{13}), \phi(P_{18})\}$, and such that the intersection point of $\Lambda_1(\mathbb{F}_4)$ and $\Lambda_2(\mathbb{F}_4)$ is either $\phi(P_1)$ or $\phi(P_2)$.

(3) For a concurrent triple (α,β,γ), let Λ_i be the $\mathbb{F}_4$-rational line passing through $\phi(T_{\alpha\beta\gamma})$ and $\phi(P_i)$ for $i = 1, 2$. Then we have $q_{\alpha\beta\gamma} = \phi^{-1}(\Lambda_1(\mathbb{F}_4) + \Lambda_2(\mathbb{F}_4))$.

Let PG' be the subgroup of $\mathrm{PGL}(3,\mathbb{F}_4)$ consisting of $g \in \mathrm{PGL}(3,\mathbb{F}_4)$ satisfying

$$\{g(\phi(P_{12})), g(\phi(P_{13})), g(\phi(P_{18}))\} = \{\phi(P_{12}), \phi(P_{13}), \phi(P_{18})\},$$

and let PG be the subgroup $\phi^{-1} \circ \mathrm{PG}' \circ \phi$ of $\mathfrak{S}(\mathcal{P})$. The order of PG is 288. Let $F' \in \mathfrak{S}(\mathbb{P}^2(\mathbb{F}_4))$ be the element of order 2 obtained by the conjugation $\omega \mapsto \bar{\omega}$ of $\mathbb{F}_4$ over $\mathbb{F}_2$. We then put

$$F := \phi^{-1} \circ F' \circ \phi = (P_1P_2)(P_3P_5)(P_4P_6)(P_7P_{14})(P_8P_{15})(P_{10}P_{16})(P_{17}P_{21}) \in \mathfrak{S}(\mathcal{P}).$$

We also put

$$T := (P_1P_2).$$

Proposition 4.1. *The group* $\mathrm{Aut}(\boldsymbol{C}_A)$ *is of order* 1152, *and is generated by* PG, F *and* T.

Proof. Since the actions of PG' and F' on $\mathbb{P}^2(\mathbb{F}_4)$ leave the set

$$\{[0,1,0],[1,1,0],[1,0,0]\} = \{\phi(P_{12}), \phi(P_{13}), \phi(P_{18})\}$$

invariant, and preserve the line-point incidence configuration, we see that $\mathrm{PG} \subset \mathrm{Aut}(\boldsymbol{C}_A)$ and $F \in \mathrm{Aut}(\boldsymbol{C}_A)$. It is obvious that $T \in \mathrm{Aut}(\boldsymbol{C}_A)$. By direct calculation, we see that the subgroup of $\mathfrak{S}(\mathcal{P})$ generated by PG, F and T is of order 1152.

Every automorphism of $\boldsymbol{C}_A$ leaves each of the sets $\{P_1, P_2\}$ and $\{P_{12}, P_{13}, P_{18}\}$ invariant. Hence, we have a homomorphism

$$\mathrm{Aut}(\boldsymbol{C}_A) \to \mathfrak{S}(\{P_1, P_2\}) \times \mathfrak{S}(\{P_{12}, P_{13}, P_{18}\}). \tag{4.1}$$

Since PG acts on $\{P_{12}, P_{13}, P_{18}\}$ as the full-symmetric group, and since T is contained in $\mathrm{Aut}(\boldsymbol{C}_A)$, the homomorphism (4.1) is surjective. Let K denote the kernel of (4.1). We have a homomorphism

$$K \to \mathfrak{S}_4 \times \mathfrak{S}_4, \qquad g \mapsto (\sigma, \sigma'), \tag{4.2}$$

where σ and σ' are given by

$$g(l_{12,\alpha}) = l_{12,\sigma(\alpha)}, \qquad g(l_{13,\beta}) = l_{13,\sigma'(\beta)}.$$

We also have a homomorphism

$$\mathfrak{S}_4 \times \mathfrak{S}_4 \to \mathfrak{S}(\mathcal{P}), \qquad (\sigma, \sigma') \mapsto g_{\sigma,\sigma'}, \tag{4.3}$$

where $g_{\sigma,\sigma'}$ is given by

$$g_{\sigma,\sigma'}(P_i) = P_i \quad \text{if } P_i \in m,$$
$$g_{\sigma,\sigma'}(T_{\alpha\beta\gamma}) = T_{\sigma(\alpha)\sigma'(\beta)\gamma'},$$
$$\text{where } (\alpha, \beta, \gamma) \text{ and } (\sigma(\alpha), \sigma'(\beta), \gamma') \text{ are concurrent triples.}$$

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Since the composite of (4.2) and (4.3) is the identity of K, the homomorphism (4.2) is injective. For each pair (σ, σ') of $\mathfrak{S}_4 \times \mathfrak{S}_4$, we check whether $g_{\sigma,\sigma'}$ is in $\mathrm{Aut}(\boldsymbol{C}_A)$, i.e. whether $g_{\sigma,\sigma'}$ satisfies the following (see Proposition 3.5):

$$g_{\sigma,\sigma'}(W_1(\boldsymbol{C}_A)) = W_1(\boldsymbol{C}_A) \quad \text{and} \quad g_{\sigma,\sigma'}(W_2(\boldsymbol{C}_A)) = W_2(\boldsymbol{C}_A). \tag{4.4}$$

Among $(4!)^2 = 576$ pairs, exactly 96 pairs satisfy (4.4). Hence, $\mathrm{Aut}(\boldsymbol{C}_A)$ is of order

$$|K|\,|\mathfrak{S}_2|\,|\mathfrak{S}_3| = 96 \times 12 = 1152,$$

and is generated by PG, F and T. □

For a parameter λ of the affine line $\mathbb{A}^1$, let

$$\gamma_\lambda : \mathcal{P} \to \mathbb{P}^2$$

be the map given in Table 1. Note that γ_ω coincides with ϕ defined above.

We denote by $\tilde{T}$ the subgroup $\{1, T\}$ of $\mathrm{Aut}(\boldsymbol{C}_A)$.

Proposition 4.2. *The map $\lambda \mapsto \gamma_\lambda$ induces an isomorphism from $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ to $\mathrm{PGL}(3, k) \setminus \mathcal{G}_A / \tilde{T}$.*

Proof. First note that γ_λ is injective if and only if

$$\lambda \neq 0 \quad \text{and} \quad \lambda \neq 1. \tag{4.5}$$

From now on, we assume (4.5).

We make the following claim.

Claim 4.3. *Let γ' be an arbitrary element of $\mathcal{G}_A$. Then there exists a unique triple*

$$(g, t, \lambda) \in \mathrm{PGL}(3, k) \times \tilde{T} \times (k \setminus \{0, 1, \omega, \bar{\omega}\})$$

such that

$$g \circ \gamma' \circ t = \gamma_\lambda.$$

Because the points $\gamma'(P_{18})$, $\gamma'(P_{19})$, $\gamma'(P_{20})$ of $\gamma'(l_{18,1})$ are on a line and the points $\gamma'(P_{12})$, $\gamma'(P_{13})$, $\gamma'(P_{18})$ of $\gamma'(m)$ are on another line, there exists a unique element $g \in \mathrm{PGL}(3, k)$ such that $\gamma := g \circ \gamma'$ satisfies the following:

$$\left.\begin{aligned} \gamma(P_{18}) &= [1, 0, 0], \\ \gamma(P_{12}) &= [0, 1, 0], \\ \gamma(P_{13}) &= [1, 1, 0], \\ \gamma(P_{20}) &= [0, 0, 1], \\ \gamma(P_{19}) &= [1, 0, 1]. \end{aligned}\right\} \tag{4.6}$$

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Let $L_{12,\alpha}$, $L_{13,\beta}$ and $L_{18,\gamma}$ be the lines containing $\gamma(l_{12,\alpha})$, $\gamma(l_{13,\beta})$ and $\gamma(l_{18,\gamma})$, respectively. We put $x := X/Z$, $y := Y/Z$. Then the defining equations of these lines can be written as

$$\left.\begin{aligned} L_{12,\alpha}: &\quad x + a_\alpha = 0,\\ L_{13,\beta}: &\quad x + y + b_\beta = 0,\\ L_{18,\gamma}: &\quad y + c_\gamma = 0. \end{aligned}\right\} \tag{4.7}$$

From (4.6), we have

$$a_1 = 0, \qquad a_2 = 1, \qquad b_3 = 1, \qquad b_4 = 0, \qquad c_1 = 0. \tag{4.8}$$

The condition that (α, β, γ) is a concurrent triple is equivalent to

$$a_\alpha + b_\beta + c_\gamma = 0.$$

By solving the linear equations corresponding to the 16 concurrent triples and combining the result with (4.8), we obtain the following solutions:

$$\left.\begin{aligned} (a_1, a_2, a_3, a_4) &= (0, 1, \lambda, 1 + \lambda),\\ (b_1, b_2, b_3, b_4) &= (1 + \lambda, \lambda, 1, 0),\\ (c_1, c_2, c_3, c_4) &= (0, 1, \lambda, 1 + \lambda), \end{aligned}\right\} \tag{4.9}$$

where λ is a parameter. The coordinates of the points $T_{\alpha\beta\gamma}$ are given by $[a_\alpha, c_\gamma, 1]$. Using Table 12, we see that $\gamma(P_i) = \gamma_\lambda(P_i)$ holds for every i except for $i = 1$ and $i = 2$. The line M containing $\gamma(m)$ is defined by $Z = 0$. Hence, we can put

$$\gamma(P_1) = [1, \tau_1, 0], \qquad \gamma(P_2) = [1, \tau_2, 0].$$

By the algorithm in Remark 3.7, we see that a conic curve containing $\gamma(q_{114})$ exists if and only if the following hold:

$$\left.\begin{aligned} (1 + \tau_2 + {\tau_2}^2)(\lambda + 1)^2\lambda^2 &= 0,\\ (\tau_1 + \tau_2)(1 + \tau_2 + {\tau_2}^2)(\lambda + 1)\lambda &= 0,\\ (\tau_1 + \tau_2)(\tau_1 + \tau_2 + 1)(\lambda + 1)\lambda &= 0. \end{aligned}\right\} \tag{4.10}$$

Here we have used the Buchberger algorithm to calculate the Gröbner basis of the ideal in $k[\lambda, \tau_1, \tau_2]$ generated by 6×6-minors of the 8×6-matrix corresponding to the eight points in $\gamma(q_{114})$. Replacing γ by $\gamma \circ T$ if necessary, we have

$$\tau_1 = \omega \quad \text{and} \quad \tau_2 = \bar{\omega}$$

by (4.5), (4.10) and $\tau_1 \neq \tau_2$. Then the conic curve containing $\gamma(q_{114})$ is defined by

$$X^2 + Y^2 + \lambda Z^2 + XY + (\lambda + 1)ZX = 0,$$

which is non-singular if and only if $\lambda^2 + \lambda + 1 \neq 0$ (see Remark 3.8). Thus, we have proved the existence and the uniqueness of the triple (g, t, λ) satisfying $g \circ \gamma' \circ t = \gamma_\lambda$.

Table 16. *Defining equations of the conic curves* Q'_i

Q'_1:	$\lambda X^2 + Y^2 + (\lambda^2+\lambda)Z^2 + YZ + \lambda^2 ZX = 0$
Q'_2:	$(\lambda+1)X^2 + Y^2 + YZ + (\lambda^2+1)ZX = 0$
Q'_3:	$(\lambda+1)X^2 + \lambda Y^2 + \lambda^2 YZ + (\lambda^2+1)ZX = 0$
Q'_4:	$\lambda X^2 + (\lambda+1)Y^2 + (\lambda^2+1)YZ + \lambda^2 ZX = 0$
Q'_5:	$X^2 + \lambda Y^2 + (\lambda^2+\lambda)Z^2 + \lambda^2 YZ + ZX = 0$
Q'_6:	$X^2 + (\lambda+1)Y^2 + (\lambda^2+1)YZ + ZX = 0$
Q'_7:	$X^2 + (\lambda+1)Y^2 + (\lambda^2+\lambda)Z^2 + (\lambda^2+1)YZ + ZX = 0$
Q'_8:	$X^2 + \lambda Y^2 + \lambda^2 YZ + ZX = 0$
Q'_9:	$\lambda X^2 + (\lambda+1)Y^2 + (\lambda^2+\lambda)Z^2 + (\lambda^2+1)YZ + \lambda^2 ZX = 0$
Q'_{10}:	$(\lambda+1)X^2 + \lambda Y^2 + (\lambda^2+\lambda)Z^2 + \lambda^2 YZ + (\lambda^2+1)ZX = 0$
Q'_{11}:	$(\lambda+1)X^2 + Y^2 + (\lambda^2+\lambda)Z^2 + YZ + (\lambda^2+1)ZX = 0$
Q'_{12}:	$\lambda X^2 + Y^2 + YZ + \lambda^2 ZX = 0$

In particular, for each double coset in $\mathrm{PGL}(3,k)\setminus\mathcal{G}_A/\tilde{T}$, there exists a unique $\lambda \in k\setminus\{0,1,\omega,\bar{\omega}\}$ such that γ_λ is contained in the coset.

Conversely, let λ be an element of $k\setminus\{0,1,\omega,\bar{\omega}\}$. We will show that γ_λ is in $\mathcal{G}_A$. The points $\gamma_\lambda(\mathcal{P})$ coincide with $Z(\mathrm{d}GA[\lambda])$, where $GA[\lambda]$ is given in Theorem 1.6. Indeed, we can check that

$$\frac{\partial GA[\lambda]}{\partial X}(\gamma_\lambda(P_i)) = \frac{\partial GA[\lambda]}{\partial Y}(\gamma_\lambda(P_i)) = \frac{\partial GA[\lambda]}{\partial Z}(\gamma_\lambda(P_i)) = 0$$

holds for $i = 1,\dots,21$. For each linear word l of $\boldsymbol{C}_A$, there exists a line containing $\gamma_\lambda(l)$. The defining equations of them are given by (4.7) and (4.9). (The line M containing $\gamma_\lambda(m)$ is defined by $Z=0$.) For each quadratic word q'_i of $\boldsymbol{C}_A$ (respectively $q_{\alpha\beta\gamma}$), there exists a non-singular conic curve Q'_i (respectively $Q_{\alpha\beta\gamma}$) containing $\gamma_\lambda(q'_i)$ (respectively, $\gamma_\lambda(q_{\alpha\beta\gamma})$). The defining equations of them are given in Tables 16 and 17. Hence, $\gamma_\lambda \in \mathcal{G}_A$ by Proposition 3.6. □

Remark 4.4. The polynomial $GA[\lambda]$ defines the nodal splitting curve

$$M \cup L_{12,1} \cup L_{18,1} \cup L_{13,3} \cup Q_{242}$$

(see Proposition 2.8).

Remark 4.5. When $\lambda \in \{\omega,\bar{\omega}\}$, the set $\gamma_\lambda(\mathcal{P})$ coincides with $\mathbb{P}^2(\mathbb{F}_4)$, and the point $[GA[\lambda]] \in \mathfrak{M}$ is the Dolgachev–Kondo point.

Let $k(\lambda)$ be the rational function field with variable λ. For each $\sigma \in \mathrm{Aut}(\boldsymbol{C}_A)$, we calculate the unique triple

$$(g_\sigma, t_\sigma, \lambda^\sigma) \in \mathrm{PGL}(3, k(\lambda)) \times \tilde{T} \times k(\lambda)$$

such that

$$g_\sigma \circ (\gamma_\lambda \circ \sigma) \circ t_\sigma = \gamma_{\lambda^\sigma}$$

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Table 17. *Defining equations of the conic curves $Q_{\alpha\beta\gamma}$*

Q_{114}:	$X^2+Y^2+XY+\lambda Z^2+(\lambda+1)ZX=0$
Q_{123}:	$X^2+Y^2+XY+(\lambda+1)Z^2+\lambda ZX=0$
Q_{132}:	$X^2+Y^2+XY+(\lambda^2+\lambda)Z^2+ZX=0$
Q_{141}:	$X^2+Y^2+XY+(\lambda^2+\lambda+1)Z^2=0$
Q_{213}:	$X^2+Y^2+XY+YZ+\lambda ZX=0$
Q_{224}:	$X^2+Y^2+XY+YZ+(\lambda+1)ZX=0$
Q_{231}:	$X^2+Y^2+XY+(\lambda^2+\lambda)Z^2+YZ=0$
Q_{242}:	$X^2+Y^2+XY+(\lambda^2+\lambda)Z^2+YZ+ZX=0$
Q_{312}:	$X^2+Y^2+XY+\lambda YZ+ZX=0$
Q_{321}:	$X^2+Y^2+XY+(\lambda+1)Z^2+\lambda YZ=0$
Q_{334}:	$X^2+Y^2+XY+\lambda YZ+(\lambda+1)ZX=0$
Q_{343}:	$X^2+Y^2+XY+(\lambda+1)Z^2+\lambda YZ+\lambda ZX=0$
Q_{411}:	$X^2+Y^2+XY+\lambda Z^2+(\lambda+1)YZ=0$
Q_{422}:	$X^2+Y^2+XY+(\lambda+1)YZ+ZX=0$
Q_{433}:	$X^2+Y^2+XY+(\lambda+1)YZ+\lambda ZX=0$
Q_{444}:	$X^2+Y^2+XY+\lambda Z^2+(\lambda+1)YZ+(\lambda+1)ZX=0$

holds (see Claim 4.3). The calculation is done as follows: g_σ is the unique linear automorphism of $\mathbb{P}^2$ characterized by

$$\begin{aligned}
g_\sigma(\gamma_\lambda(\sigma(P_{18}))) &= [1,0,0],\\
g_\sigma(\gamma_\lambda(\sigma(P_{12}))) &= [0,1,0],\\
g_\sigma(\gamma_\lambda(\sigma(P_{13}))) &= [1,1,0],\\
g_\sigma(\gamma_\lambda(\sigma(P_{20}))) &= [0,0,1],\\
g_\sigma(\gamma_\lambda(\sigma(P_{19}))) &= [1,0,1];
\end{aligned}$$

$t_\sigma \in \tilde{T}$ is given by

$$t_\sigma = \begin{cases} \mathrm{id} & \text{if } g_\sigma(\gamma_\lambda(\sigma(P_1))) = [1,\omega,0],\\ T & \text{if } g_\sigma(\gamma_\lambda(\sigma(P_1))) = [1,\bar{\omega},0]; \end{cases}$$

and λ^σ is the rational function of the parameter λ satisfying

$$g_\sigma(\gamma_\lambda(\sigma(P_{10}))) = [0,\lambda^\sigma,1].$$

The map $\sigma \mapsto t_\sigma$ is a homomorphism from $\mathrm{Aut}(\boldsymbol{C}_A)$ to $\tilde{T}$. We put

$$N_A := \mathrm{Ker}(\mathrm{Aut}(\boldsymbol{C}_A) \to \tilde{T}).$$

From the proof of Proposition 4.2, we obtain the following corollary.

Corollary 4.6. *The space* $\mathrm{PGL}(3,k) \setminus \mathcal{G}_A$ *has exactly two connected components, each of which is isomorphic to* $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$*. Set-theoretically, they are given by*

$$(\mathrm{PGL}(3,k) \setminus \mathcal{G}_A)^+ := \{[\gamma_\alpha] \mid \alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}\}$$

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and

$$(\mathrm{PGL}(3,k)\setminus\mathcal{G}_A)^- := \{[\gamma_\alpha \circ T] \mid \alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}\}.$$

The group N_A acts on $(\mathrm{PGL}(3,k)\setminus\mathcal{G}_A)^+$, and the moduli curve $\mathfrak{M}_A$ is equal to the quotient space $(\mathrm{PGL}(3,k)\setminus\mathcal{G}_A)^+/N_A$.

Let

$$p_A : \mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} \cong (\mathrm{PGL}(3,k)\setminus\mathcal{G}_A)^+ \to \mathfrak{M}_A = (\mathrm{PGL}(3,k)\setminus\mathcal{G}_A)^+/N_A$$

denote the natural projection. For $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, let $P[\alpha]$ be the point of $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$ given by $\lambda = \alpha$. Then $p_A(P[\alpha]) \in \mathfrak{M}_A$ corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$.

Proposition 4.7. *The set $p_A^{-1}(p_A(P[\alpha]))$ is equal to*

$$\{P[\alpha], P[1/\alpha], P[\alpha+1], P[1/(\alpha+1)], P[\alpha/(\alpha+1)], P[(\alpha+1)/\alpha]\}, \tag{4.11}$$

and $\mathrm{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ is equal to the group (1.1).

Proof. The set $\{\lambda^\sigma \mid \sigma \in N_A\} \subset k(\lambda)$ coincides with the group Γ_A given in Theorem 1.6. The fibre $p_A^{-1}(p_A(P[\alpha]))$ is therefore equal to (4.11). Note that the fibre $p_A^{-1}(p_A(P[\alpha]))$ consists of six distinct points for any $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, i.e. the action of Γ_A on $(\mathrm{PGL}(3,k)\setminus\mathcal{G}_A)^+$ is free. Hence, for any $\alpha \in k\setminus\{0,1,\omega,\bar{\omega}\}$ and any $\sigma \in \mathrm{Aut}(\boldsymbol{C}_A)$, the projective equivalence classes $[\gamma_\alpha]$ and

$$[\gamma_\alpha \circ \sigma] = [\gamma_{\alpha^\sigma} \circ t_\sigma] \in \mathrm{PGL}(3,k) \setminus \mathcal{G}_C$$

coincide if and only if $t_\sigma = \mathrm{id}$ and $\lambda^\sigma = \lambda$ hold. Therefore, using Corollary 2.21, we can obtain $\mathrm{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ from the subgroup

$$\{g_\sigma \mid t_\sigma = \mathrm{id} \text{ and } \lambda^\sigma = \lambda\} \subset \mathrm{PGL}(3, k(\lambda))$$

by substituting α for λ. □

Corollary 4.8. *We have $\mathfrak{M}_A = \mathrm{Spec}\, k[J_A, 1/J_A]$, where $J_A = (\lambda^2+\lambda+1)/\lambda^2(\lambda+1)^2$. The morphism p_A is an étale Galois covering with Galois group Γ_A, which is isomorphic to $\mathfrak{S}_3$.*

5. The moduli curve corresponding to the code C_B

In this section, we prove Theorem 1.7.

Let AF be the affine plane over $\mathbb{F}_3$, $P(\mathrm{AF})$ the set of rational points of AF, and $L(\mathrm{AF})$ the set of rational affine lines of AF. Each element of $P(AF)$ is expressed by a pair aa' of elements of $\mathbb{F}_3$, and each element of $L(\mathrm{AF})$ is expressed as a subset $\{aa', bb', cc'\}$ of $P(\mathrm{AF})$ with cardinality 3. We have

$$|P(\mathrm{AF})| = 9 \quad \text{and} \quad |L(\mathrm{AF})| = 12.$$

Table 18. *C-points* $C(aa')$ *and* T*-points* $T(\ell)$ *for* $\ell = \{aa', bb', cc'\}$

aa'	00	01	02	10	11	12	20	21	22
$C(aa')$	P_{17}	P_{13}	P_5	P_{10}	P_8	P_6	P_2	P_3	P_1

aa'	bb'	cc'	$T(\ell)$
00	01	02	P_{21}
00	10	20	P_{20}
00	11	22	P_{19}
00	12	21	P_{18}
01	10	22	P_{16}
01	11	21	P_{15}
01	12	20	P_{14}
02	10	21	P_{11}
02	11	20	P_9
02	12	22	P_7
10	11	12	P_{12}
20	21	22	P_4

The incidence relation

$$\{(p, \ell) \in P(\mathrm{AF}) \times L(\mathrm{AF}) \mid p \in \ell\}$$

is called the *Hesse configuration* [**4**]. The automorphism group

$$G_{\mathrm{Hesse}} := \{\sigma \in \mathfrak{S}(P(\mathrm{AF})) \mid \sigma(\ell) \in L(\mathrm{AF}) \text{ for all } \ell \in L(\mathrm{AF})\}$$

of this configuration is isomorphic to the group of affine transformations of AF defined over $\mathbb{F}_3$. In particular, the order of G_{Hesse} is 432.

We define injective maps

$$C : P(\mathrm{AF}) \to \mathcal{P} \quad \text{and} \quad T : L(\mathrm{AF}) \to \mathcal{P}$$

by Table 18. Then $\mathcal{P}$ is a disjoint union of $C(P(\mathrm{AF}))$ and $T(L(\mathrm{AF}))$. A point $P \in \mathcal{P}$ is called a *C-point* or a *T-point* according to whether $P \in C(P(\mathrm{AF}))$ or $P \in T(L(\mathrm{AF}))$. The code $\boldsymbol{C}_B$ is described as follows.

The linear words of $\boldsymbol{C}_B$ are precisely the words

$$l_{aa'} := \{C(aa'), T(\ell_1), T(\ell_2), T(\ell_3), T(\ell_4)\}, \quad aa' \in P(\mathrm{AF}),$$

where $\ell_1, \dots, \ell_4 \in L(\mathrm{AF})$ are the four affine lines passing through the point $aa' \in P(\mathrm{AF})$.

There are two types of quadratic words.

(I) Let $\ell = \{aa', bb', cc'\}$ be an element of $L(\mathrm{AF})$. There exists a unique pair of distinct affine lines

$$\ell_1 = \{a_1a_1', b_1b_1', c_1c_1'\} \neq \ell, \qquad \ell_2 = \{a_2a_2', b_2b_2', c_2c_2'\} \neq \ell$$

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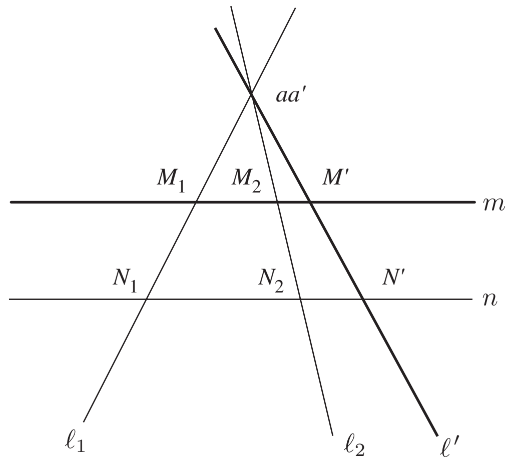

Figure 1. Intersection points.

that are parallel to ℓ. Then the word

$$q_\ell := \{C(a_1a_1'), C(b_1b_1'), C(c_1c_1'), C(a_2a_2'), C(b_2b_2'), C(c_2c_2'), T(\ell_1), T(\ell_2)\}$$

is a quadratic word of $\boldsymbol{C}_B$.

(II) Let ℓ_1 and ℓ_2 be two distinct elements of $L(\mathrm{AF})$ that are not parallel, and let $aa' \in P(\mathrm{AF})$ be the intersection point of ℓ_1 and ℓ_2. Then there exists a pair $\{m, n\}$ of elements of $L(\mathrm{AF})$ with the following properties:

(i) m and n are parallel,

(ii) $aa' \notin m$, $aa' \notin n$,

(iii) none of the pairs (ℓ_1, m), (ℓ_2, m), (ℓ_1, n), (ℓ_2, n) are parallel.

For such a pair $\{m, n\}$, there exists a unique line $\ell' \in L(\mathrm{AF})$ such that

(a) $aa' \in \ell'$,

(b) is distinct from ℓ_1 and ℓ_2,

(c) intersects both of m and n.

We denote the intersection points of these affine lines as in Figure 1. Then the word

$$q'_{\ell_1,\ell_2} := \{C(M_1), C(M_2), C(N_1), C(N_2), T(M_1N'), T(M_2N'), T(N_1M'), T(N_2M')\}$$

is a quadratic word of $\boldsymbol{C}_B$, where $MN \in L(\mathrm{AF})$ denotes the affine line containing the points M and N. For each (ℓ_1, ℓ_2), there exist exactly two pairs satisfying (i), (ii) and (iii). However, the word q'_{ℓ_1,ℓ_2} is independent of the choice of the pair.

There exist 12 quadratic words of type I, and 54 quadratic words of type II. The quadratic words of $\boldsymbol{C}_B$ are precisely these 66 words. The linear and quadratic words of $\boldsymbol{C}_B$ are explicitly presented in Tables 19–21.

The following proposition can be checked easily.

Table 19. *Linear words of $\boldsymbol{C}_B$*

$$
\begin{aligned}
l_{00} &= \{17, 18, 19, 20, 21\}\\
l_{01} &= \{13, 14, 15, 16, 21\}\\
l_{02} &= \{5, 7, 9, 11, 21\}\\
l_{10} &= \{10, 11, 12, 16, 20\}\\
l_{11} &= \{8, 9, 12, 15, 19\}\\
l_{12} &= \{6, 7, 12, 14, 18\}\\
l_{20} &= \{2, 4, 9, 14, 20\}\\
l_{21} &= \{3, 4, 11, 15, 18\}\\
l_{22} &= \{1, 4, 7, 16, 19\}
\end{aligned}
$$

Table 20. *Quadratic words of type I in $\boldsymbol{C}_B$*

$$
\begin{aligned}
q_{00,01,02} &= \{1, 2, 3, 4, 6, 8, 10, 12\}\\
q_{00,10,20} &= \{1, 3, 5, 6, 7, 8, 13, 15\}\\
q_{00,11,22} &= \{2, 3, 5, 6, 10, 11, 13, 14\}\\
q_{00,12,21} &= \{1, 2, 5, 8, 9, 10, 13, 16\}\\
q_{01,10,22} &= \{2, 3, 5, 6, 8, 9, 17, 18\}\\
q_{01,11,21} &= \{1, 2, 5, 6, 7, 10, 17, 20\}\\
q_{01,12,20} &= \{1, 3, 5, 8, 10, 11, 17, 19\}\\
q_{02,10,21} &= \{1, 2, 6, 8, 13, 14, 17, 19\}\\
q_{02,11,20} &= \{1, 3, 6, 10, 13, 16, 17, 18\}\\
q_{02,12,22} &= \{2, 3, 8, 10, 13, 15, 17, 20\}\\
q_{10,11,12} &= \{1, 2, 3, 4, 5, 13, 17, 21\}\\
q_{20,21,22} &= \{5, 6, 8, 10, 12, 13, 17, 21\}
\end{aligned}
$$

Proposition 5.1. *Let $\ell = \{aa', bb', cc'\}$ be an element of $L(\mathrm{AF})$. Then the quadratic word q_ℓ of type I is disjoint from the three linear words $l_{aa'}$, $l_{bb'}$ and $l_{cc'}$ containing $T(\ell) \in \mathcal{P}$.*

We define a homomorphism

$$\Psi : G_{\mathrm{Hesse}} \to \mathfrak{S}(\mathcal{P})$$

by

$$\Psi(g)(C(aa')) := C(g(aa')), \qquad \Psi(g)(T(\ell)) := T(g(\ell)).$$

It is obvious that Ψ is injective.

Proposition 5.2. *The automorphism group $\mathrm{Aut}(\boldsymbol{C}_B)$ of the code $\boldsymbol{C}_B$ coincides with the image of Ψ.*

Proof. The above description of the linear and quadratic words in $\boldsymbol{C}_B$ shows that every element in the image of Ψ preserves the sets of these words. Since $\boldsymbol{C}_B$ is generated by the word $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$ and these words, the image of Ψ is contained in $\mathrm{Aut}(\boldsymbol{C}_B)$.

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Table 21. *Quadratic words of type II in* $\boldsymbol{C}_B$

$T(\ell)$	$T(\ell')$	$q'_{\ell,\ell'}$
18	19	$\{1,3,6,8,9,11,14,16\}$
18	20	$\{2,3,6,7,9,10,15,16\}$
18	21	$\{3,4,5,6,9,12,13,16\}$
19	20	$\{1,2,7,8,10,11,14,15\}$
19	21	$\{1,4,5,8,11,12,13,14\}$
20	21	$\{2,4,5,7,10,12,13,15\}$
11	12	$\{3,4,5,6,8,14,19,21\}$
11	16	$\{1,3,5,9,13,14,18,19\}$
11	20	$\{2,3,5,7,14,15,17,19\}$
12	16	$\{1,4,6,8,9,13,18,21\}$
12	20	$\{2,4,6,7,8,15,17,21\}$
16	20	$\{1,2,7,9,13,15,17,18\}$
4	9	$\{1,3,5,8,12,16,18,21\}$
4	14	$\{1,3,6,11,12,13,19,21\}$
4	20	$\{1,3,7,10,12,15,17,21\}$
9	14	$\{5,6,8,11,13,16,18,19\}$
9	20	$\{5,7,8,10,15,16,17,18\}$
14	20	$\{6,7,10,11,13,15,17,19\}$
14	15	$\{2,3,6,7,8,11,19,20\}$
14	16	$\{1,2,6,9,10,11,18,19\}$
14	21	$\{2,4,5,6,11,12,17,19\}$
15	16	$\{1,3,7,8,9,10,18,20\}$
15	21	$\{3,4,5,7,8,12,17,20\}$

Suppose that $\sigma \in \mathrm{Aut}(\boldsymbol{C}_B)$ is given. A point $P \in \mathcal{P}$ is a C-point if and only if there exists exactly one linear word in $\boldsymbol{C}_B$ that contains P. Hence, σ preserves the set of C-points. Via the bijection $C : P(\mathrm{AF}) \cong \mathrm{Im}\, C$, we obtain a unique element $\tilde{\sigma} \in \mathfrak{S}(P(\mathrm{AF}))$ such that $\sigma \circ C = C \circ \tilde{\sigma}$ holds. When $P = C(aa')$, the unique linear word in $\boldsymbol{C}_B$ containing P is just $l_{aa'}$. The Hesse configuration on AF is recovered from $\boldsymbol{C}_B$ as follows: a set $\{aa', bb', cc'\}$ of cardinality 3 is an element of $L(\mathrm{AF})$ if and only if the three linear words $l_{aa'}$, $l_{bb'}$, $l_{cc'}$ have a point in common. In this case, the common point of $l_{aa'}$, $l_{bb'}$, $l_{cc'}$ is just $T(\{aa', bb', cc'\})$. Therefore, we see that $\tilde{\sigma} \in G_{\mathrm{Hesse}}$, and that $\sigma \circ T = T \circ \tilde{\sigma}$ holds. Thus, $\sigma = \Psi(\tilde{\sigma})$. □

Let λ be a parameter of the affine line $\mathbb{A}^1$, and let $\gamma_\lambda : \mathcal{P} \to \mathbb{P}^2$ be the map given in Table 2. We also denote by $\tilde{T} = \langle T \rangle$ the subgroup of $\mathrm{Aut}(\boldsymbol{C}_B)$ of order 2 generated by

$$T := (P_2P_5)(P_3P_6)(P_4P_7)(P_{10}P_{13})(P_{11}P_{14})(P_{12}P_{15})(P_{20}P_{21}),$$

which corresponds to the automorphism of the Hesse configuration given by $aa' \mapsto a'a$.

Proposition 5.3. *The map* $\lambda \mapsto \gamma_\lambda$ *induces an isomorphism from* $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ *to* $\mathrm{PGL}(3, k) \setminus \mathcal{G}_B / \tilde{T}$.

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Table 21. (*Cont.*) *Quadratic words of type II in* $\boldsymbol{C}_B$

$T(\ell)$	$T(\ell')$	$q'_{\ell,\ell'}$
16	21	$\{1,4,5,9,10,12,17,18\}$
9	12	$\{2,4,5,6,10,16,18,21\}$
9	15	$\{2,3,5,7,13,16,18,20\}$
9	19	$\{1,2,5,11,14,16,17,18\}$
12	15	$\{3,4,6,7,10,13,20,21\}$
12	19	$\{1,4,6,10,11,14,17,21\}$
15	19	$\{1,3,7,11,13,14,17,20\}$
4	11	$\{1,2,5,10,12,14,19,21\}$
4	15	$\{1,2,7,8,12,13,20,21\}$
4	18	$\{1,2,6,9,12,16,17,21\}$
11	15	$\{5,7,8,10,13,14,19,20\}$
11	18	$\{5,6,9,10,14,16,17,19\}$
15	18	$\{6,7,8,9,13,16,17,20\}$
7	9	$\{1,2,6,8,15,16,18,20\}$
7	11	$\{1,3,6,10,14,15,19,20\}$
7	21	$\{1,4,6,12,13,15,17,20\}$
9	11	$\{2,3,8,10,14,16,18,19\}$
9	21	$\{2,4,8,12,13,16,17,18\}$
11	21	$\{3,4,10,12,13,14,17,19\}$
7	12	$\{1,4,5,8,10,15,20,21\}$
7	14	$\{1,2,5,11,13,15,19,20\}$
7	18	$\{1,3,5,9,15,16,17,20\}$
12	14	$\{2,4,8,10,11,13,19,21\}$
12	18	$\{3,4,8,9,10,16,17,21\}$
14	18	$\{2,3,9,11,13,16,17,19\}$
4	7	$\{2,3,5,6,12,15,20,21\}$
4	16	$\{2,3,9,10,12,13,18,21\}$
4	19	$\{2,3,8,11,12,14,17,21\}$
7	16	$\{5,6,9,10,13,15,18,20\}$
7	19	$\{5,6,8,11,14,15,17,20\}$
16	19	$\{8,9,10,11,13,14,17,18\}$

Proof. First note that γ_λ is injective if and only if

$$\lambda \neq 0, \quad \lambda \neq 1 \quad \text{and} \quad \lambda \neq \bar{\omega}$$

hold.

Suppose that $\lambda \neq 0, 1, \omega$ and $\bar{\omega}$. Then γ_λ is injective, and the image $\gamma_\lambda(\mathcal{P})$ coincides with $Z(\mathrm{d}GB[\lambda])$, where $GB[\lambda]$ is given in Theorem 1.7. Moreover, for each linear word $l_{aa'}$ (respectively, each quadratic word q_ℓ of type I) (respectively, each quadratic word

Table 22. *Defining equations of the lines $L_{aa'}$*

aa'	defining equation
00	$X = 0$
01	$Y = 0$
02	$X + \omega Y = 0$
10	$Z = 0$
11	$X + Y + Z = 0$
12	$\omega X + \omega Y + Z = 0$
20	$\omega X + Z = 0$
21	$X + \omega Y + Z = 0$
22	$Y + Z = 0$

Table 23. *Defining equations of the conic curves of type I*

aa'	bb'	cc'	defining equation of $Q_{aa',bb',cc'}$
00	01	02	$(\lambda+\bar\omega)X^2+\bar\omega Y^2+(\lambda+\omega)Z^2+\lambda XY$
00	10	20	$(\omega\lambda+1)X^2+(\bar\omega\lambda+1)Y^2+\omega\lambda Z^2+ZX$
00	11	22	$(\bar\omega\lambda+\omega)X^2+\bar\omega Y^2+\omega\lambda Z^2+(\lambda+1)XY+(\lambda+1)ZX$
00	12	21	$Y^2+\lambda Z^2+(\omega\lambda+1)XY+(\lambda+\bar\omega)ZX$
01	10	22	$(\bar\omega\lambda+1)X^2+Y^2+\bar\omega\lambda Z^2+(\lambda+\bar\omega)YZ$
01	11	21	$(\omega\lambda+1)X^2+\lambda Z^2+XY+YZ$
01	12	20	$(\lambda+\bar\omega)X^2+Y^2+\lambda Z^2+(\omega\lambda+\omega)XY+(\lambda+1)YZ$
02	11	20	$\omega Y^2+\lambda Z^2+(\bar\omega\lambda+\omega)XY+(\omega\lambda+1)YZ+(\lambda+\bar\omega)ZX$
02	12	22	$(\omega\lambda+1)X^2+\omega\lambda Z^2+XY+\omega YZ+ZX$
10	11	12	$(\lambda+\bar\omega)X^2+Y^2+\lambda YZ+\lambda ZX$
20	21	22	$(\lambda+\bar\omega)X^2+\bar\omega Y^2+\lambda XY+\bar\omega\lambda YZ+\lambda ZX$

$q'_{\ell,\ell'}$ of type II) of the code $\boldsymbol{C}_B$, there exists a line $L_{aa'}$ containing $\gamma_\lambda(\ell_{aa'})$ (respectively, a conic curve Q_ℓ containing $\gamma_\lambda(q_\ell)$) (respectively, a conic curve $Q'_{\ell,\ell'}$ containing $\gamma_\lambda(q'_{\ell,\ell'})$) given in Tables 22–24. The conic curves in Tables 23 and 24 are non-singular because $\lambda \notin \{0, 1, \omega, \bar\omega\}$. Hence, γ_λ is in $\mathcal{G}_B$ by Proposition 3.6.

Conversely, let γ' be an arbitrary element of $\mathcal{G}_B$. We make the following claim.

Claim 5.4. *There exists a unique triple*

$$(g, t, \lambda) \in \mathrm{PGL}(3, k) \times \tilde{T} \times (k \setminus \{0, 1, \omega, \bar\omega\})$$

such that $g \circ \gamma' \circ t = \gamma_\lambda$ holds.

The points $\gamma'(P_{15})$, $\gamma'(P_{16})$, $\gamma'(P_{21})$ of $\gamma'(\ell_{01})$ are on a line, and the points $\gamma'(P_{12})$, $\gamma'(P_{16})$, $\gamma'(P_{20})$ of $\gamma'(\ell_{10})$ are on another line. Hence, there exists a unique element

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Table 24. *Defining equations of the conic curves of type II*

$T(\ell)$	$T(\ell')$	defining equation of $Q'_{\ell,\ell'}$
18	19	$\bar{\omega}\lambda Y^2 + \bar{\omega}Z^2 + \omega\lambda XY + ZX$
18	20	$Y^2 + (\lambda+1)Z^2 + (\omega\lambda+1)XY + (\lambda+1)ZX$
18	21	$(\lambda+1)Y^2 + \lambda Z^2 + (\lambda+1)XY + (\lambda+\bar{\omega})ZX$
19	20	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}Y^2 + (\omega\lambda+1)Z^2 + (\lambda+1)XY + (\lambda+\bar{\omega})ZX$
19	21	$(\bar{\omega}\lambda+\omega)X^2 + (\lambda+\bar{\omega})Y^2 + \omega\lambda Z^2 + (\omega\lambda+1)XY + (\lambda+1)ZX$
20	21	$(\omega\lambda+1)X^2 + Y^2 + \omega\lambda Z^2 + \omega\lambda XY + ZX$
11	12	$\omega X^2 + (\bar{\omega}\lambda+\omega)Y^2 + (\bar{\omega}\lambda+\omega)YZ + ZX$
11	16	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}\lambda Y^2 + \omega\lambda Z^2 + \lambda YZ + (\lambda+1)ZX$
11	20	$\bar{\omega}\lambda X^2 + \omega Y^2 + \omega\lambda Z^2 + \omega(\lambda+1)YZ + \lambda ZX$
12	16	$(\lambda+\bar{\omega})X^2 + \bar{\omega}(\lambda+1)Y^2 + \omega(\lambda+1)YZ + \lambda ZX$
12	20	$(\lambda+1)X^2 + Y^2 + \lambda YZ + (\lambda+1)ZX$
16	20	$(\omega\lambda+1)X^2 + \bar{\omega}Y^2 + \omega\lambda Z^2 + (\bar{\omega}\lambda+\omega)YZ + ZX$
4	9	$\bar{\omega}(\lambda+1)Y^2 + \bar{\omega}(\lambda+1)XY + \omega(\lambda+1)YZ + ZX$
4	14	$(\lambda+\bar{\omega})X^2 + (\omega\lambda+1)Y^2 + (\bar{\omega}\lambda+\omega)XY + (\omega\lambda+1)YZ + \lambda ZX$
4	20	$(\lambda+\bar{\omega})X^2 + \bar{\omega}Y^2 + \lambda XY + \bar{\omega}\lambda YZ + (\lambda+\bar{\omega})ZX$
9	14	$\omega\lambda Y^2 + \lambda Z^2 + \lambda XY + \bar{\omega}\lambda YZ + (\lambda+\bar{\omega})ZX$
9	20	$\omega Y^2 + \lambda Z^2 + (\bar{\omega}\lambda+\omega)XY + (\omega\lambda+1)YZ + \lambda ZX$
14	20	$(\omega\lambda+1)X^2 + \omega Y^2 + \omega\lambda Z^2 + \bar{\omega}(\lambda+1)XY + \omega(\lambda+1)YZ + ZX$
14	15	$\lambda X^2 + (\lambda+\bar{\omega})Z^2 + \omega\lambda XY + (\lambda+\bar{\omega})YZ$
14	16	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}Y^2 + \omega Z^2 + (\lambda+1)XY + YZ$
14	21	$\bar{\omega}X^2 + Y^2 + \lambda Z^2 + \omega XY + (\lambda+1)YZ$
15	16	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}(\lambda+1)Z^2 + \omega XY + (\lambda+1)YZ$
15	21	$(\lambda+1)X^2 + \lambda Z^2 + (\lambda+1)XY + YZ$
16	21	$(\omega\lambda+1)X^2 + Y^2 + \bar{\omega}\lambda Z^2 + \omega\lambda XY + (\lambda+\bar{\omega})YZ$
9	12	$\bar{\omega}Y^2 + (\lambda+\bar{\omega})XY + \omega YZ + (\lambda+1)ZX$
9	15	$\lambda Z^2 + (\bar{\omega}\lambda+\bar{\omega})XY + \omega\lambda YZ + (\lambda+\bar{\omega})ZX$
9	19	$Y^2 + \bar{\omega}\lambda Z^2 + \bar{\omega}XY + (\lambda+\bar{\omega})YZ + \lambda ZX$
12	15	$(\lambda+\bar{\omega})X^2 + \bar{\omega}XY + (\lambda+\bar{\omega})YZ + \lambda ZX$
12	19	$(\omega\lambda+1)X^2 + \omega Y^2 + \bar{\omega}(\lambda+1)XY + \omega\lambda YZ + (\lambda+\bar{\omega})ZX$
15	19	$(\bar{\omega}\lambda+\omega)X^2 + \omega\lambda Z^2 + (\lambda+\bar{\omega})XY + \omega YZ + (\lambda+1)ZX$
4	11	$(\omega\lambda+1)X^2 + Y^2 + \omega\lambda XY + YZ + (\lambda+\bar{\omega})ZX$
4	15	$(\lambda+\bar{\omega})X^2 + (\lambda+\bar{\omega})XY + \bar{\omega}(\lambda+1)YZ + \lambda ZX$
4	18	$\omega Y^2 + \omega XY + \omega\lambda YZ + (\lambda+1)ZX$
11	15	$(\bar{\omega}\lambda+\omega)X^2 + \omega\lambda Z^2 + \omega XY + \omega\lambda YZ + (\lambda+1)ZX$
11	18	$\bar{\omega}Y^2 + \bar{\omega}\lambda Z^2 + (\lambda+\bar{\omega})XY + \bar{\omega}(\lambda+1)YZ + \lambda ZX$
15	18	$\lambda Z^2 + \omega\lambda XY + YZ + (\lambda+\bar{\omega})ZX$
7	9	$(\lambda+1)Z^2 + \bar{\omega}\lambda XY + \omega(\lambda+1)YZ + (\lambda+1)ZX$
7	11	$(\bar{\omega}\lambda+\omega)X^2 + (\omega\lambda+1)Z^2 + \omega XY + (\omega\lambda+1)YZ + (\lambda+\bar{\omega})ZX$

Table 24. (*Cont.*) *Defining equations of the conic curves of type II*

$T(\ell)$	$T(\ell')$	defining equation of $Q'_{\ell,\ell'}$
7	21	$(\omega\lambda+1)X^2+\omega\lambda Z^2+(\omega\lambda+1)XY+\omega YZ+ZX$
9	11	$Y^2+\bar{\omega}Z^2+(\omega\lambda+1)XY+\omega YZ+ZX$
9	21	$\omega Y^2+\lambda Z^2+\omega XY+(\omega\lambda+1)YZ+(\lambda+\bar{\omega})ZX$
11	21	$(\bar{\omega}\lambda+\omega)X^2+\omega Y^2+\omega\lambda Z^2+\bar{\omega}\lambda XY+\omega(\lambda+1)YZ+(\lambda+1)ZX$
7	12	$(\lambda+\bar{\omega})X^2+\bar{\omega}XY+(\lambda+1)YZ+(\lambda+\bar{\omega})ZX$
7	14	$(\omega\lambda+1)X^2+\omega\lambda Z^2+(0\bar{\omega}\lambda+\omega)XY+\omega\lambda YZ+ZX$
7	18	$\lambda Z^2+\omega(1\lambda+1)XY+YZ+\lambda ZX$
12	14	$(2\lambda+\bar{\omega})X^2+Y^2+\omega(3\lambda+1)XY+YZ+\lambda ZX$
12	18	$\omega Y^2+(4\bar{\omega}\lambda+\omega)XY+\omega\lambda YZ+ZX$
14	18	$Y^2+\lambda Z^2+\bar{\omega}XY+(5\lambda+1)YZ+(6\lambda+\bar{\omega})ZX$
4	7	$(7\lambda+1)X^2+(8\lambda+1)XY+(9\bar{\omega}\lambda+\omega)YZ+(0\lambda+1)ZX$
4	16	$(1\lambda+\bar{\omega})X^2+\bar{\omega}Y^2+\lambda XY+\omega YZ+\lambda ZX$
4	19	$\omega X^2+\bar{\omega}Y^2+XY+\bar{\omega}\lambda YZ+ZX$
7	16	$(2\omega\lambda+1)X^2+\omega\lambda Z^2+XY+\bar{\omega}\lambda YZ+ZX$
7	19	$\bar{\omega}\lambda X^2+\omega\lambda Z^2+\lambda XY+\omega YZ+\lambda ZX$
16	19	$(3\bar{\omega}\lambda+\omega)X^2+\bar{\omega}Y^2+\omega\lambda Z^2+(4\lambda+1)XY+(5\bar{\omega}\lambda+\omega)YZ+(6\lambda+1)ZX$

$g\in \mathrm{PGL}(3,k)$ such that $\gamma:=g\circ\gamma'$ satisfies

$$\left.\begin{aligned}\gamma(P_{16})&=[1,0,0],\\ \gamma(P_{12})&=[1,1,0],\\ \gamma(P_{20})&=[0,1,0],\\ \gamma(P_{15})&=[1,0,1],\\ \gamma(P_{21})&=[0,0,1].\end{aligned}\right\}\tag{5.1}$$

For $aa'\in P(\mathrm{AF})$, let $L_{aa'}\subset\mathbb{P}^2$ be the line containing $\gamma(l_{aa'})$, and let

$$\xi_{aa'}X+\eta_{aa'}Y+\zeta_{aa'}Z=0$$

be the defining equation of $L_{aa'}$. By (5.1), we can put

$$\begin{array}{lll}\xi_{00}=1, & \eta_{00}=0, & \zeta_{00}=0,\\ \xi_{01}=0, & \eta_{01}=1, & \zeta_{01}=0,\\ \xi_{02}=1, & & \zeta_{02}=0,\\ \xi_{10}=0, & \eta_{10}=0, & \zeta_{10}=1,\\ \xi_{11}=1, & \eta_{11}=1, & \zeta_{11}=1,\\ \xi_{12}=1, & \eta_{12}=1, & \\ \xi_{20}=1, & \eta_{20}=0, & \\ \xi_{21}=1, & & \zeta_{21}=1,\\ \xi_{22}=0, & \eta_{22}=1. & \end{array}$$

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Table 25. *Basis* $\{F_1 = 0, F_2 = 0\}$ *of the pencil* PQ_ℓ *and the member* $Q_\ell = \{F_1 + \beta_\ell F_2 = 0\}$

ℓ	F_1	F_2	β_ℓ
00,01,02	$\omega X^2 + \omega Y^2 + Z^2$	$\bar\omega X^2 + \omega Y^2 + XY$	$(\lambda^2 + \bar\omega)/\lambda^2$
00,10,20	$X^2 + \omega Y^2 + Z^2$	$X^2 + Y^2 + ZX$	$(\omega\lambda^2 + 1)/\lambda^2$
00,11,22	$\bar\omega X^2 + \omega Y^2 + Z^2$	$\omega X^2 + \bar\omega Y^2 + XY + ZX$	$(\bar\omega\lambda^2 + \omega)/\lambda^2$
00,12,21	$\omega Y^2 + Z^2$	$\omega Y^2 + \omega XY + ZX$	$(\lambda^2 + \bar\omega)/\lambda^2$
01,10,22	$\omega X^2 + \bar\omega Y^2 + Z^2$	$\omega X^2 + \omega Y^2 + YZ$	$(\omega\lambda + 1)/\lambda$
01,11,21	$\omega X^2 + Z^2$	$X^2 + XY + YZ$	$1/\lambda$
01,12,20	$\omega X^2 + Y^2 + Z^2$	$\bar\omega X^2 + Y^2 + \omega XY + YZ$	$(\lambda + 1)/\lambda$
02,10,21	$\bar\omega X^2 + Y^2 + Z^2$	$\omega X^2 + \omega Y^2 + \omega YZ + ZX$	$(\bar\omega\lambda + \bar\omega)/\lambda$
02,11,20	$\bar\omega Y^2 + Z^2$	$\bar\omega Y^2 + \bar\omega XY + \omega YZ + ZX$	$(\lambda + \bar\omega)/\lambda$
02,12,22	$X^2 + Z^2$	$X^2 + XY + \omega YZ + ZX$	$\bar\omega/\lambda$
10,11,12	$\bar\omega X^2 + Y^2$	$X^2 + YZ + ZX$	λ
20,21,22	$X^2 + Y^2$	$X^2 + XY + \bar\omega YZ + ZX$	$\omega\lambda$

The three lines $L_{aa'}$, $L_{bb'}$ and $L_{cc'}$ are concurrent if $\{aa', bb', cc'\} \in L(\mathrm{AF})$. Hence, we obtain a system of equations

$$\det \begin{bmatrix} \xi_{aa'} & \eta_{aa'} & \zeta_{aa'} \\ \xi_{bb'} & \eta_{bb'} & \zeta_{bb'} \\ \xi_{cc'} & \eta_{cc'} & \zeta_{cc'} \end{bmatrix} = 0 \quad \text{for every } \{aa', bb', cc'\} \in L(\mathrm{AF}). \tag{5.2}$$

A Gröbner basis of the ideal generated by the left-hand side of (5.2) in the polynomial ring $k[\eta_{02}, \eta_{21}, \zeta_{12}, \zeta_{20}, \zeta_{22}]$ is calculated as follows:

$$\langle 1 + \zeta_{22}, 1 + \zeta_{20} + \eta_{21}, 1 + \zeta_{12} + \eta_{21}, \eta_{02} + \eta_{21}, 1 + \eta_{21} + \eta_{21}^2 \rangle.$$

Hence, there are two solutions of this system of equations:

$$\eta_{21} = \eta_{02} = \omega, \qquad \zeta_{12} = \zeta_{20} = \bar\omega, \qquad \zeta_{22} = 1,$$

or

$$\eta_{21} = \eta_{02} = \bar\omega, \qquad \zeta_{12} = \zeta_{20} = \omega, \qquad \zeta_{22} = 1,$$

which are conjugate over $\mathbb{F}_2$. If the latter holds, then we replace γ with $g_0 \circ \gamma \circ T$, where

$$g_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so that we can assume the former always holds. The image of the T-points by γ is therefore equal to the ones given in Table 2, and the lines $L_{aa'}$ are given by equations in Table 22.

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We next determine the coordinates of the image of C-points by γ. The point $\gamma(C(00)) = \gamma(P_{17})$ is on the line $L_{00} = \{X = 0\}$ and is different from $\gamma(P_{20}) = [0, 1, 0]$ and $\gamma(P_{21}) = [0, 0, 1]$. Hence, we can put

$$\gamma(P_{17}) = [0, \lambda, 1], \tag{5.3}$$

where λ is a non-zero parameter. Let ℓ, ℓ_1 and ℓ_2 be three distinct elements of $L(\mathrm{AF})$ that are parallel to each other. The conic curves Q satisfying the following conditions form a pencil PQ_ℓ:

(i) Q contains $\gamma(T(\ell_1))$ and $\gamma(T(\ell_2))$;

(ii) Q is tangent to the lines $L_{aa'}$, $L_{bb'}$, $L_{cc'}$, where $\ell = \{aa', bb', cc'\}$ (recall Definition 3.9 and Remark 3.10).

Using the coordinates of the points $\gamma(T(\ell))$ and the defining equations of the nine lines $L_{aa'}$ determined so far, we can calculate this pencil explicitly. By Proposition 5.1, the conic curve Q_ℓ containing $\gamma(q_\ell)$ is a non-singular member of the pencil PQ_ℓ. Starting from (5.3), we can determine the coordinates of $\gamma(C(aa'))$, and see that they coincide with Table 2. For example, consider $\ell = \{01, 10, 22\} \in L(\mathrm{AF})$. We have

$$\ell_1 = \{02, 11, 20\}, \qquad \ell_2 = \{00, 12, 21\}.$$

The pencil of conic curves passing through the points

$$\gamma(T(\ell_1)) = \gamma(P_9) = [1, \bar{\omega}, \omega], \qquad \gamma(T(\ell_2)) = \gamma(P_{18}) = [0, 1, \omega],$$

and tangent to the lines

$$L_{01} = \{Y = 0\}, \qquad L_{10} = \{Z = 0\}, \qquad L_{22} = \{Y + Z = 0\}$$

is spanned by the two conic curves defined by

$$\omega X^2 + \bar{\omega} Y^2 + Z^2 = 0 \quad \text{and} \quad \omega X^2 + \omega Y^2 + YZ = 0.$$

Because the conic curve Q_ℓ passes through $\gamma(P_{17}) = [0, \lambda, 1]$, it is defined by

$$\lambda(\omega X^2 + \bar{\omega} Y^2 + Z^2) + (\omega\lambda + 1)(\omega X^2 + \omega Y^2 + YZ) = 0.$$

The intersection points of Q_ℓ with the line $L_{12} = \{\omega X + \omega Y + Z = 0\}$ are $\gamma(T(\ell_2)) = \gamma(P_{18}) = [0, 1, \omega]$ and $\gamma(C(12)) = \gamma(P_6)$. Hence, we obtain

$$\gamma(C(12)) = \gamma(P_6) = [\omega\lambda + 1, \lambda + 1, \lambda].$$

See Table 25 for details of the calculation.

Thus, we have proved that γ is equal to γ_λ. Because γ_λ is injective, λ is not among $\{0, 1, \bar{\omega}\}$.

There exists a unique conic curve containing $\gamma_\lambda(q)$ for each quadratic word q of $\boldsymbol{C}_B$, and the defining equations of those conic curves are given in Tables 23 and 24. The smoothness of these curves implies that $\lambda \neq \omega$. Thus, we have proved Claim 5.4. □

Remark 5.5. The polynomial $GB[\lambda]$ is the defining equation of the nodal splitting curve

$$L_{00} \cup L_{01} \cup L_{10} \cup L_{11} \cup Q'_{\ell,\ell'},$$

where $T(\ell) = P_{16}$ and $T(\ell') = P_{19}$ (see Proposition 2.8).

Remark 5.6. Consider the projective plane $(\mathbb{P}^2)^\vee$ of lines on $\mathbb{P}^2$. Let $[U, V, W]$ be the homogeneous coordinates of $(\mathbb{P}^2)^\vee$ dual to the homogeneous coordinates $[X, Y, Z]$ of $\mathbb{P}^2$. Let E_λ be the cubic curve in $(\mathbb{P}^2)^\vee$ defined by

$$\bar{\omega}U^2W + UW^2 + \omega\lambda UV^2 + (\omega\lambda + 1)V^2W + (\omega\lambda + 1)VW^2 + \bar{\omega}\lambda U^2V + (\omega + \lambda)UVW = 0.$$

The points $\gamma_\lambda(C(P(\mathrm{AF})))$ then correspond to the nine flex tangents to E_λ, and the points $\gamma_\lambda(T(L(\mathrm{AF})))$ correspond to the twelve lines containing three flex points of E_λ.

Remark 5.7. When $\lambda = \omega$, the set $\gamma_\lambda(\mathcal{P})$ coincides with $\mathbb{P}^2(\mathbb{F}_4)$, and the point $[GB[\lambda]] \in \mathfrak{M}$ is equal to the Dolgachev–Kondo point.

For each $\sigma \in \mathrm{Aut}(\boldsymbol{C}_B)$, we calculate the unique triple

$$(g_\sigma, t_\sigma, \lambda^\sigma) \in \mathrm{PGL}(3, k(\lambda)) \times \tilde{T} \times k(\lambda)$$

such that $g_\sigma \circ (\gamma_\lambda \circ \sigma) \circ t_\sigma = \gamma_{\lambda^\sigma}$ holds. The map $\sigma \mapsto t_\sigma$ is a homomorphism from $\mathrm{Aut}(\boldsymbol{C}_B)$ to $\tilde{T}$. We put $N_B := \mathrm{Ker}(\mathrm{Aut}(\boldsymbol{C}_B) \to \tilde{T})$.

Corollary 5.8. *The space* $\mathrm{PGL}(3, k) \setminus \mathcal{G}_B$ *has exactly two connected components, each of which is isomorphic to* $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$. *One of them is given, set-theoretically, by*

$$(\mathrm{PGL}(3, k) \setminus \mathcal{G}_B)^+ := \{[\gamma_\alpha] \mid \alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}\},$$

and the other is equal to $((\mathrm{PGL}(3, k)\setminus\mathcal{G}_B)^+)\cdot T$. *The group* N_B *acts on* $(\mathrm{PGL}(3, k)\setminus\mathcal{G}_B)^+$, *and the moduli curve* $\mathfrak{M}_B$ *is equal to the quotient space* $(\mathrm{PGL}(3, k) \setminus \mathcal{G}_B)^+/N_B$.

Consider the natural projection

$$p_B : \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\} \cong (\mathrm{PGL}(3, k) \setminus \mathcal{G}_B)^+ \to \mathfrak{M}_B = (\mathrm{PGL}(3, k) \setminus \mathcal{G}_B)^+/N_B.$$

For $\alpha \in k\setminus\{0, 1, \omega, \bar{\omega}\}$, let $P[\alpha]$ denote the point of $\mathbb{A}^1\setminus\{0, 1, \omega, \bar{\omega}\}$ given by $\lambda = \alpha$. Then $p_B(P[\alpha]) \in \mathfrak{M}_B$ corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$. The following is proved in the same way as Proposition 4.7.

Proposition 5.9. *The fibre* $p_B^{-1}(p_B(P[\alpha]))$ *is equal* $\{P[\varphi]\}$, *where* φ *runs through the set* Γ_B *in Theorem 1.7 with* λ *replaced by* α. *The group* $\mathrm{Aut}(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$ *is equal to the subgroup of* $\mathrm{PGL}(3, k)$ *generated by the elements in (1.2).*

Corollary 5.10. *We have* $\mathfrak{M}_B = \mathrm{Spec}\, k[J_B, 1/J_B]$, *where*

$$J_B = \frac{(\lambda + \omega)^{12}}{\lambda^3(\lambda + 1)^3(\lambda + \bar{\omega})^3}.$$

The morphism p_B *is an étale Galois covering with Galois group* Γ_B, *which is isomorphic to the alternating group* $\mathfrak{A}_4$.

Indeed the group Γ_B acts on the set $\{0, 1, \bar{\omega}, \infty\}$ as $\mathfrak{A}_4$.

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Table 26. *Linear words of $\boldsymbol{C}_C$*

$l_1 := \{17, 18, 19, 20, 21\}$
$l_2 := \{13, 14, 15, 16, 21\}$
$l_3 := \{9, 10, 11, 12, 21\}$
$l_4 := \{5, 6, 7, 8, 21\}$
$l_5 := \{1, 2, 3, 4, 21\}$

Table 27. *Bijection ϕ from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4)$*

$\phi(P_1) = [1, 1, 0]$	$\phi(P_8) = [1, \bar{\omega}, \bar{\omega}]$	$\phi(P_{15}) = [1, 0, \bar{\omega}]$
$\phi(P_2) = [1, 1, 1]$	$\phi(P_9) = [1, \omega, \bar{\omega}]$	$\phi(P_{16}) = [1, 0, \omega]$
$\phi(P_3) = [1, 1, \omega]$	$\phi(P_{10}) = [1, \omega, 0]$	$\phi(P_{17}) = [0, 1, 1]$
$\phi(P_4) = [1, 1, \bar{\omega}]$	$\phi(P_{11}) = [1, \omega, \omega]$	$\phi(P_{18}) = [0, 1, 0]$
$\phi(P_5) = [1, \bar{\omega}, \omega]$	$\phi(P_{12}) = [1, \omega, 1]$	$\phi(P_{19}) = [0, 1, \bar{\omega}]$
$\phi(P_6) = [1, \bar{\omega}, 0]$	$\phi(P_{13}) = [1, 0, 1]$	$\phi(P_{20}) = [0, 1, \omega]$
$\phi(P_7) = [1, \bar{\omega}, 1]$	$\phi(P_{14}) = [1, 0, 0]$	$\phi(P_{21}) = [0, 0, 1]$

6. The moduli curve corresponding to the code $\boldsymbol{C}_C$

In this section, we prove Theorem 1.8.

The linear words of $\boldsymbol{C}_C$ are listed in Table 26.

The list of quadratic words in $\boldsymbol{C}_C$ is omitted. The point P_{21} is special because every linear word contains it. The following proposition can be checked directly by computer.

Proposition 6.1. *Let $\phi : \mathcal{P} \xrightarrow{\sim} \mathbb{P}^2(\mathbb{F}_4)$ be the bijection given in Table 27.*

(1) *The linear words of $\boldsymbol{C}_C$ are precisely the words $\phi^{-1}(\Lambda(\mathbb{F}_4))$, where Λ are $\mathbb{F}_4$-rational lines passing through*

$$O := [0, 0, 1] = \phi(P_{21}).$$

(2) *The quadratic words of $\boldsymbol{C}_C$ are precisely the words $\phi^{-1}(\Lambda(\mathbb{F}_4) + \Lambda'(\mathbb{F}_4))$, where Λ and Λ' are distinct $\mathbb{F}_4$-rational lines that do not pass through O.*

Note that ϕ embeds $\boldsymbol{C}_C$ into the Dolgachev–Kondo code $\boldsymbol{C}_{\mathrm{DK}}$.

Corollary 6.2. *For each quadratic word q in $\boldsymbol{C}_C$, there exists a unique linear word l in $\boldsymbol{C}_C$ such that $q \cap l = \emptyset$.*

From Remark 3.11, we obtain the following corollary.

Corollary 6.3. *Let l and l' be distinct linear words of $\boldsymbol{C}_C$, and let $A_1, A_2 \in l$ (respectively, $B_1, B_2 \in l'$) be distinct points not equal to P_{21}. Then there are exactly two quadratic words q and q' in $\boldsymbol{C}_C$ containing the points $\{A_1, A_2, B_1, B_2\}$. Moreover, if a linear word $l'' \in \boldsymbol{C}_C$ is disjoint from q, then l'' is also disjoint from q'.*

https://doi.org/10.1017/S0013091504000562 Published online by Cambridge University Press

For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_4$, we denote by $\Lambda[\alpha_1\alpha_2\alpha_3]$ the $\mathbb{F}_4$-rational line defined by

$$\alpha_1 X + \alpha_2 Y + \alpha_3 Z = 0,$$

and by $q[\alpha_1\alpha_2\alpha_3, \beta_1\beta_2\beta_3] \in \boldsymbol{C}_C$ the quadratic word

$$\phi^{-1}(\Lambda[\alpha_1\alpha_2\alpha_3](\mathbb{F}_4) + \Lambda[\beta_1\beta_2\beta_3](\mathbb{F}_4)).$$

We put

$$\mathrm{LG}' := \{g \in \mathrm{PGL}(3, \mathbb{F}_4) \mid g(O) = O\}.$$

The automorphism group $\mathrm{Aut}(\boldsymbol{C}_C)$ of the code $\boldsymbol{C}_C$ contains a subgroup

$$\mathrm{LG} := \phi^{-1} \circ \mathrm{LG}' \circ \phi.$$

The order of LG is 2880. The group $\mathrm{Aut}(\boldsymbol{C}_C)$ also contains the permutation

$$T := (P_3P_4)(P_5P_9)(P_6P_{10})(P_7P_{12})(P_8P_{11})(P_{15}P_{16})(P_{19}P_{20})$$

of $\mathcal{P}$ that corresponds, via the bijection ϕ, to the action of the conjugation $\omega \mapsto \bar{\omega}$ over $\mathbb{F}_2$ on $\mathbb{P}^2(\mathbb{F}_4)$. It can be checked easily by computer that the following permutation is also contained in $\mathrm{Aut}(\boldsymbol{C}_C)$:

$$S := (P_1P_3)(P_2P_4)(P_5P_7)(P_6P_8)(P_9P_{11})(P_{10}P_{12}).$$

The automorphisms T and S of $\boldsymbol{C}_C$ generate a subgroup isomorphic to the dihedral group of order 8 in $\mathrm{Aut}(\boldsymbol{C}_C)$. An ordered quartet

$$(R_1, R_2, R'_1, R'_2)$$

of points in $\mathcal{P} \setminus \{P_{21}\}$ is called a *marking quartet* if P_{21}, R_1, R_2 are in a linear word, and P_{21}, R'_1, R'_2 are in another linear word. There are 2880 marking quartets, and the action of LG on the set of marking quartets is simply transitive.

Proposition 6.4. *The group* $\mathrm{Aut}(\boldsymbol{C}_C)$ *is generated by* LG, T *and* S, *and the order of* $\mathrm{Aut}(\boldsymbol{C}_C)$ *is* 23 040.

Proof. Let σ be an arbitrary element of $\mathrm{Aut}(\boldsymbol{C}_C)$. Because $(P_{17}, P_{18}, P_{13}, P_{14})$ and $(\sigma(P_{17}), \sigma(P_{18}), \sigma(P_{13}), \sigma(P_{14}))$ are marking quartets, there exists an element $\tau \in \mathrm{LG}$ such that $\tau\sigma(P_i) = P_i$ for $i = 13, 14, 17, 18, 21$. Because $\tau\sigma(l_1) = l_1$ and $\tau\sigma(l_2) = l_2$, we have

$$\{\tau\sigma(P_{19}), \tau\sigma(P_{20})\} = \{P_{19}, P_{20}\} \quad \text{and} \quad \{\tau\sigma(P_{15}), \tau\sigma(P_{16})\} = \{P_{15}, P_{16}\}.$$

If $\tau\sigma(P_{19}) = P_{20}$, then we replace τ by $T\tau$. Therefore, modulo the subgroup generated by LG and T, we can assume that σ has the following properties:

(σ (i)) σ fixes each of the seven points $P_{13}, P_{14}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21}$,

(σ (ii)) $\{\sigma(P_{15}), \sigma(P_{16})\} = \{P_{15}, P_{16}\}$.

Table 28. *List of the triples* $(f, \{q, q'\}, l_\nu)$

f	$[q, q']$	l_ν
$\{13,14,17,18\}$	$q[101,011] = \{7,8,11,12,13,14,17,18\}$ $q[111,001] = \{5,6,9,10,13,14,17,18\}$	l_5
$\{13,14,17,19\}$	$q[1\bar{\omega}1,011] = \{2,3,10,11,13,14,17,19\}$ $q[111,0\bar{\omega}1] = \{1,4,9,12,13,14,17,19\}$	l_4
$\{13,14,17,20\}$	$q[1\omega 1,011] = \{2,4,6,8,13,14,17,20\}$ $q[111,0\omega 1] = \{1,3,5,7,13,14,17,20\}$	l_3
$\{13,14,18,19\}$	$q[101,0\bar{\omega}1] = \{2,4,5,7,13,14,18,19\}$ $q[1\bar{\omega}1,001] = \{1,3,6,8,13,14,18,19\}$	l_3
$\{13,14,18,20\}$	$q[101,0\omega 1] = \{2,3,9,12,13,14,18,20\}$ $q[1\omega 1,001] = \{1,4,10,11,13,14,18,20\}$	l_4
$\{13,14,19,20\}$	$q[1\bar{\omega}1,0\omega 1] = \{7,8,9,10,13,14,19,20\}$ $q[1\omega 1,0\bar{\omega}1] = \{5,6,11,12,13,14,19,20\}$	l_5

Consider, for example, a set of four points $\{P_{13}, P_{14}, P_{17}, P_{18}\}$, each of which is fixed by σ. The two quadratic words containing them are

$$q[101,011] = \{2,7,12,13,18\} + \{2,8,11,14,17\} = \{7,8,11,12,13,14,17,18\},$$
$$q[111,001] = \{1,5,9,13,17\} + \{1,6,10,14,18\} = \{5,6,9,10,13,14,17,18\}.$$

Both of $q[101,011]$ and $q[111,001]$ are disjoint from l_5. By Corollary 6.2, we have $\sigma(l_5) = l_5$. Considering other sets of four points fixed by σ, we can show that $\sigma(l_4) = l_4$ and $\sigma(l_3) = l_3$. In Table 28, we list the triples $(f, \{q, q'\}, l_\nu)$, where f is a set of four points pointwise fixed by σ, $\{q, q'\}$ is the pair of quadratic words containing f, and l_ν is the linear word disjoint from both of q and q'.

Therefore, we have the following:

(σ (iii)) σ leaves each of the sets

$$\{P_1, P_2, P_3, P_4\}, \qquad \{P_5, P_6, P_7, P_8\}, \qquad \{P_9, P_{10}, P_{11}, P_{12}\}$$

invariant.

Let us consider the quadratic words $q_1 := q[101,011]$ and $q_2 := q[111,001]$ again. Since

$$\{\sigma(q_1 \cap l_4), \sigma(q_2 \cap l_4)\} = \{q_1 \cap l_4, q_2 \cap l_4\},$$

the action of σ on $\{P_5, P_6, P_7, P_8\}$ preserves the decomposition

$$\{P_5, P_6, P_7, P_8\} = \{P_5, P_6\} \cup \{P_7, P_8\};$$

that is, $\{\sigma(P_5), \sigma(P_6)\}$ is either $\{P_5, P_6\}$ or $\{P_7, P_8\}$. By the same argument applied to the pairs $\{q, q'\}$ of quadratic words in Table 28, we see the following:

(σ (iv)) σ preserves the decompositions

$$\{P_1, P_2, P_3, P_4\} = \{P_1, P_4\} \cup \{P_2, P_3\} = \{P_1, P_3\} \cup \{P_2, P_4\},$$
$$\{P_5, P_6, P_7, P_8\} = \{P_5, P_6\} \cup \{P_7, P_8\} = \{P_5, P_7\} \cup \{P_6, P_8\},$$

and

$$\{P_9, P_{10}, P_{11}, P_{12}\} = \{P_9, P_{10}\} \cup \{P_{11}, P_{12}\} = \{P_9, P_{12}\} \cup \{P_{10}, P_{11}\}.$$

The two quadratic words containing $\{P_{13}, P_{16}, P_{17}, P_{18}\}$ are

$$q[\omega 11, 101] = \{2, 4, 10, 12, 13, 16, 17, 18\}$$

and

$$q[111, \omega 01] = \{1, 3, 9, 11, 13, 16, 17, 18\},$$

both of which are disjoint from l_4. On the other hand, the two quadratic words containing $\{P_{13}, P_{15}, P_{17}, P_{18}\}$ are

$$q[\bar{\omega}11, 101] = \{2, 3, 6, 7, 13, 15, 17, 18\}$$

and

$$q[111, \bar{\omega}01] = \{1, 4, 5, 8, 13, 15, 17, 18\},$$

both of which are disjoint from l_3. Since σ fixes each of l_4 and l_3, we see that the property (σ (ii)) of σ can be strengthened to the following:

(σ (ii))$'$ $\sigma(P_{15}) = P_{15}$, $\sigma(P_{16}) = P_{16}$.

Using computer, we can easily list all $4^3 = 64$ permutations σ satisfying (σ (i)), (σ (ii))$'$, (σ (iii)) and (σ (iv)). We can check that exactly four of them id, S,

$$(ST)^2 = (P_1P_2)(P_3P_4)(P_5P_6)(P_7P_8)(P_9P_{10})(P_{11}P_{12})$$

and

$$(ST)^2S = (P_1P_4)(P_2P_3)(P_5P_8)(P_6P_7)(P_9P_{12})(P_{10}P_{11})$$

preserve the set of quadratic words in $\boldsymbol{C}_C$. Hence, by Proposition 3.5, $\mathrm{Aut}(\boldsymbol{C}_C)$ is generated by LG, S and T. It can be checked by computer that the order of $\mathrm{Aut}(\boldsymbol{C}_C)$ is 23 040. □

Let λ be a parameter of the affine line $\mathbb{A}^1$ and let $\gamma_\lambda : \mathcal{P} \to \mathbb{P}^2$ be the map given in Table 3. When $\lambda = 0$, the map γ_λ is equal to ϕ. Let $\tilde{T}$ denote the subgroup of $\mathrm{Aut}(\boldsymbol{C}_C)$ generated by the involution T.

Proposition 6.5. *The map* $\lambda \mapsto \gamma_\lambda$ *induces an isomorphism from* $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ *to* $\mathrm{PGL}(3, k) \setminus \mathcal{G}_C/\tilde{T}$.

Table 29. *Parametric presentation of* γ

$\gamma(P_1)=[1,t_5,s_{5,1}]$	$\gamma(P_8)=[1,t_4,s_{4,4}]$	$\gamma(P_{15})=[\alpha_2,0,1]$
$\gamma(P_2)=[1,t_5,s_{5,2}]$	$\gamma(P_9)=[1,t_3,s_{3,1}]$	$\gamma(P_{16})=[\beta_2,0,1]$
$\gamma(P_3)=[1,t_5,s_{5,3}]$	$\gamma(P_{10})=[1,t_3,s_{3,2}]$	$\gamma(P_{17})=[0,1,1]$
$\gamma(P_4)=[1,t_5,s_{5,4}]$	$\gamma(P_{11})=[1,t_3,s_{3,3}]$	$\gamma(P_{18})=[0,1,0]$
$\gamma(P_5)=[1,t_4,s_{4,1}]$	$\gamma(P_{12})=[1,t_3,s_{3,4}]$	$\gamma(P_{19})=[0,\alpha_1,1]$
$\gamma(P_6)=[1,t_4,s_{4,2}]$	$\gamma(P_{13})=[1,0,1]$	$\gamma(P_{20})=[0,\beta_1,1]$
$\gamma(P_7)=[1,t_4,s_{4,3}]$	$\gamma(P_{14})=[1,0,0]$	$\gamma(P_{21})=[0,0,1]$

Proof. First note that γ_λ is injective for every λ.

Claim 6.6. *Let γ' be an arbitrary element of $\mathcal{G}_C$. Then there exists a unique triple*

$$(g,t,\lambda)\in \mathrm{PGL}(3,k)\times \tilde{T}\times (k\setminus\{0,1,\omega,\bar{\omega}\})$$

such that $g\circ\gamma'\circ t=\gamma_\lambda$.

Since $\gamma'(P_{21}), \gamma'(P_{13}), \gamma'(P_{14})$ are on a line, and $\gamma'(P_{21}), \gamma'(P_{17}), \gamma'(P_{18})$ are on another line, there exists a unique $g\in\mathrm{PGL}(3,k)$ such that $\gamma := g\circ\gamma'$ satisfies

$$\gamma(P_{21})=[0,0,1]=O,\qquad \gamma(P_{17})=[0,1,1],\qquad \gamma(P_{18})=[0,1,0],$$
$$\gamma(P_{13})=[1,0,1],\qquad \gamma(P_{14})=[1,0,0].$$

The X-coordinate of $\gamma(P_i)$ is not 0 for $i=1,\dots,16$, because otherwise $\gamma(P_i)$, $\gamma(P_{17})$ and $\gamma(P_{21})$ would be collinear, and hence there would exist a linear word of $\boldsymbol{C}_C$ containing $\{P_i,P_{17},P_{21}\}$ by Proposition 2.9. Therefore, there exist parameters α_1, α_2, β_1, β_2, t_i, s_{ij}, $i=3,4,5$, $j=1,\dots,4$, such that γ is given by Table 29.

The lines L_ν containing the points $\gamma(l_\nu)$ are defined by

$$\begin{aligned} L_1&=\{X=0\},\\ L_2&=\{Y=0\},\\ L_3&=\{Y=t_3X\},\\ L_4&=\{Y=t_4X\},\\ L_5&=\{Y=t_5X\}.\end{aligned}$$

Claim 6.7. $t_5=1$.

Consider the quadratic word

$$q_1:=\{7,8,11,12,13,14,17,18\}=\{2,7,12,13,18\}+\{2,8,11,14,17\},$$

which passes through the four points P_{13}, P_{14}, P_{17}, P_{18}, and is disjoint from the linear word l_5. The conic curves containing the points $\gamma(P_{13})=[1,0,1]$, $\gamma(P_{14})=[1,0,0]$, $\gamma(P_{17})=[0,1,1]$ and $\gamma(P_{18})=[0,1,0]$ form a pencil

$$\sigma Z(X+Y+Z)+XY=0,\quad \sigma\in\mathbb{P}^1.$$

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The conic curve $Q_1 \subset \mathbb{P}^2$ containing $\gamma(q_1)$ is a member of this pencil. Since Q_1 is non-singular, the value of the parameter σ corresponding to Q_1 is not 0 nor ∞. Since Q_1 is tangent to the line $L_5 = \{Y = t_5 X\}$, we have $\sigma(1 + t_5) = 0$. Hence, $t_5 = 1$.

From the quadratic words that

(i) contain exactly one of $\{P_{17}, P_{18}\}$,

(ii) contain exactly one of $\{P_{13}, P_{14}\}$,

(iii) are disjoint from l_5,

we obtain the following relations.

Claim 6.8.

$$\alpha_1 = \alpha_2 \,(=: \alpha), \qquad \beta_1 = \beta_2 \,(=: \beta), \qquad \alpha + \beta = \alpha\beta.$$

Consider, for example, the quadratic word

$$q_2 := \{6, 8, 9, 11, 14, 16, 17, 19\} = \{2, 6, 9, 16, 19\} + \{2, 8, 11, 14, 17\}.$$

Since the conic curve Q_2 containing $\gamma(q_2)$ passes through the points $\gamma(P_{14}) = [1, 0, 0]$ and $\gamma(P_{17}) = [0, 1, 1]$ and is tangent to $L_5 = \{X = Y\}$, it is a member of the web

$$\sigma_1(Y^2 + Z^2) + \sigma_2 XY + \sigma_3(Y^2 + YZ + ZX) = 0, \quad [\sigma_1, \sigma_2, \sigma_3] \in \mathbb{P}^2,$$

of conic curves. Since $\gamma(P_{16}) = [\beta_2, 0, 1] \in Q_2$, we have $\beta_2 = \sigma_1/\sigma_3$. Since $\gamma(P_{19}) = [0, \alpha_1, 1] \in Q_2$ and $\alpha_1 \neq 1$, we have $\alpha_1 = \sigma_1/(\sigma_1 + \sigma_3)$. Therefore, we obtain a relation $\alpha_1 + \beta_2 + \alpha_1\beta_2 = 0$.

From the quadratic words that contain exactly three of P_{17}, P_{18}, P_{13}, P_{14}, we obtain the following relations.

Claim 6.9.

$$\begin{aligned} \alpha_1 + t_3 &= 0, & \beta_1 + t_4 &= 0; \\ 1 + \alpha_2 t_4 &= 0, & 1 + \beta_2 t_3 &= 0; \\ 1 + \alpha_2 + \alpha_2 t_3 &= 0, & 1 + \beta_2 + \beta_2 t_4 &= 0; \\ \alpha_1 + t_4 + \alpha_1 t_4 &= 0, & \beta_1 + t_3 + \beta_1 t_3 &= 0. \end{aligned}$$

Consider, for example, the quadratic word

$$q_3 := \{2, 4, 10, 12, 13, 16, 17, 18\} = \{4, 7, 10, 16, 17\} + \{2, 7, 12, 13, 18\}.$$

Since the conic curve Q_3 containing $\gamma(q_3)$ passes through the points $\gamma(P_{13}) = [1, 0, 1]$, $\gamma(P_{17}) = [0, 1, 1]$ and $\gamma(P_{18}) = [0, 1, 0]$, it is a member of the web

$$\sigma_1(X^2 + Z^2 + YZ) + \sigma_2(X^2 + ZX) + \sigma_3 XY = 0, \quad [\sigma_1, \sigma_2, \sigma_3] \in \mathbb{P}^2,$$

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of conic curves. Since $\gamma(P_{16}) = [\beta_2, 0, 1]$ is contained in Q_3, we obtain ${\beta_2}^2(\sigma_1 + \sigma_2) + \beta_2\sigma_2 + \sigma_1 = 0$. Since Q_3 is tangent to the line $L_4 = \{Y = t_4X\}$, we have $t_4\sigma_1 + \sigma_2 = 0$. Combining these two relations and $\beta_2 \neq 1$, we obtain a relation $1 + \beta_2 + \beta_2 t_4 = 0$.

Combining Claims 6.7–6.9, we obtain the following two possibilities for the parameters:

$$\begin{array}{lllll} \alpha_1 = \alpha_2 = \omega, & \beta_1 = \beta_2 = \bar{\omega}, & t_3 = \omega, & t_4 = \bar{\omega}, & t_5 = 1; \\ \alpha_1 = \alpha_2 = \bar{\omega}, & \beta_1 = \beta_2 = \omega, & t_3 = \bar{\omega}, & t_4 = \omega, & t_5 = 1. \end{array}$$

If the latter holds, then we replace γ by $\gamma \circ T$ so that we assume that the former always holds.

Next we put

$$P_1 = [1, 1, \lambda],$$

where $\lambda = s_{5,1}$ is a parameter. Using quadratic words that

(i) contain exactly four points among $l_1 \cup l_2$,

(ii) are not disjoint from l_5,

we obtain the following claim.

Claim 6.10.

$$\begin{array}{llll} s_{5,1} = \lambda, & s_{5,2} = \lambda + 1, & s_{5,3} = \lambda + \omega, & s_{5,4} = \lambda + \bar{\omega}, \\ s_{4,1} = \omega\lambda + \omega, & s_{4,2} = \omega\lambda, & s_{4,3} = \omega\lambda + 1, & s_{4,4} = \omega\lambda + \bar{\omega}, \\ s_{3,1} = \bar{\omega}\lambda + \bar{\omega}, & s_{3,2} = \bar{\omega}\lambda, & s_{3,3} = \bar{\omega}\lambda + \omega, & s_{3,4} = \bar{\omega}\lambda + 1. \end{array}$$

Consider, for example, the quadratic word

$$q_4 := \{1, 2, 11, 12, 14, 16, 17, 20\} = \{1, 8, 12, 16, 20\} + \{2, 8, 11, 14, 17\},$$

which is disjoint from l_4. Because there exists a conic curve Q_4 that contains $\gamma(q_4)$ and is tangent to the line $L_4 = \{Y = \bar{\omega}X\}$, the following matrix $\tilde{M}$ is of rank less than 6:

$$\tilde{M} := \begin{bmatrix} 1 & 1 & \lambda^2 & 1 & \lambda & \lambda \\ 1 & 1 & {s_{5,2}}^2 & 1 & s_{5,2} & s_{5,2} \\ 1 & \bar{\omega} & {s_{3,3}}^2 & \omega & \omega s_{3,3} & s_{3,3} \\ 1 & \bar{\omega} & {s_{3,4}}^2 & \omega & \omega s_{3,4} & s_{3,4} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \omega & 0 & 1 & 0 & 0 & \bar{\omega} \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & \omega & 1 & 0 & \bar{\omega} & 0 \\ 0 & 0 & 0 & 0 & \bar{\omega} & 1 \end{bmatrix}.$$

Indeed, if the equation

$$a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ + a_6ZX = 0$$

defines a conic curve containing $\gamma(q_4)$ and tangent to L_4, then $\boldsymbol{a} = [a_1, a_2, \dots, a_6]^{\mathrm{T}}$ is a non-zero solution of $\tilde{M}\boldsymbol{x} = \mathbf{1}$. (The condition $\bar{\omega}a_5 + a_6 = 0$ is equivalent to the condition that the conic curve is tangent to L_4.) Let $\tilde{M}[i_1, \dots, i_6]$ denote the submatrix of $\tilde{M}$ consisting of i_jth rows of $\tilde{M}$. Because

$$\det \tilde{M}[1, 2, 5, 7, 8, 9] = (s_{5,2} + \lambda)(s_{5,2} + \lambda + 1)$$

and $s_{5,2} \neq s_{5,1} = \lambda$, we obtain $s_{5,2} = \lambda + 1$. Because

$$\det \tilde{M}[1, 3, 5, 7, 8, 9] = (s_{3,3} + \bar{\omega}\lambda + 1)(s_{3,3} + \bar{\omega}\lambda + \omega),$$

we obtain

$$s_{3,3} = \bar{\omega}\lambda + 1 \quad \text{or} \quad \bar{\omega}\lambda + \omega.$$

Continuing the same calculations, we get the relations in Claim 6.10.

Thus, we have proved Claim 6.6.

Conversely, suppose that $\lambda \in k \backslash \{0, 1, \omega, \bar{\omega}\}$ is given. Then $\gamma_\lambda(\mathcal{P})$ is equal to $Z(\mathrm{d}GC[\lambda])$, where $GC[\lambda]$ is given in Theorem 1.8. Moreover, for every linear word l of $\boldsymbol{C}_C$, there exists a line containing $\gamma_\lambda(l)$, and for every quadratic word q of $\boldsymbol{C}_C$, there exists a unique conic curve containing $\gamma_\lambda(q)$. The defining equations of the 120 conic curves are omitted. These conic curves are non-singular because $\lambda \notin \{0, 1, \omega, \bar{\omega}\}$. Hence, γ_λ is in $\mathcal{G}_B$ by Proposition 3.6. □

Remark 6.11. When $\lambda \in \{0, 1, \omega, \bar{\omega}\}$, the set $\gamma_\lambda(\mathcal{P})$ coincides with $\mathbb{P}^2(\mathbb{F}_4)$, and the point $[GC[\lambda]] \in \mathfrak{M}$ is equal to the Dolgachev–Kondo point.

For each $\sigma \in \mathrm{Aut}(\boldsymbol{C}_C)$, we calculate the unique triple

$$(g_\sigma, t_\sigma, \lambda^\sigma) \in \mathrm{PGL}(3, k(\lambda)) \times \tilde{T} \times k(\lambda)$$

such that $g_\sigma \circ (\gamma_\lambda \circ \sigma) \circ t_\sigma = \gamma_{\lambda^\sigma}$ holds. The map $\sigma \mapsto t_\sigma$ is a homomorphism from $\mathrm{Aut}(\boldsymbol{C}_C)$ to $\tilde{T}$. We put $N_C := \mathrm{Ker}(\mathrm{Aut}(\boldsymbol{C}_C) \to \tilde{T})$.

Corollary 6.12. *The space* $\mathrm{PGL}(3, k) \setminus \mathcal{G}_C$ *has exactly two connected components*

$$(\mathrm{PGL}(3, k) \setminus \mathcal{G}_C)^+ := \{[\gamma_\alpha] \mid \alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}\}$$

and

$$((\mathrm{PGL}(3, k) \setminus \mathcal{G}_C)^+) \cdot T,$$

each of which is isomorphic to $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$. *The group* N_C *acts on* $(\mathrm{PGL}(3, k) \setminus \mathcal{G}_C)^+$, *and the moduli curve* $\mathfrak{M}_C$ *is equal to the quotient space* $(\mathrm{PGL}(3, k) \setminus \mathcal{G}_C)^+/N_C$.

Consider the natural projection

$$p_C : \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\} \cong (\mathrm{PGL}(3, k) \setminus \mathcal{G}_C)^+ \to \mathfrak{M}_C = (\mathrm{PGL}(3, k) \setminus \mathcal{G}_C)^+/N_C.$$

For $\alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}$, let $P[\alpha]$ denote the point of $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ given by $\lambda = \alpha$. Then $p_C(P[\alpha]) \in \mathfrak{M}_C$ corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_{GC[\alpha]}, \mathcal{L}_{GC[\alpha]})$.

Proposition 6.13. *We have*

$$p_C^{-1}(p_C(P[\alpha])) = \{P[u\alpha + v] \mid u \in \mathbb{F}_4^{\times},\ v \in \mathbb{F}_4\}.$$

The group $\mathrm{Aut}(X_{GC[\alpha]}, \mathcal{L}_{GC[\alpha]})$ *is equal to the subgroup (1.3) of* $\mathrm{PGL}(3,k)$*.*

Corollary 6.14. *We have* $\mathfrak{M}_C = \operatorname{Spec} k[J_C, 1/J_C]$*, where* $J_C := (\lambda^4 + \lambda)^3$*. The morphism* p_C *is an étale Galois covering with Galois group* Γ_C*.*

7. Cremona transformations by quintic curves

7.1. Preliminaries

Let Σ_1 and Σ_2 be disjoint sets of reduced points of $\mathbb{P}^2$ with $|\Sigma_1| = n_1$ and $|\Sigma_2| = n_2$, and let $\mathcal{I}_{\Sigma_1} \subset \mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{I}_{\Sigma_2} \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaves defining Σ_1 and Σ_2. We define $\tilde{\Sigma}$ to be the zero-dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf

$$\mathcal{I}_{\tilde{\Sigma}} := \mathcal{I}_{\Sigma_1}\mathcal{I}_{\Sigma_2}^2.$$

The length of $\mathcal{O}_{\tilde{\Sigma}}$ is $n_1 + 3n_2$. Let d be a positive integer. The linear system $|\mathcal{I}_{\tilde{\Sigma}}(d)|$ consists of plane curves of degree d that pass through the points of $\Sigma_1 \cup \Sigma_2$ and are singular at each point of Σ_2.

Proposition 7.1. *Suppose that the linear system* $|\mathcal{I}_{\tilde{\Sigma}}(d)|$ *is of dimension greater than or equal to* 1 *and has no fixed components. If*

$$\dim |\mathcal{I}_{\tilde{\Sigma}}(d)| > \tfrac{1}{2}(d+2)(d+1) - (n_1 + 3n_2) - 1, \tag{7.1}$$

then there exists a projective plane curve of degree $d-3$ *that passes through all the points of* Σ_2*.*

Corollary 7.2. *Suppose that the linear system* $|\mathcal{I}_{\tilde{\Sigma}}(d)|$ *is of dimension greater than or equal to* 1 *and has no fixed components. If* $d \leqslant 3$ *and* $n_2 > 0$*, then the dimension of the linear system* $|\mathcal{I}_{\tilde{\Sigma}}(d)|$ *is equal to* $\frac{1}{2}(d+2)(d+1) - (n_1 + 3n_2) - 1$*.*

Proof. We follow the argument in [**6**, pp. 712–714]. From the exact sequence

$$0 \to \mathcal{I}_{\tilde{\Sigma}}(d) \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_{\tilde{\Sigma}}(d) \to 0,$$

we obtain

$$h^0(\mathbb{P}^2, \mathcal{I}_{\tilde{\Sigma}}(d)) = \tfrac{1}{2}(d+2)(d+1) - (n_1 + 3n_2) + h^1(\mathbb{P}^2, \mathcal{I}_{\tilde{\Sigma}}(d)). \tag{7.2}$$

Let $\beta : S \to \mathbb{P}^2$ denote the blowing up of $\mathbb{P}^2$ at the points of $\Sigma_1 \cup \Sigma_2$. We put

$$\Delta_1 := \beta^{-1}(\Sigma_1), \qquad \Delta_2 := \beta^{-1}(\Sigma_2),$$

both of which are considered to be reduced divisors. Let $H \subset S$ be the pull-back of a general line on $\mathbb{P}^2$. We put

$$L := \beta^*\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_S(-\Delta_1 - 2\Delta_2) = \mathcal{O}_S(dH - \Delta_1 - 2\Delta_2).$$

Because $K_S = -3H + \Delta_1 + \Delta_2$, we have

$$L^2 = d^2 - n_1 - 4n_2, \qquad LK_S = -3d + n_1 + 2n_2.$$

The complete linear system $|H| = |\beta^* \mathcal{O}_{\mathbb{P}^2}(1)|$ on S is fixed-component free. Since the intersection number $H(K_S - L) = -d - 3$ is negative, we have

$$h^2(S, L) = h^0(S, K_S - L) = 0.$$

By the Riemann–Roch theorem, we obtain

$$h^0(S, L) = \tfrac{1}{2}(d+2)(d+1) - (n_1 + 3n_2) + h^1(S, L). \tag{7.3}$$

There exists a canonical isomorphism

$$|\mathcal{I}_{\tilde{\Sigma}}(d)| \cong |L| \tag{7.4}$$

that maps a member C of $|\mathcal{I}_{\tilde{\Sigma}}(d)|$ to the member $\beta^* C - \Delta_1 - 2\Delta_2$ of $|L|$. From (7.2)–(7.4), we obtain

$$h^1(\mathbb{P}^2, \mathcal{I}_{\tilde{\Sigma}}(d)) = h^1(S, L). \tag{7.5}$$

Using the assumption (7.1) and the equalities (7.2) and (7.5), we obtain

$$h^1(S, L) > 0. \tag{7.6}$$

Since $|\mathcal{I}_{\tilde{\Sigma}}(d)|$ is of dimension greater than or equal to 1 and has no fixed components, we obtain by the isomorphism (7.4) global sections s and s' of L such that the subscheme $R = \{s = s' = 0\}$ of S is of dimension 0. Let $\mathcal{I}_R \subset \mathcal{O}_S$ be the ideal sheaf defining R. From the Koszul complex

$$0 \to \mathcal{O}_S(K_S - L) \xrightarrow{(s,s')} \mathcal{O}_S(K_S) \oplus \mathcal{O}_S(K_S) \xrightarrow{(-s',s)^{\mathrm{T}}} \mathcal{I}_R(K_S + L) \to 0$$

and $h^0(S, \mathcal{O}_S(K_S)) = h^1(S, \mathcal{O}_S(K_S)) = 0$, we obtain

$$h^1(S, L) = h^1(S, \mathcal{O}_S(K_S - L)) = h^0(\mathcal{I}_R(K_S + L)).$$

From (7.6), we see that the linear system $|\mathcal{I}_R(K_S + L)|$ is non-empty. Since $K_S + L = \beta^* \mathcal{O}_{\mathbb{P}^2}(d-3) \otimes \mathcal{O}_S(-\Delta_2)$, a member of $|\mathcal{I}_R(K_S + L)|$ is mapped by β to a projective plane curve of degree $d - 3$ that passes through the points of Σ_2. □

Definition 7.3. Let F be an effective divisor of $\mathbb{P}^2$. We put

$$\Sigma_1' := (\Sigma_1 \setminus (\Sigma_1 \cap F)) \cup (\Sigma_2 \cap F^0), \qquad \Sigma_2' := \Sigma_2 \setminus (\Sigma_2 \cap F),$$

where F^0 is the locus of all $p \in \mathrm{Supp}(F)$ at which F is reduced and non-singular. We then define $\tilde{\Sigma} \setminus F$ to be the zero-dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf

$$\mathcal{I}_{\tilde{\Sigma} \setminus F} := \mathcal{I}_{\Sigma_1'} \mathcal{I}_{\Sigma_2'}^2.$$

If F is a fixed component of $|\mathcal{I}_{\tilde{\Sigma}}(d)|$, then $C \mapsto C - F$ gives an isomorphism

$$|\mathcal{I}_{\tilde{\Sigma}}(d)| \cong |\mathcal{I}_{\tilde{\Sigma} \setminus F}(d - \deg F)|$$

of linear systems. By the definition, we have

$$\tilde{\Sigma} \setminus (F_1 + F_2) = (\tilde{\Sigma} \setminus F_1) \setminus F_2 \tag{7.7}$$

for any effective (not necessarily distinct) divisors F_1 and F_2 of $\mathbb{P}^2$.

7.2. A homaloidal system of quintic curves

Let $\Sigma = \{p_1, \dots, p_6\}$ be a set of distinct six points of $\mathbb{P}^2$ satisfying the following conditions:

(Σ1) no three points of Σ are collinear;

(Σ2) there are no conic curves containing Σ.

These are equivalent to the following:

(Σ3) for each $p_i \in \Sigma$, there exists a non-singular conic curve $N_i' \subset \mathbb{P}^2$ that contains $\Sigma \setminus \{p_i\}$ and does not contain p_i.

Proposition 7.4. *The linear system $|\mathcal{I}^2_\Sigma(5)|$ of quintic curves that pass through the points of Σ and are singular at each point of Σ is of dimension* 2, *and has no fixed components.*

Proof. Because each point of Σ imposes three linear conditions on $|\mathcal{O}_{\mathbb{P}^2}(5)|$, we have $\dim |\mathcal{I}_{\tilde{\Sigma}}(5)| \geqslant 2$.

Suppose that $|\mathcal{I}^2_\Sigma(5)|$ has a fixed component. Let F be the fixed component, and let

$$F = F_1 + \cdots + F_N$$

be the decomposition into the reduced irreducible components of F, where non-reduced components are expressed by repetition of summation. We have

$$\deg F = \deg F_1 + \cdots + \deg F_N > 0.$$

As in the previous subsection, we denote by $\tilde{\Sigma}$ the zero-dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf $\mathcal{I}^2_\Sigma$. We will consider the linear system

$$|\mathcal{I}_{\tilde{\Sigma}\setminus F}(5 - \deg F)|,$$

which has no fixed components and is of dimension equal to $\dim |\mathcal{I}_{\tilde{\Sigma}}(5)| \geqslant 2$. For $\nu = 0, \dots, N$, we define reduced zero-dimensional subschemes $\Sigma_1^{(\nu)}$ and $\Sigma_2^{(\nu)}$ of $\mathbb{P}^2$ by

$$\mathcal{I}_{\tilde{\Sigma}\setminus(F_1+\cdots+F_\nu)} = \mathcal{I}_{\Sigma_1^{(\nu)}}\mathcal{I}^2_{\Sigma_2^{(\nu)}}.$$

Then

$$|\Sigma_1^{(\nu+1)}| = |\Sigma_1^{(\nu)}| - i + j - k, \qquad |\Sigma_2^{(\nu+1)}| = |\Sigma_2^{(\nu)}| - j,$$

where

$$i := |\Sigma_1^{(\nu)} \cap F_{\nu+1}|, \quad j := |\Sigma_2^{(\nu)} \cap F_{\nu+1}|, \quad k := |\Sigma_2^{(\nu)} \cap \operatorname{Sing} F_{\nu+1}|.$$

The integers i, j and k are subject to the following conditions:

(i) $i + j \leqslant 2$ and $k = 0$ if $\deg F_{\nu+1} = 1$, because of (Σ1);

(ii) $i + j \leqslant 5$ and $k = 0$ if $\deg F_{\nu+1} = 2$, because of (Σ2);

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(iii) $k \leqslant 1$ if $\deg F_{\nu+1} = 3$, because an irreducible cubic curve has at most one singular point;

(iv) $k \leqslant 4$ if $\deg F_{\nu+1} = 4$, because if $k \geqslant 5$, there would exist a conic curve C with $CF_{\nu+1} \geqslant 10$.

Since

$$|\mathcal{I}^2_{\tilde{\Sigma}}(5)| \cong |\mathcal{I}_{\Sigma_1^{(N)}} \mathcal{I}^2_{\Sigma_2^{(N)}}(5 - \deg F)|$$

is of dimension greater than or equal to 2 and fixed-component free, we have

$$\deg F = 4 \implies |\Sigma_1^{(N)}| \leqslant 1 \text{ and } |\Sigma_2^{(N)}| = 0,$$
$$\deg F = 3 \implies |\Sigma_1^{(N)}| \leqslant 4 \text{ and } |\Sigma_2^{(N)}| = 0.$$

We put

$$\delta := \tfrac{1}{2}(6 - \deg F)(7 - \deg F) - (|\Sigma_1^{(N)}| + 3|\Sigma_2^{(N)}|) - 1.$$

From Corollary 7.2, we also have

$$\deg F \geqslant 2 \text{ and } 2 > \delta \implies |\Sigma_2^{(N)}| = 0.$$

Using these considerations, we see that the triple $(|\Sigma_1^{(N)}|, |\Sigma_2^{(N)}|, \deg F)$ is one of the following:

$$(0, 6, 1), \qquad (1, 5, 1), \qquad (2, 4, 1).$$

For these triples, however, we have $|\mathcal{I}_{\tilde{\Sigma}\setminus F}(5 - \deg F)| = \emptyset$, because otherwise there would exist an irreducible quartic curve C_4 and a conic curve C_2 such that $C_4 C_2 > 8$. Thus, we have proved that $|\mathcal{I}_{\tilde{\Sigma}}(5)|$ is fixed-component free.

If $\dim |\mathcal{I}_{\tilde{\Sigma}}(5)| > 2$, then, by Proposition 7.1, there would exist a conic curve that contains Σ, which contradicts (Σ2). □

Remark 7.5. Recall from (Σ3) that $N'_i \subset \mathbb{P}^2$ is the conic curve such that $N'_i \cap \Sigma = \Sigma \setminus \{p_i\}$. Let Q be a general member of $|\mathcal{I}^2_{\Sigma}(5)|$. Since $N'_i Q = 10$ for each i, the multiplicity of Q at each point of Σ is 2.

Let $\beta : S \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the points of Σ, and let M_i be the exceptional (reduced) divisor $\beta^{-1}(p_i)$. We put

$$L := \beta^* \mathcal{O}_{\mathbb{P}^2}(5) \otimes \mathcal{O}_S(-2M_1 - \cdots - 2M_6).$$

Let N_i be the strict transform of N'_i by β. We have $L^2 = 1$, $N_i L = 0$ and $N_i^2 = -1$.

Proposition 7.6. *The complete linear system $|L|$ on S has no base points, and the morphism $\Phi_{|L|} : S \to \mathbb{P}^2$ defined by $|L|$ is the contraction of the curves $N_1, \ldots, N_6$. Let p'_i be the image of N_i by $\Phi_{|L|}$. Then $\Sigma' = \{p'_1, \ldots, p'_6\}$ satisfies the condition (Σ3).*

Proof. By Proposition 7.4, the complete linear system $|L|$ on S is of dimension 2 and has no fixed components. Suppose that $|L|$ has a base point $p \in S$. Let $\tilde{\beta} : \tilde{S} \to S$ be the blowing up of S at p, and let M' be the exceptional divisor $\tilde{\beta}^{-1}(p)$ of $\tilde{\beta}$. Since $L^2 = 1$, the complete linear system $|\tilde{L}|$ of the line bundle $\tilde{L} := \tilde{\beta}^* L \otimes \mathcal{O}_{\tilde{S}}(-M')$ is of dimension 2 and has no fixed components. We have $K_{\tilde{S}}\tilde{L} = -2$, and hence $h^2(\tilde{S}, \tilde{L}) = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \tilde{L})) = 0$ follows. By the Riemann–Roch theorem, we have $h^1(\tilde{S}, \tilde{L}) = h^1(\tilde{S}, K_{\tilde{S}} - \tilde{L}) = 1$. Using the argument of the Koszul complex as in the proof of Proposition 7.1, we see that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \tilde{L})) > 0$. Hence, there exists a conic curve in $\mathbb{P}^2$ that passes through the points of Σ, which contradicts ($\Sigma 2$). Thus, $|L|$ has no base points.

Since $L^2 = 1$, the morphism $\Phi_{|L|}$ is of degree 1. Because $N_i L = 0$, the curves N_i are contracted by $\Phi_{|L|}$. Let C be a reduced irreducible curve on S that is contracted by $\Phi_{|L|}$. Because $M_i L = 2$, we have $C \neq M_i$ and, hence, $C' := \beta(C) \subset \mathbb{P}^2$ is a reduced irreducible curve. We will show that C' is equal to one of the conic curves N_i'. Let d be the degree of C'. We have

$$\beta^*(C') = C + m_1 M_1 + \cdots + m_6 M_6,$$

where m_j is the multiplicity of C' at p_j. The condition $CL = 0$ implies that

$$5d = 2(m_1 + \cdots + m_6).$$

If C' is not equal to N_i' for any i, then

$$C' N_i' = 2d \geqslant (m_1 + \cdots + m_6) - m_i = \tfrac{5}{2} d - m_i$$

holds for each i. Hence, $2m_i \geqslant d$ for $i = 1, \dots, 6$. Therefore, $5d = 2\sum m_j \geqslant 6d$, which is absurd. Thus, we have proved that $\Phi_{|L|}$ is the contraction of the (-1)-curves $N_1, \dots, N_6$.

Since $M_i L = 2$, the image of M_i by $\Phi_{|L|}$ is a non-singular conic curve. Because $M_i N_j = 0$ if and only if $i = j$, the conic curve $\Phi_{|L|}(M_i)$ satisfies

$$\Phi_{|L|}(M_i) \cap \Sigma' = \Sigma' \setminus \{p_i'\}.$$

Hence, Σ' satisfies ($\Sigma 3$). □

Corollary 7.7. *The rational map*

$$\mathrm{CT}_{\Sigma} : \mathbb{P}^2 \cdots \to \mathbb{P}^2,$$

defined by the linear system $|\mathcal{I}_{\Sigma}^2(5)|$, *is birational, and the inverse map is given by* $\mathrm{CT}_{\Sigma'}$.

We will write

$$\beta' : S \to \mathbb{P}^2$$

instead of $\Phi_{|L|}$. Let H and H' be the pull-backs of a general line of $\mathbb{P}^2$ by β and β', respectively. We put

$$M_i' := \beta'(M_i),$$

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which is a non-singular conic curve containing $\Sigma' \setminus \{p'_i\}$ and not passing through p'_i. We also put

$$U := S \setminus \Big(\bigcup_{i=1}^{6} N_i \cup \bigcup_{i=1}^{6} M_i\Big).$$

The morphisms β and β' induce isomorphisms

$$\mathbb{P}^2 \setminus \cup N'_i \cong U \cong \mathbb{P}^2 \setminus \cup M'_i. \tag{7.8}$$

The Picard group $\operatorname{Pic} S$ of S is a free $\mathbb{Z}$-module of rank 7, and is generated by the linear equivalence classes $[H], [M_1], \dots, [M_6]$, or by the linear equivalence classes $[H'], [N_1], \dots, [N_6]$. They are related by

$$[H'] = 5[H] - 2\sum_{j=1}^{6}[M_j], \qquad [N_i] = 2[H] - \sum_{j=1}^{6}[M_j] + [M_i], \quad i = 1, \dots, 6.$$

In particular, we have

$$3[H] - \sum_{j=1}^{6}[M_j] = 3[H'] - \sum_{j=1}^{6}[N_j] \tag{7.9}$$

in $\operatorname{Pic} S$.

7.3. Cremona transformations of supersingular $K3$ surfaces

Let G be a homogeneous polynomial in $\mathcal{U}$, and let $\Sigma = \{p_1, \dots, p_6\}$ be a subset of $Z(\mathrm{d}G)$ with $|\Sigma| = 6$. We assume that Σ satisfies the condition $(\Sigma 1)$ and

$(\Sigma 2)'$ for each $p_i \in \Sigma$, the non-singular conic curve N'_i containing $\Sigma \setminus \{p_i\}$ satisfies $N'_i \cap Z(\mathrm{d}G) = \Sigma \setminus \{p_i\}$.

Then the subset

$$Z' := \mathrm{CT}_\Sigma(Z(\mathrm{d}G) \setminus \Sigma) \cup \Sigma'$$

of $\mathbb{P}^2$ is well defined and consists of 21 points.

Proposition 7.8. *There exists $G' \in \mathcal{U}$ such that $Z' = Z(\mathrm{d}G')$.*

For the proof of Proposition 7.8, we first prove the following lemma.

Lemma 7.9. *There exists $G_1 \in \mathcal{U}$ that satisfies $Z(\mathrm{d}G) = Z(\mathrm{d}G_1)$ and $G_1(p_i) = 0$ for each $p_i \in \Sigma$.*

Proof. By $(\Sigma 1)$ and $(\Sigma 2)'$, the points of Σ impose independent linear conditions on the linear system $|\mathcal{O}_{\mathbb{P}^2}(3)|$ (see [**6**, p. 715]). Hence, there exists a homogeneous cubic polynomial H such that $(G + H^2)(p_i) = 0$ holds for each $p_i \in \Sigma$. Then $G_1 := G + H^2$ satisfies $Z(\mathrm{d}G) = Z(\mathrm{d}G_1)$. □

We replace G by G_1 in Lemma 7.9. Then the sextic curve D defined by $G = 0$ is reduced and has an ordinary node at each point of Σ. Hence,

$$\tilde{D} := \beta^* D - 2\sum_{j=1}^{6} M_j$$

is a reduced effective divisor of S, and it contains no M_j.

Proof of Proposition 7.8. Let D' be the image of $\tilde{D}$ by β'. Since $\tilde{D}H' = 6$, D' is a reduced curve of degree 6. Let $G' = 0$ be the defining equation of D'. We will show that $Z' = Z(\mathrm{d}G')$. It is enough to show that $Z(\mathrm{d}G')$ is of dimension 0 and that $Z' \subseteq Z(\mathrm{d}G')$.

Since $\tilde{D}N_j = 2$ for each N_j, we have

$$\tilde{D} := \beta'^* D' - 2\sum_{j=1}^{6} N_j.$$

Because $\tilde{D}$ is effective, we have $\operatorname{Sing} D' \supset \Sigma'$. Hence, $\Sigma' \subseteq Z(\mathrm{d}G')$. We put

$$\sqrt{L(H)} := \mathcal{O}_S\biggl(3H - \sum_{j=1}^{6} M_j\biggr) = \mathcal{O}_S\biggl(3H' - \sum_{j=1}^{6} N_j\biggr).$$

Let $\tilde{G}$ be a global section of

$$\sqrt{L(H)}^{\otimes 2} = L(H) = \mathcal{O}_S\Bigl(6H - 2\sum M_j\Bigr) = \mathcal{O}_S\Bigl(6H' - 2\sum N_j\Bigr)$$

such that $\tilde{G} = 0$ defines $\tilde{D}$. Let m and n be global sections of $\mathcal{O}_S(\sum M_j)$ and $\mathcal{O}_S(\sum N_j)$ such that $\sum M_j = \{m = 0\}$ and $\sum N_j = \{n = 0\}$. We can choose these in such a way that

$$\beta^* G = \tilde{G} m^2 \quad \text{and} \quad \beta'^* G' = \tilde{G} n^2 \tag{7.10}$$

hold. We define isomorphisms

$$\beta^* \mathcal{O}_{\mathbb{P}^2}(3) \mid U \cong \sqrt{L(H)} \mid U \cong \beta'^* \mathcal{O}_{\mathbb{P}^2}(3) \mid U \tag{7.11}$$

of line bundles on U by multiplications by m and n. We can define $\mathrm{d}\tilde{G}$, $\mathrm{d}(\beta^* G)$ and $\mathrm{d}(\beta'^* G')$ as global sections of the vector bundles

$$\Omega_S^1 \otimes \sqrt{L(H)}^{\otimes 2}, \quad \Omega_S^1 \otimes \beta^* \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2} \quad \text{and} \quad \Omega_S^1 \otimes \beta'^* \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2},$$

respectively. By (7.10) and (7.11), we get

$$\begin{aligned}
\beta^{-1}(Z(\mathrm{d}G) \setminus \Sigma) &= \beta^{-1}(Z(\mathrm{d}G) \cap (\mathbb{P}^2 \setminus \cup N_j')) \\
&= \beta^{-1}(Z(\mathrm{d}G)) \cap U \\
&= Z(\mathrm{d}(\beta^* G \mid U)) \\
&= Z(\mathrm{d}(\tilde{G}) \mid U) \\
&= Z(\mathrm{d}(\beta'^* G' \mid U)) \\
&= \beta'^{-1}(Z(\mathrm{d}G')) \cap U \\
&= \beta'^{-1}(Z(\mathrm{d}G') \cap (\mathbb{P}^2 \setminus \cup M_j')).
\end{aligned}$$

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Hence, we get

$$Z(\mathrm{d}G') \cap (\mathbb{P}^2 \setminus \cup M'_j) = \mathrm{CT}_\Sigma(Z(\mathrm{d}G) \setminus \Sigma).$$

In particular, we have $\mathrm{CT}_\Sigma(Z(\mathrm{d}G) \setminus \Sigma) \subset Z(\mathrm{d}G')$.

If $\dim Z(\mathrm{d}G') > 0$, then one of the conic curves M'_j is contained in $Z(\mathrm{d}G')$. Suppose that $M'_k \subset Z(\mathrm{d}G')$. Then $M_k \subset Z(\mathrm{d}\tilde{G})$ holds. We choose affine coordinates (x, y) of $\mathbb{P}^2$ such that $p_k = \beta(M_k)$ is the origin. Let

$$g(x, y) = \sum_{i+j \leqslant 6} a_{ij} x^i y^j$$

be the inhomogeneous polynomial corresponding to G. Since $p_k = (0, 0) \in \Sigma$ is contained in $\operatorname{Sing} D$, we have

$$a_{ij} = 0 \quad \text{for } i + j \leqslant 1.$$

Let the blowing up β be given by

$$(u, v) \mapsto (x, y) = (uv, v)$$

around a point of M_k. Then $\tilde{G}$ is written in terms of the coordinates (u, v) as

$$\tilde{g}(u, v) = \frac{\beta^* g}{v^2} = \sum_{2 \leqslant i+j \leqslant 6} a_{ij} u^i v^{i+j-2}.$$

Since $\mathrm{d}\tilde{G}$ is zero along the curve $M_k = \{v = 0\}$, we have

$$\frac{\partial \tilde{g}}{\partial u}(u, 0) = a_{11} = 0.$$

This contradicts the fact that p_k is a reduced point of $Z(\mathrm{d}G)$. □

Proposition 7.10. *Let G and G' be as above. Then the Cremona transformation CT_Σ of $\mathbb{P}^2$ lifts to an isomorphism*

$$\widetilde{\mathrm{CT}}_\Sigma : X_G \xrightarrow{\sim} X_{G'}$$

of supersingular $K3$ surfaces.

Proof. Let $\tilde{Y}$ be the subvariety of the total space of the line bundle $\sqrt{L(H)}$ defined by $W^2 = \tilde{G}$, where $\tilde{G}$ is the global section of $\sqrt{L(H)}^{\otimes 2} = L(H)$ introduced in the proof of Proposition 7.8, and W is the fibre coordinate of $\sqrt{L(H)}$. From (7.10) and (7.11), we obtain the isomorphisms

$$X_G \mid (\mathbb{P}^2 \setminus \cup N'_j) \cong \tilde{Y} \mid U \cong X_{G'} \mid (\mathbb{P}^2 \setminus \cup M'_j)$$

that are compatible with the isomorphisms (7.8). Since $K3$ surfaces are minimal, the isomorphism between Zariski open subsets of X_G and $X_{G'}$ extends to an isomorphism between X_G and $X_{G'}$. □

Remark 7.11. We describe the action of $\widetilde{\mathrm{CT}}_\Sigma$ on the numerical Néron–Severi lattices of the supersingular $K3$ surfaces. We number the points of $Z(\mathrm{d}G)$ and $Z(\mathrm{d}G')$ in such a way that

$$\Sigma = \{p_1, \dots, p_6\}, \qquad Z(\mathrm{d}G) = \Sigma \cup \{p_7, \dots, p_{21}\},$$
$$\Sigma' = \{p'_1, \dots, p'_6\}, \qquad Z(\mathrm{d}G') = \Sigma' \cup \{p'_7, \dots, p'_{21}\},$$

where $p'_i = \mathrm{CT}_\Sigma(p_i)$ for $i = 7, \dots, 21$. Let $E_i \subset X_G$ be the (-2)-curve that is contracted to $p_i \in \mathbb{P}^2$, and $E'_i \subset X_{G'}$ be the (-2)-curve that is contracted to $p'_i \in \mathbb{P}^2$. Then $\mathrm{NS}(X_G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by $[E_1], \dots, [E_{21}], [\mathcal{L}_G]$, and $\mathrm{NS}(X_{G'}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by $[E'_1], \dots, [E'_{21}], [\mathcal{L}_{G'}]$. Since $\mathrm{CT}_\Sigma(p_i) = p'_i$ for $i > 6$, we have

$$\widetilde{\mathrm{CT}}^*_\Sigma([E'_i]) = [E_i] \quad \text{for } i > 6.$$

The exceptional curve N_i on S contracted to p'_i by $\beta' : S \to \mathbb{P}^2$ is mapped by $\beta : S \to \mathbb{P}^2$ to the non-singular conic curve N'_i such that $N'_i \cap Z(\mathrm{d}G) = \Sigma \setminus \{p_i\}$. Hence,

$$\widetilde{\mathrm{CT}}^*_\Sigma([E'_i]) = 2[\mathcal{L}_G] - \sum_{j=1}^{6} [E_j] + [E_i] \quad \text{for } i = 1, \dots, 6.$$

The pull-back of a general line of $\mathbb{P}^2$ by $\mathrm{CT}_\Sigma : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ is a quintic curve Q such that $Q \cap Z(\mathrm{d}G) = \Sigma$ and that the multiplicity of Q at each point of Σ is 2 (see Remark 7.5). Thus,

$$\widetilde{\mathrm{CT}}^*_\Sigma([\mathcal{L}_{G'}]) = 5[\mathcal{L}_G] - 2\sum_{j=1}^{6} [E_j].$$

These formulae completely describe the homomorphism $\widetilde{\mathrm{CT}}^*_\Sigma \otimes_{\mathbb{Z}} \mathbb{Q}$ from $\mathrm{NS}(X_{G'}) \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathrm{NS}(X_G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 7.12. Suppose that the point $[G'] \in \mathfrak{M}$ corresponding to $G' \in \mathcal{U}$ in Proposition 7.8 coincides with the point $[G] \in \mathfrak{M}$. Then the Cremona transformation $\widetilde{\mathrm{CT}}_\Sigma$ defines a right coset in $\mathrm{Aut}(X_G)$ with respect to the subgroup $\mathrm{Aut}(X_G, \mathcal{L}_G) \subset \mathrm{Aut}(X_G)$. Indeed, the assumption $[G] = [G']$ implies the existence of a linear isomorphism $g : \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^2$ such that $g(Z(\mathrm{d}G')) = Z(\mathrm{d}G)$. Let $\hat{g} \in \mathrm{GL}(3, k)$ be a lift of $g \in \mathrm{PGL}(3, k)$. There then exists $c \in k^\times$ and $H \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ such that $\hat{g}^* G = cG' + H^2$. Let X_G and $X_{G'}$ be defined by $W^2 = G(X, Y, Z)$ and $W'^2 = G'(X, Y, Z)$, respectively. We have a lift $\tilde{g} : X_{G'} \xrightarrow{\sim} X_G$ of g given by

$$\tilde{g}^* W = \sqrt{c}\, W' + H.$$

The composite $\tilde{g} \circ \widetilde{\mathrm{CT}}_\Sigma$ is an automorphism of X_G. Since the linear isomorphism g is unique up to the group

$$\{h \in \mathrm{PGL}(3, k) \mid h(Z(\mathrm{d}G)) = Z(\mathrm{d}G)\} = \mathrm{Aut}(X_G, \mathcal{L}_G),$$

the automorphism $\tilde{g} \circ \widetilde{\mathrm{CT}}_\Sigma \in \mathrm{Aut}(X_G)$ is also unique up to $\mathrm{Aut}(X_G, \mathcal{L}_G)$.

https://doi.org/10.1017/S0013091504000562 Published online by Cambridge University Press

8. The isomorphism correspondences by Cremona transformations

8.1. The action of Cremona transformations on the moduli space

Let $\boldsymbol{C}$ be a code satisfying the conditions in Theorem 2.4(ii). For $\gamma \in \mathcal{G}_{\boldsymbol{C}}$, we denote by $G_\gamma \in \mathcal{U}$ a homogeneous polynomial such that $\gamma(\mathcal{P}) = Z(\mathrm{d}G_\gamma)$. Let $c \in \mathrm{Pow}(\mathcal{P})$ be a word of weight 6. Recall from Definition 1.11 that $\gamma(c)$ is a centre of Cremona transformation for G_γ if no three points of $\gamma(c)$ are collinear and there are no non-singular conic curves $C \subset \mathbb{P}^2$ such that $|C \cap \gamma(c)| \geqslant 5$ and $|C \cap \gamma(\mathcal{P})| \geqslant 6$. By Propositions 2.9, 2.10 and 2.17, we see that the following conditions on a word $c \in \mathrm{Pow}(\mathcal{P})$ of weight 6 are equivalent:

(i) the word c satisfies

  (a) $|c \cap l| \leqslant 2$ for any linear word l of $\boldsymbol{C}$,

  (b) $|c \cap q| \leqslant 4$ for any quadratic word q of $\boldsymbol{C}$;

(ii) there exists $\gamma \in \mathcal{G}_{\boldsymbol{C}}$ such that $\gamma(c)$ is a centre of Cremona transformation for G_γ;

(iii) for arbitrary $\gamma \in \mathcal{G}_{\boldsymbol{C}}$, $\gamma(c)$ is a centre of Cremona transformation for G_γ.

Definition 8.1. A word $c \in \mathrm{Pow}(\mathcal{P})$ of weight 6 is called a *centre of Cremona transformation with respect to* $\boldsymbol{C}$ if the above conditions are satisfied.

Let c be a centre of Cremona transformation with respect to $\boldsymbol{C}$. For $\gamma \in \mathcal{G}_{\boldsymbol{C}}$, we put

$$\Sigma := \gamma(c),$$

and consider the Cremona transformation CT_Σ. We put

$$Z'_{\gamma,c} := \{\mathrm{CT}_\Sigma(\gamma(P)) \mid P \in \mathcal{P} \setminus c\} \cup \{p'_1, \dots, p'_6\},$$

where p'_i is the image of the strict transform $N_i \subset S$ of the conic curve N'_i that contains $\gamma(c) \setminus \{p_i\}$. By Proposition 7.8, there exists a polynomial $G'_{\gamma,c} \in \mathcal{U}$ such that $Z'_{\gamma,c} = Z(\mathrm{d}G'_{\gamma,c})$. Even though the polynomial $G'_{\gamma,c}$ is not uniquely determined, the corresponding point $[G'_{\gamma,c}] \in \mathfrak{M}$ is uniquely determined by c and γ. The map $\gamma \mapsto [G'_{\gamma,c}]$ gives a morphism from $\mathcal{G}_{\boldsymbol{C}}$ to $\mathfrak{M}$. It is obvious that this morphism descends to the morphism

$$\mathrm{ct}_c : \mathrm{PGL}(3,k) \setminus \mathcal{G}_{\boldsymbol{C}} \to \mathfrak{M}.$$

8.2. The case where the Artin invariant is 2

Let T be A, B or C, and let $c \in \mathrm{Pow}(\mathcal{P})$ be a centre of Cremona transformation with respect to $\boldsymbol{C}_T$. The image by ct_c of the connected component

$$(\mathrm{PGL}(3,k) \setminus \mathcal{G}_T)^+ = \{[\gamma_\lambda] \mid \lambda \in \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}\}$$

of $\mathrm{PGL}(3,k) \setminus \mathcal{G}_T$ is a connected component of $\mathfrak{M}_2 = \mathfrak{M}_A \sqcup \mathfrak{M}_B \sqcup \mathfrak{M}_C$ and, hence, there exists $T' \in \{A, B, C\}$ such that ct_c yields a morphism

$$\mathrm{ct}^+_{T,c} : (\mathrm{PGL}(3,k) \setminus \mathcal{G}_T)^+ \to \mathfrak{M}_{T'}.$$

https://doi.org/10.1017/S0013091504000562 Published online by Cambridge University Press

Table 30. *Isomorphism correspondences by Cremona transformations*

T	c	$\lvert N_T \cdot c\rvert$	T'	$D_{T,T'}[c]$
A	$\{1,2,8,10,15,16\}$	12	A	$\Delta = 0$
A	$\{1,7,8,15,18,21\}$	144	A	$D1 = 0$
A	$\{2,4,8,11,14,16\}$	576	B	$D2 = 0$
A	$\{1,4,6,9,12,20\}$	72	A	$D3 = 0$
A	$\{2,8,10,12,14,21\}$	72	A	$D3 = 0$
A	$\{5,8,9,10,14,16\}$	48	A	$\Delta = 0$
A	$\{4,9,12,16,17,18\}$	24	A	$\Delta = 0$
A	$\{1,2,9,10,16,19\}$	36	A	$\Delta = 0$
A	$\{7,12,13,14,19,20\}$	36	A	$\Delta = 0$
A	$\{2,6,9,10,13,21\}$	48	C	$D4 = 0$
A	$\{2,5,11,13,17,21\}$	576	A	$D1 = 0$
B	$\{2,7,8,9,10,17\}$	216	B	$\Delta = 0$
B	$\{1,2,11,12,13,18\}$	72	B	$D5 = 0$
B	$\{4,5,6,10,13,19\}$	54	B	$\Delta = 0$
B	$\{4,7,12,15,20,21\}$	6	B	$\Delta = 0$
B	$\{1,2,6,10,14,16\}$	54	B	$\Delta = 0$
B	$\{3,5,14,16,19,20\}$	108	B	$\Delta = 0$
B	$\{1,3,8,12,13,17\}$	108	B	$\Delta = 0$
B	$\{1,5,6,16,20,21\}$	216	A	$D6 = 0$
B	$\{2,6,9,13,16,18\}$	36	B	$\Delta = 0$
B	$\{3,7,8,10,19,21\}$	216	B	$D5 = 0$
B	$\{2,5,9,16,18,19\}$	108	B	$\Delta = 0$
B	$\{1,3,5,15,19,21\}$	72	B	$D5 = 0$
B	$\{2,6,7,16,20,21\}$	108	B	$\Delta = 0$
C	$\{3,5,9,13,17,21\}$	960	A	$D7 = 0$
C	$\{3,5,10,14,17,21\}$	64	C	$D8 = 0$
C	$\{1,5,8,10,14,18\}$	960	C	$\Delta = 0$
C	$\{1,2,5,8,18,19\}$	240	C	$\Delta = 0$

$$\Delta := J_T + J_{T'}$$

$$\begin{aligned} D1 := {} & J_{T'}^6 J_T^2 + J_{T'}^4 J_T^4 + J_{T'}^2 J_T^6 + J_{T'}^4 J_T^3 + J_{T'}^3 J_T^4 + J_{T'}^4 J_T^2 + J_{T'}^3 J_T^3 + J_{T'}^2 J_T^4 \\ & + J_{T'}^4 J_T + J_{T'} J_T^4 + J_{T'}^3 J_T + J_{T'}^2 J_T^2 + J_{T'} J_T^3 + J_{T'}^3 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_T^3 \end{aligned}$$

$$D2 := J_T^4 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_{T'} J_T + J_{T'}$$

$$D3 := J_{T'}^3 J_T^2 + J_{T'}^2 J_T^3 + J_{T'}^2 J_T^2 + J_{T'} J_T + 1$$

$$D4 := J_{T'}^2 J_T^4 + J_{T'} J_T^2 + J_{T'} J_T + J_{T'} + J_T$$

$$D5 := J_{T'}^4 J_T + J_{T'}^3 J_T^2 + J_{T'}^2 J_T^3 + J_{T'} J_T^4 + J_{T'}^3 J_T + J_{T'}^2 J_T^2 + J_{T'} J_T^3 + 1$$

$$D6 := J_{T'}^4 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_{T'} J_T + J_T$$

$$D7 := J_{T'}^4 J_T^2 + J_{T'}^2 J_T + J_{T'} J_T + J_{T'} + J_T$$

$$D8 := J_{T'}^4 J_T^4 + J_{T'}^3 J_T + J_{T'}^2 J_T^2 + J_{T'} J_T^3 + J_{T'}^3 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_T^3$$

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Table 31. *Generators of* $H^0(\mathbb{P}^2, \mathcal{I}^2_\Sigma(5))$

$$\begin{aligned}
F_1 := {} & X^4Z + X^3Z^2\lambda^2\omega + X^3Z^2\lambda\omega + X^2Y^2Z\omega + X^2YZ^2\lambda^2\omega + X^2YZ^2\lambda\omega \\
& + X^2YZ^2\omega + X^2Z^3\lambda^4\omega + X^2Z^3\lambda^4 + X^2Z^3\lambda^2 + X^2Z^3\lambda\omega + X^2Z^3 \\
& + XY^2Z^2\lambda^2\omega + XY^2Z^2\lambda\omega + XY^2Z^2\omega + XYZ^3\omega + Y^3Z^2\lambda^2\omega + Y^3Z^2\lambda\omega \\
& + Y^2Z^3\lambda^4\omega + Y^2Z^3\lambda^4 + Y^2Z^3\lambda^2 + Y^2Z^3\lambda\omega
\end{aligned}$$

$$\begin{aligned}
F_2 := {} & XY^3Z\lambda^2\omega + X^3Z^2\lambda^2 + X^2YZ^2\lambda + X^2Z^3\lambda + XY^2Z^2\lambda^2 + XYZ^3 + X^2Y^2Z \\
& + X^2Z^3\lambda^4 + Y^3Z^2\lambda^4 + Y^2Z^3\lambda^4 + Y^2Z^3\lambda + X^2YZ^2\lambda^4 + X^3Z^2\lambda \\
& + XY^2Z^2\lambda + X^4Y + XY^2Z^2 + Y^3Z^2\lambda + Y^2Z^3\lambda^3\omega + X^2Z^3\lambda^5\omega \\
& + X^2Z^3\lambda^3\omega + Y^2Z^3\lambda^5\omega + Y^2Z^3\lambda^2\omega + Y^2Z^3\lambda\omega + Y^2Z^3\lambda^6\omega + X^2Y^3\omega \\
& + X^3YZ\lambda^2\omega + X^2Z^3\lambda^4\omega + X^2YZ^2\lambda\omega + Y^3Z^2\lambda\omega + XY^2Z^2\omega + X^2Z^3\lambda^6\omega \\
& + XY^3Z\omega + Y^3Z^2\lambda^4\omega + XY^3Z\lambda\omega + Y^2Z^3\lambda^4\omega + X^3YZ\lambda\omega + X^2Y^2Z\lambda\omega \\
& + X^2Y^2Z\omega + X^2Y^2Z\lambda^2\omega + X^2YZ^2\lambda^4\omega
\end{aligned}$$

$$\begin{aligned}
F_3 := {} & XY^3Z\lambda^2\omega + X^2YZ^2\lambda^2\omega + X^3Y^2\omega + X^3Z^2 + X^2YZ^2\lambda + X^2Z^3\lambda + XYZ^3 \\
& + X^2YZ^2 + X^2Y^2Z + X^2Z^3\lambda^4 + Y^3Z^2\lambda^2 + Y^2Z^3\lambda^4 + Y^2Z^3\lambda + XY^2Z^2\lambda^4 \\
& + X^3Z^2\lambda^4 + X^3Z^2\lambda + XY^2Z^2\lambda + X^2YZ^2\lambda^2 + XY^2Z^2 + Y^3Z^2\lambda \\
& + Y^2Z^3\lambda^3\omega + X^5 + X^2Z^3\lambda^5\omega + X^2Z^3\lambda^3\omega + Y^2Z^3\lambda^5\omega + X^2Z^3\lambda^2\omega \\
& + Y^2Z^3\lambda\omega + X^3Z^2\lambda^4\omega + Y^2Z^3\lambda^6\omega + X^3YZ\omega + X^3YZ\lambda^2\omega + X^2YZ^2\lambda\omega \\
& + Y^3Z^2\lambda\omega + Y^3Z^2\lambda^2\omega + XY^2Z^2\omega + XYZ^3\omega + X^2Z^3\lambda^6\omega + XY^3Z\lambda\omega \\
& + XY^2Z^2\lambda^4\omega + X^3YZ\lambda\omega + X^3Z^2\lambda^2\omega + X^2Y^2Z\lambda\omega + XY^2Z^2\lambda^2\omega + X^2Y^2Z\lambda^2\omega
\end{aligned}$$

Using $\mathrm{ct}^+_{T,c}$ and the quotient morphism

$$p_T : (\mathrm{PGL}(3,k) \setminus \mathcal{G}_T)^+ \to \mathfrak{M}_T$$

by $N_T = \mathrm{Ker}(\mathrm{Aut}(\boldsymbol{C}_T) \to \tilde{T})$, we obtain an irreducible isomorphism correspondence

$$D_{T,T'}[c] := \{(p_T([\gamma]), \mathrm{ct}^+_{T,c}([\gamma])) \mid [\gamma] \in (\mathrm{PGL}(3,k) \setminus \mathcal{G}_T)^+\} \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}.$$

For $\sigma \in N_T$, we have

$$\mathrm{ct}_c([\gamma \circ \sigma]) = \mathrm{ct}_{\sigma(c)}([\gamma]).$$

Hence, the type T' and the correspondence $D_{T,T'}[c]$ depends only on the orbit of c under the action of N_T. We present in Table 30 the decomposition of the set of centres of Cremona transformation with respect to $\boldsymbol{C}_T$ into the orbits under the action of N_T. For each orbit, the type T' and the defining equation of the isomorphism correspondence $D_{T,T'}[c]$ are also given.

We will explain the algorithm for obtaining the defining equation of $D_{T,T'}[c]$. For example, consider the case where $T = A$ and $c = \{P_1, P_4, P_6, P_9, P_{12}, P_{20}\}$. The six points $\Sigma = \gamma_\lambda(c) = \{p_1, \dots, p_6\}$ are as follows:

$$\begin{aligned}
p_1 &:= \gamma_\lambda(P_1) = [1, \omega, 0], & p_4 &:= \gamma_\lambda(P_9) = [1, 1, 1], \\
p_2 &:= \gamma_\lambda(P_4) = [1+\lambda, \lambda, 1], & p_5 &:= \gamma_\lambda(P_{12}) = [0, 1, 0], \\
p_3 &:= \gamma_\lambda(P_6) = [\lambda, 1+\lambda, 1], & p_6 &:= \gamma_\lambda(P_{20}) = [0, 0, 1].
\end{aligned}$$

Table 32. *Points* q_i

$$\begin{aligned}
q_1 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_2)) &&= [0,1,\omega]\\
q_2 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_3)) &&= [\omega,\lambda^2+\bar\omega\lambda+1,\lambda(\lambda+\bar\omega)]\\
q_3 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_5)) &&= [\omega,\lambda^2+\bar\omega\lambda+\bar\omega,(\lambda+1)(\lambda+\omega)]\\
q_4 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_7)) &&= [\omega,\lambda^2+\omega\lambda+\omega,(\lambda+\bar\omega)^2]\\
q_5 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_8)) &&= [\omega,(\lambda+1)^2,\lambda^2+\omega\lambda+1]\\
q_6 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{10})) &&= [\omega,(\lambda+\omega)^2,(\lambda+1)(\lambda+\bar\omega)]\\
q_7 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{11})) &&= [\omega,(\lambda+\omega)(\lambda+\bar\omega),\lambda^2+\lambda+\omega]\\
q_8 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{13})) &&= [0,1,1]\\
q_9 &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{14})) &&= [\omega,\lambda^2+\omega\lambda+1,(\lambda+\omega)^2]\\
q_{10} &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{15})) &&= [\omega,\lambda^2,\lambda^2+\omega\lambda+\omega]\\
q_{11} &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{16})) &&= [\omega,(\lambda+\bar\omega)^2,\lambda(\lambda+\omega)]\\
q_{12} &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{17})) &&= [\omega,\lambda(\lambda+\omega),\lambda^2]\\
q_{13} &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{18})) &&= [0,0,1]\\
q_{14} &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{19})) &&= [\omega,\lambda^2+\lambda+\bar\omega,\lambda(\lambda+1)]\\
q_{15} &:= \mathrm{CT}_\Sigma(\gamma_\lambda(P_{21})) &&= [\omega,(\lambda+1)(\lambda+\bar\omega),(\lambda+1)^2]
\end{aligned}$$

$$\begin{aligned}
q_{16} &:= \beta'(N_1) = [0,1,0]\\
q_{17} &:= \beta'(N_2) = [\omega,(\lambda+\bar\omega)\lambda,\lambda^2+\bar\omega\lambda+\bar\omega]\\
q_{18} &:= \beta'(N_3) = [\omega,(\lambda+1)(\lambda+\omega),\lambda^2+\bar\omega\lambda+1]\\
q_{19} &:= \beta'(N_4) = [\omega,\lambda^2+\lambda+\omega,\lambda^2+\lambda+\bar\omega]\\
q_{20} &:= \beta'(N_5) = [0,1,\bar\omega]\\
q_{21} &:= \beta'(N_6) = [\omega,\lambda(\lambda+1),(\lambda+\omega)(\lambda+\bar\omega)]
\end{aligned}$$

Solving linear equations, we see that the three-dimensional linear space $H^0(\mathbb{P}^2,\mathcal{I}^2_\Sigma(5))$ is generated by the homogeneous quintic polynomials in Table 31.

The Cremona transformation $\mathrm{CT}_\Sigma : \mathbb{P}^2 \cdots\to \mathbb{P}^2$ is given by

$$[X,Y,Z]\mapsto[F_1,F_2,F_3].$$

The points $\gamma_\lambda(P_i)$, $P_i\notin c$, is mapped by CT_Σ to the points in Table 32.

The conic curve $N_1'\subset\mathbb{P}^2$ containing $\Sigma\setminus\{p_1\}$ is defined by

$$E_1 := X^2+(\lambda^2+\lambda)YZ+(\lambda^2+\lambda+1)ZX=0.$$

Let V_1 be the vector space of cubic homogeneous polynomials C such that E_1C is a member of $H^0(\mathbb{P}^2,\mathcal{I}^2_\Sigma(5))$. Then we have $\dim V_1=2$, and the image of the linear map $V_1\to H^0(\mathbb{P}^2,\mathcal{I}^2_\Sigma(5))$ given by $C\mapsto E_1C$ is spanned by F_1 and F_3. Hence, the image $\beta'(N_1)$ of the strict transform $N_1\subset S$ of N_1' is $[0,1,0]$. In the same way, we calculate $\beta'(N_i)$ as in Table 32.

The set LW of collinear 5-tuples of the points in $Z'=\{q_1,\dots,q_{21}\}$ and the set QW of 8-tuples of the points in Z' that are on a non-singular conic curve are given in Table (33), where $\{1,3,5,11,17\}$ means $\{q_1,q_3,q_5,q_{11},q_{17}\}$, for example. Since $|LW|=13$ and $|QW|=28$, we see that the type T' of the target moduli curve is A. Let σ be the

Table 33. *Sets LW and QW*

$$LW = \{\{1,3,5,11,17\}, \{4,5,6,8,12\}, \{1,2,6,10,18\}, \{2,3,8,19,21\}, \{1,4,9,14,21\}, \{8,9,10,11,15\}, \{6,7,11,20,21\}, \{1,7,12,15,19\}, \{3,9,12,18,20\}, \{1,8,13,16,20\}, \{7,8,14,17,18\}, \{5,10,14,19,20\}, \{2,4,15,17,20\}\}$$

$$QW = \{\{2,3,4,5,9,10,13,16\}, \{2,3,4,7,10,11,12,14\}, \{2,3,5,6,7,9,14,15\}, \{2,4,5,7,9,11,18,19\}, \{3,4,5,7,10,15,18,21\}, \{3,4,6,7,9,10,17,19\}, \{2,5,6,7,13,16,17,19\}, \{2,3,6,11,12,13,15,16\}, \{3,4,6,11,14,15,18,19\}, \{3,5,6,13,14,16,18,21\}, \{2,5,7,9,10,12,17,21\}, \{4,6,7,10,13,14,15,16\}, \{3,4,7,12,13,16,17,21\}, \{5,7,9,11,12,13,14,16\}, \{2,6,9,11,12,14,17,19\}, \{4,6,9,11,13,16,17,18\}, \{2,7,9,13,15,16,18,21\}, \{5,6,9,15,17,18,19,21\}, \{3,7,10,11,13,16,18,19\}, \{6,9,10,12,13,16,19,21\}, \{3,6,10,12,14,15,17,21\}, \{2,5,11,12,14,15,18,21\}, \{4,10,11,12,17,18,19,21\}, \{2,10,11,13,14,16,17,21\}, \{4,5,11,13,15,16,19,21\}, \{2,4,12,13,14,16,18,19\}, \{5,10,12,13,15,16,17,18\}, \{3,9,13,14,15,16,17,19\}\}$$

following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 13 & 16 & 2 & 6 & 21 & 4 & 18 & 17 & 3 & 7 & 12 \end{pmatrix.$$
$$\begin{matrix} 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 1 & 8 & 9 & 11 & 15 & 14 & 20 & 5 & 19 & 10 \end{pmatrix}.$$

Then the map

$$\gamma' : \mathcal{P} \to \mathbb{P}^2$$

defined by $\gamma'(P_i) = q_{\sigma(i)}$ yields bijections from the set of linear words in $\boldsymbol{C}_A$ (see Table 10) to LW and from the set of quadratic words in $\boldsymbol{C}_A$ (see Tables 13 and 14) to QW. Hence, the map γ' is an element of $\mathcal{G}_A$. We make the linear change of homogeneous coordinates of $\mathbb{P}^2$ so that

$$\gamma'(P_{18}) = q_{20} = [1,0,0], \qquad \gamma'(P_{12}) = q_1 = [0,1,0], \qquad \gamma'(P_{13}) = q_8 = [1,1,0],$$
$$\gamma'(P_{20}) = q_{19} = [0,0,1], \qquad \gamma'(P_{19}) = q_5 = [1,0,1]$$

hold (see (4.6)); that is, we multiply the vectors $\gamma'(P_i) = q_{\sigma(i)}$ by the matrix

$$\begin{bmatrix} \lambda^2+\lambda & 1 & \bar{\omega} \\ \lambda^2+\lambda+1 & \bar{\omega} & 1 \\ \omega\lambda+1 & 0 & 0 \end{bmatrix}$$

from the left. Then we have

$$\gamma'(P_1) = q_{13} = [1, \omega, 0].$$

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Therefore, the projective equivalence class $[\gamma'] \in \mathrm{PGL}(3,k) \setminus \mathcal{G}_A$ of γ' is contained in the connected component $(\mathrm{PGL}(3,k) \setminus \mathcal{G}_A)^+$, because otherwise we would have $\gamma'(P_1) = [1, \bar{\omega}, 0]$. Since

$$\gamma'(P_{10}) = q_7 = [0, 1, \lambda + \bar{\omega}],$$

the point $[\gamma']$ corresponds to $1/(\lambda + \bar{\omega})$ under the isomorphism $(\mathrm{PGL}(3,k) \setminus \mathcal{G}_A)^+ \cong \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$. Substituting $1/(\lambda + \bar{\omega})$ for λ in

$$J_A = \frac{(\lambda^2 + \lambda + 1)^3}{\lambda^2(\lambda + 1)^2},$$

we see that the J_A-invariant of $[\gamma']$ is equal to

$$J'_A = \frac{\lambda^3(\lambda + 1)^3}{(\lambda^2 + \lambda + 1)^2}.$$

Eliminating λ from J_A and J'_A, we obtain the defining equation

$$1 + J_A J'_A + J_A^2 J_A'^2 + J_A^3 J_A'^2 + J_A^2 J_A'^3 = 0$$

of the isomorphism correspondence given by the Cremona transformation with the centre $c = \{P_1, P_4, P_6, P_9, P_{12}, P_{20}\}$.

Putting

$$\begin{aligned}
D_{A,A,1} &:= \{D3 = 0\},\\
D_{A,A,2} &:= \{D1 = 0\},\\
D_{B,B,1} &:= \{D5 = 0\},\\
D_{C,C,1} &:= \{D8 = 0\},\\
D_{A,B,1} &:= \{D2 = 0\} = {D_{B,A,1}}^{\mathrm{T}} = \{D6 = 0\}^{\mathrm{T}},\\
D_{A,C,1} &:= \{D4 = 0\} = {D_{C,A,1}}^{\mathrm{T}} = \{D7 = 0\}^{\mathrm{T}},
\end{aligned}$$

we obtain Theorem 1.13. The composite $D_1 * D_2$ of correspondences

$$D_1 = \{f_1(J_T, J_{T'}) = 0\} \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$$

and

$$D_2 = \{f_2(J_{T'}, J_{T''}) = 0\} \subset \mathfrak{M}_{T'} \times \mathfrak{M}_{T''}$$

is obtained by eliminating the variable $J_{T'}$ from $f_1(J_T, J_{T'}) = f_2(J_{T'}, J_{T''}) = 0$. Starting from the eight isomorphism correspondences above and making composites, we obtain irreducible isomorphism correspondences listed in Table 4, which have the relations given in Appendix A. This table also shows that the isomorphism correspondences Δ_A, Δ_B, Δ_C, and those in Table 4, are closed under compositions of correspondences.

Appendix A. Relations between non-trivial isomorphism correspondences

$$
\begin{aligned}
D_{A,A,1} * D_{A,A,1} &= \Delta_A + D_{A,A,2},\\
D_{A,A,1} * D_{A,A,2} &= D_{A,A,1} + D_{A,A,2},\\
D_{A,A,2} * D_{A,A,1} &= D_{A,A,1} + D_{A,A,2},\\
D_{A,A,2} * D_{A,A,2} &= \Delta_A + D_{A,A,1} + D_{A,A,2},\\
D_{A,A,1} * D_{A,B,1} &= D_{A,B,2},\\
D_{A,A,1} * D_{A,B,2} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,A,2} * D_{A,B,1} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,A,2} * D_{A,B,2} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,A,1} * D_{A,C,1} &= D_{A,C,2},\\
D_{A,A,1} * D_{A,C,2} &= D_{A,C,1} + D_{A,C,2},\\
D_{A,A,2} * D_{A,C,1} &= D_{A,C,1} + D_{A,C,2},\\
D_{A,A,2} * D_{A,C,2} &= D_{A,C,1} + D_{A,C,2},
\end{aligned}
$$

$$
\begin{aligned}
D_{B,B,1} * D_{B,B,1} &= \Delta_B + D_{B,B,1},\\
D_{B,B,1} * D_{B,A,1} &= D_{B,A,1} + D_{B,A,2},\\
D_{B,B,1} * D_{B,A,2} &= D_{B,A,1} + D_{B,A,2},\\
D_{B,B,1} * D_{B,C,1} &= D_{B,C,2},\\
D_{B,B,1} * D_{B,C,2} &= D_{B,C,1} + D_{B,C,2},
\end{aligned}
$$

$$
\begin{aligned}
D_{C,C,1} * D_{C,C,1} &= \Delta_C + D_{C,C,1},\\
D_{C,C,1} * D_{C,B,1} &= D_{C,B,2},\\
D_{C,C,1} * D_{C,B,2} &= D_{C,B,1} + D_{C,B,2},\\
D_{C,C,1} * D_{C,A,1} &= D_{C,A,1} + D_{C,A,2},\\
D_{C,C,1} * D_{C,A,2} &= D_{C,A,1} + D_{C,A,2},
\end{aligned}
$$

$$
\begin{aligned}
D_{A,B,1} * D_{B,B,1} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,B,2} * D_{B,B,1} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,B,1} * D_{B,A,1} &= \Delta_A + D_{A,A,2},\\
D_{A,B,1} * D_{B,A,2} &= D_{A,A,1} + D_{A,A,2},\\
D_{A,B,2} * D_{B,A,1} &= D_{A,A,1} + D_{A,A,2},\\
D_{A,B,2} * D_{B,A,2} &= \Delta_A + D_{A,A,1} + D_{A,A,2},\\
D_{A,B,1} * D_{B,C,1} &= D_{A,C,1},\\
D_{A,B,1} * D_{B,C,2} &= D_{A,C,1} + D_{A,C,2},\\
D_{A,B,2} * D_{B,C,1} &= D_{A,C,2},\\
D_{A,B,2} * D_{B,C,2} &= D_{A,C,1} + D_{A,C,2},
\end{aligned}
$$

$$D_{B,A,1} * D_{A,A,1} = D_{B,A,2},$$
$$D_{B,A,1} * D_{A,A,2} = D_{B,A,1} + D_{B,A,2},$$
$$D_{B,A,2} * D_{A,A,1} = D_{B,A,1} + D_{B,A,2},$$
$$D_{B,A,2} * D_{A,A,2} = D_{B,A,1} + D_{B,A,2},$$
$$D_{B,A,1} * D_{A,B,1} = \Delta_B + D_{B,B,1},$$
$$D_{B,A,1} * D_{A,B,2} = D_{B,B,1},$$
$$D_{B,A,2} * D_{A,B,1} = D_{B,B,1},$$
$$D_{B,A,2} * D_{A,B,2} = \Delta_B + D_{B,B,1},$$
$$D_{B,A,1} * D_{A,C,1} = D_{B,C,1} + D_{B,C,2},$$
$$D_{B,A,1} * D_{A,C,2} = D_{B,C,2},$$
$$D_{B,A,2} * D_{A,C,1} = D_{B,C,2},$$
$$D_{B,A,2} * D_{A,C,2} = D_{B,C,1} + D_{B,C,2},$$

$$D_{B,C,1} * D_{C,C,1} = D_{B,C,2},$$
$$D_{B,C,2} * D_{C,C,1} = D_{B,C,1} + D_{B,C,2},$$
$$D_{B,C,1} * D_{C,B,1} = \Delta_B,$$
$$D_{B,C,1} * D_{C,B,2} = D_{B,B,1},$$
$$D_{B,C,2} * D_{C,B,1} = D_{B,B,1},$$
$$D_{B,C,2} * D_{C,B,2} = \Delta_B + D_{B,B,1},$$
$$D_{B,C,1} * D_{C,A,1} = D_{B,A,1},$$
$$D_{B,C,1} * D_{C,A,2} = D_{B,A,2},$$
$$D_{B,C,2} * D_{C,A,1} = D_{B,A,1} + D_{B,A,2},$$
$$D_{B,C,2} * D_{C,A,2} = D_{B,A,1} + D_{B,A,2},$$

$$D_{C,B,1} * D_{B,B,1} = D_{C,B,2},$$
$$D_{C,B,2} * D_{B,B,1} = D_{C,B,1} + D_{C,B,2},$$
$$D_{C,B,1} * D_{B,A,1} = D_{C,A,1},$$
$$D_{C,B,1} * D_{B,A,2} = D_{C,A,2},$$
$$D_{C,B,2} * D_{B,A,1} = D_{C,A,1} + D_{C,A,2},$$
$$D_{C,B,2} * D_{B,A,2} = D_{C,A,1} + D_{C,A,2},$$
$$D_{C,B,1} * D_{B,C,1} = \Delta_C,$$
$$D_{C,B,1} * D_{B,C,2} = D_{C,C,1},$$
$$D_{C,B,2} * D_{B,C,1} = D_{C,C,1},$$
$$D_{C,B,2} * D_{B,C,2} = \Delta_C + D_{C,C,1},$$

$$D_{C,A,1} * D_{A,A,1} = D_{C,A,2},$$
$$D_{C,A,1} * D_{A,A,2} = D_{C,A,1} + D_{C,A,2},$$

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$$
\begin{aligned}
D_{C,A,2} * D_{A,A,1} &= D_{C,A,1} + D_{C,A,2},\\
D_{C,A,2} * D_{A,A,2} &= D_{C,A,1} + D_{C,A,2},\\
D_{C,A,1} * D_{A,B,1} &= D_{C,B,1} + D_{C,B,2},\\
D_{C,A,1} * D_{A,B,2} &= D_{C,B,2},\\
D_{C,A,2} * D_{A,B,1} &= D_{C,B,2},\\
D_{C,A,2} * D_{A,B,2} &= D_{C,B,1} + D_{C,B,2},\\
D_{C,A,1} * D_{A,C,1} &= \Delta_C + D_{C,C,1},\\
D_{C,A,1} * D_{A,C,2} &= D_{C,C,1},\\
D_{C,A,2} * D_{A,C,1} &= D_{C,C,1},\\
D_{C,A,2} * D_{A,C,2} &= \Delta_C + D_{C,C,1},
\end{aligned}
$$

$$
\begin{aligned}
D_{A,C,1} * D_{C,C,1} &= D_{A,C,1} + D_{A,C,2},\\
D_{A,C,2} * D_{C,C,1} &= D_{A,C,1} + D_{A,C,2},\\
D_{A,C,1} * D_{C,B,1} &= D_{A,B,1},\\
D_{A,C,1} * D_{C,B,2} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,C,2} * D_{C,B,1} &= D_{A,B,2},\\
D_{A,C,2} * D_{C,B,2} &= D_{A,B,1} + D_{A,B,2},\\
D_{A,C,1} * D_{C,A,1} &= \Delta_A + D_{A,A,2},\\
D_{A,C,1} * D_{C,A,2} &= D_{A,A,1} + D_{A,A,2},\\
D_{A,C,2} * D_{C,A,1} &= D_{A,A,1} + D_{A,A,2},\\
D_{A,C,2} * D_{C,A,2} &= \Delta_A + D_{A,A,1} + D_{A,A,2}.
\end{aligned}
$$

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