A NOTE ON LIFTINGS OF HERMITIAN ELEMENTS AND UNITARIES

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Let A be a complex Banach algebra with unit 1 satisfying $\|1\| = 1$. An element u in A is said to be *unitary* if it is invertible and $\|u\| = \|u^{-1}\| = 1$. An element h in A is said to be *hermitian* if $\|\exp(ith)\| = 1$ for all real t; that is, $\exp(ith)$ is unitary for all real t. Suppose that J is a closed two-sided ideal and $\pi: A \to A/J$ is the quotient mapping. It is easy to see that if x in A is hermitian (resp. unitary), then so is $\pi(x)$ in A/J. We consider the following general question which is the converse of the above statement: given a hermitian (resp. unitary) element y in A/J, can we find a hermitian (resp. unitary) element x in A such that $\pi(x) = y$? (The author has learned that this question, in a more restrictive form, was raised by F. F. Bonsall and that some special cases were investigated; see [**1**], [**2**].) In the present note, we give a partial answer to this question under the assumption that A is finite dimensional.

For notation and terminology, we follow the book by Bonsall and Duncan [**3**]. *We shall always assume that A is finite dimensional.*

Note that there exists an idempotent e in A which is minimal with respect to the property that $\|e\| = 1$ and $1 - e \in J$; that is, if f is an idempotent such that $ef = fe = f$, $\|f\| = 1$ and $1 - f \in J$, then $e = f$. Also note that eAe is a Banach algebra with e as its unit, $eAe + J = A$ and an element h in eAe is a hermitian element in the algebra eAe if and only if $\|\exp(ith)e\| = 1$ for all real t.

THEOREM A. *If h is a hermitian element in A/J, then there is a hermitian element $\tilde{h}$ in eAe such that $\pi(\tilde{h}) = h$.*

THEOREM B. *If u is a unitary element in A/J, then there exists a unitary element $\tilde{u}$ in eAe such that $\pi(\tilde{u}) = u$.*

To prove these theorems, we need some technical lemmas. First we note that, since A is finite dimensional, for $x \in A$, the spectrum $\mathrm{Sp}(x)$ is a finite set. For $\lambda \in \mathrm{Sp}(x)$, we shall write $e(\lambda, x)$, or simply e_λ if this does not cause confusion, for the idempotent

$$\frac{1}{2\pi i}\int_{\partial D_\lambda} (\zeta - x)^{-1}\, d\zeta,$$

where D_λ is a closed disc with λ as its center and $D_\lambda \cap \mathrm{Sp}(x) = \{\lambda\}$.

LEMMA 1. *If $x \in A$ and $\lambda \in \mathrm{Sp}(x) \setminus \mathrm{Sp}(\pi(x))$, then $e_\lambda \in J$.*

Proof. Since $\lambda \notin \mathrm{Sp}(\pi(x))$, we have

$$\pi(e_\lambda) = \frac{1}{2\pi i}\int_{\partial D_\lambda} (\zeta - \pi(x))^{-1}\, d\zeta = e(\lambda, \pi(x)) = 0.$$

Therefore $e_\lambda \in J$.

LEMMA 2. *If* $x \in A$, $\|x\| = 1$, $\lambda \in \mathrm{Sp}(x)$ *and* $|\lambda| = 1$, *then* $e_\lambda x = xe_\lambda = \lambda e_\lambda$ *and* $\|e_\lambda\| = 1$.

Proof. Consider the left regular representation $\Lambda : A \to BL(A)$ defined by $\Lambda(a)z = az$. Let $T = \Lambda(x)$. Then $\mathrm{Sp}(T) = \mathrm{Sp}(x)$. Let P_λ be the spectral projection $e(\lambda, T)$. Then it is easy to show that $\Lambda(e_\lambda) = P_\lambda$. Since $\|T\| = 1$ and $|\lambda| = 1$, it follows from **[4]** that the range of P_λ is the eigenspace $\{w \in A : Tw = \lambda w\}$ and $\|P_\lambda\| \le 1$. Hence $\|e_\lambda\| = 1$. From $TP_\lambda = \lambda P_\lambda$ and the fact that Λ is one-one, we have $xe_\lambda = \lambda e_\lambda$.

Proof of Theorem A. For each real number t, let

$$K_t = \{x \in eAe : \|x\| = 1 \quad \text{and} \quad \pi(x) = \exp(ith)\}.$$

Note that K_t is non-empty for each real t. (In fact, let $y \in A$ be such that $\pi(y) = \exp(ith)$. Then

$$1 = \|\exp(ith)\| = \inf\{\|y + z\| : z \in J\}.$$

By a compactness argument, we see that there exists some $z \in J$ such that $\|y + z\| = 1$. Now one can check that $e(y + z)e \in K_t$.) It is straightforward to verify that $K_s K_t \subseteq K_{s+t}$ and that the graph of the set-valued mapping $t \to K_t$ given by

$$\{(t, x) \in \mathbf{R} \times A : x \in K_t\}$$

is closed.

Let $x \in K_0$. Then $\pi(x) = 1$. Consider the idempotent $e_1 = e(1, x)$. By Lemma 2, $xe_1 = e_1$. Hence $ee_1 = exe_1 = xe_1 = e_1$. In the same way, we obtain $e_1 e = e_1$. On the other hand, by Lemma 1, $1 - e_1 \in J$ and, by Lemma 2, $\|e_1\| = 1$. The minimality of e implies $e_1 = e$. Thus, it follows that $x = xe = xe_1 = e_1 = e$. In other words, K_0 is the singleton $\{e\}$. From the relation $K_t K_{-t} \subseteq K_0 = \{e\}$ we can show that each K_t is a singleton, say $K_t = \{x_t\}$. From the fact that the graph of $t \to K_t$ is closed it follows that $t \to x_t$ is continuous. Thus we obtain a one-parameter group $\{x_t\}$ of unitary elements in eAe with $\pi(x_t) = \exp(ith)$. Let $\tilde{h} = \lim_{t \to 0}[(x_t - e)/it]$. Then $\tilde{h}$ is a hermitian element in eAe and $\pi(\tilde{h}) = h$.

Proof of Theorem B. Let x be an element in A such that $\|x\| = 1$ and $\pi(x) = u$. For $\lambda \in \mathrm{Sp}(x)$, we write e_λ for the idempotent $e(\lambda, x)$. Since u is unitary, $\mathrm{Sp}(u)$ is contained in the unit circle. Hence, by Lemma 1, if $\lambda \in \mathrm{Sp}(x)$ and $|\lambda| < 1$, then $e_\lambda \in J$. Let $F = \mathrm{Sp}(x) \cap \{\lambda : |\lambda| = 1\}$ and $e_F = \sum_{\lambda \in F} e_\lambda$. Then $1 - e_F \in J$ and, by Lemma 2,

$$x = \sum_{\lambda \in F} \lambda e_\lambda + z,$$

where $z = x(1 - e_F) \in J$, has spectral radius less than 1. Choose an increasing sequence $\{n_k\}$ of positive integers such that

(1) $m_k = n_{k+1} - n_k \to \infty$ as $k \to \infty$, and

(2) for all $\lambda \in F$, $\lambda^{n_k} \to \mu_\lambda$ for some μ_λ as $k \to \infty$.

Then $x^{m_k+1} = \sum_{\lambda \in F} \lambda^{n_{k+1}-n_k} e_\lambda + z^{m_k}$ which tends to $e_F = \sum_{\lambda \in F} e_\lambda$ as $k \to \infty$. Since $\|x^{m_k}\| \leq 1$ for all k, we have $\|e_F\| = 1$. Note that $x^{n_{k+1}}$ and $x^{n_{k-1}}$ tend to $\tilde{u} = \sum \lambda e_\lambda$ and $\tilde{v} = \sum \bar{\lambda} e_\lambda$ respectively. Hence we obtain $\|\tilde{u}\| \leq 1$, $\|\tilde{v}\| \leq 1$. Obviously $\tilde{u}\tilde{v} = \tilde{v}\tilde{u} = e_F$. Hence $\tilde{u}$ is a unitary element in $e_F A e_F$ with $1 - e_F \in J$ and $\pi(\tilde{u}) = u$. Here the conclusion is slightly different from the statement of the theorem. This can be adjusted by choosing x at the beginning that satisfies the additional condition $x \in eAe$, from which we can deduce $e = e_F$. The proof is complete.

REMARKS. In Theorem A, the assumption that A is finite dimensional can be replaced by a weaker one that J is finite dimensional with a modified proof.

2. From the proof of Theorem B it follows that if A is a finite dimensional algebra and x is an element in A with its spectrum contained in the unit circle and $\|x\| = 1$, then x is unitary.

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