ON THE NILPOTENT RANKS OF CERTAIN SEMIGROUPS OF
TRANSFORMATIONS

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1. Introduction. Let \( P_n \) be the semigroup of all partial transformations on the set \( X_n = \{1, \ldots, n\} \). As usual, we shall say that an element \( \alpha \) in \( P_n \) is of type \((k, r)\) or belongs to the set \([k, r]\) if \(|\text{dom } \alpha| = k\) and \(|\text{im } \alpha| = r\). The completion \( \alpha^* \) of an element \( \alpha \in [n - 1, n - 1] \) is an element in \([n, n]\) defined by

\[
ia^* = j, \quad x\alpha^* = x\alpha \text{ otherwise},
\]

where \( \{i\} = X_n \setminus \text{dom } \alpha \) and \( \{j\} = X_n \setminus \text{im } \alpha \).

For \( n \) even, the subsemigroup \( \text{SI}_n \) of \( P_n \) consisting of all strictly partial one–one transformations was proved to be nilpotent-generated by Gomes and Howie [2]. If \( n \) is odd, they showed that the nilpotents in \( \text{SI}_n \) generate \( \text{SI}_n \setminus \text{W}_{n-1} \), where \( \text{W}_{n-1} \) consists of all \( \alpha \in [n - 1, n - 1] \) whose completions are odd permutations.

Simultaneously and independently, Sullivan [7] showed that the subsemigroup \( \text{SP}_n \) of \( P_n \) consisting of all strictly partial transformations of \( X_n \) is nilpotent-generated if \( n \) is even. If \( n \) is odd, the nilpotents in \( \text{SP}_n \) generate \( \text{SP}_n \setminus \text{W}_{n-1} \).

The rank of a semigroup \( S \) is the cardinality of any subset \( A \) of minimal order in \( S \) such that \( \langle A \rangle = S \). If the generating set \( A \) consists of nilpotent elements only, then we shall refer to the cardinality of \( A \) as the nilpotent rank of \( S \). Since one of the semigroups we will be considering is an inverse semigroup, we would like to clarify the notion of a generating set in an inverse semigroup. Given a subset \( A \) in an inverse semigroup \( S \), we shall always want to consider the smallest inverse subsemigroup containing \( A \). In effect this is the set of all finite products of elements of \( A \) and their inverses. Following [3], we shall use the notation \( \langle \langle A \rangle \rangle \) for this inverse subsemigroup. Accordingly, by the rank of an inverse semigroup \( S \) we shall mean the cardinality of any subset \( A \) of minimal order in \( S \) such that \( \langle \langle A \rangle \rangle = S \).

Let \( N \) and \( M \) be the sets of all nilpotent elements in \( \text{SI}_n \) and \( \text{SP}_n \) respectively. In [3], Gomes and Howie proved that the rank and the nilpotent rank of \( \langle \langle N \rangle \rangle \) are both equal to \( n + 1 \) for all \( n \), and in [1], Garba showed that the rank and the nilpotent rank of \( \langle M \rangle \) are both equal to \( n + 2 \) for all \( n \). In Section 2 we generalize the results of Gomes and Howie [3] (in line with Howie and McFadden [6]) by showing that if \( 1 \leq r \leq n - 2 \) then the rank and the nilpotent rank of the inverse semigroup

\[
U(n, r) = \{ \alpha \in \text{SI}_n : |\text{im } \alpha| \leq r \}
\]

are both equal to \( (\binom{n}{r}) + 1 \). In Section 3 we generalize the results of Garba [1] by showing that if \( 1 \leq r \leq n - 2 \) then the rank and the nilpotent rank of the semigroup

\[
V(n, r) = \{ \alpha \in \text{SP}_n : |\text{im } \alpha| \leq r \}
\]

are both equal to \( (r + 1)S(n, r + 1) \), where \( S(n, r + 1) \) is the Stirling number of the second kind, defined by

\[
S(n, 1) = S(n, n) = 1, \quad S(n, r) = S(n - 1, r - 1) + rS(n - 1, r).
\]

For standard terms in semigroup theory see [4]. In all that follows, we consider \( n \geq 3 \).

2. One-one partial transformations.

Lemma 2.1. For all \( r \leq n - 2 \), we have
\[
J_r \subseteq (N \cap J_r)^2,
\]
where \( J_r = \{ \alpha \in \text{SL}_n : |\text{im} \alpha| = r \} \) is the \( \mathcal{J} \)-class of all elements in \( \text{SL}_n \) with rank \( r \).

Proof. The result is trivial for \( r \leq 1 \). If \( r \geq 2 \) then the result follows from Remark 3.16 in [2], where the authors prove that \( J_{n-2} \subseteq (N \cap J_{n-2})^3 \), and from Lemma 4.1 in the same paper, which states that if \( J_r \subseteq (N \cap J_k)^k \) then \( J_{r-1} \subseteq (N \cap J_{r-1})^k \) for \( 2 \leq r \leq n - 1 \).

It follows from this lemma that the nilpotents in \( J_r \) generate \( U(n, r) \).

Denote by \( P_r \) the principal factor \( U(n, r) / U(n, r - 1) \). Then \( P_r \) may be thought of in the usual way as \( J_r \cup \{0\} \). Also, \( P_r \) has \( \binom{\gamma}{r} \) non-null \( \mathcal{R} \)-classes corresponding to the \( \binom{\gamma}{r} \) possible domains of cardinality \( r \), and \( \binom{\gamma}{r} \) non-null \( \mathcal{L} \)-classes corresponding to the \( \binom{\gamma}{r} \) possible images. It is a Brandt semigroup isomorphic to \( B(S_r, \{1, \ldots, m\}) \), where \( S_r \) is the symmetric group on \( X_r \) and \( m = \binom{\gamma}{r} \). Hence, since the rank of \( S_r \) is known to be 2, it follows by Theorem 3.3 in [3] that \( P_r \) has inverse semigroup rank \( \binom{\gamma}{r} + 1 \).

From [2], we borrow the notation \( \langle a_1, a_2, \ldots, a_{r+1} \rangle \) (\( 1 \leq r \leq n - 1 \)) for the nilpotent \( \alpha \) with domain \( \{a_1, \ldots, a_r\} \) and image \( \{a_2, \ldots, a_{r+1}\} \) for which \( a_i \alpha = a_{i+1} \) \((i = 1, \ldots, r)\). We shall refer to these type of nilpotents as primitive in the next section.

Theorem 2.2. Let \( n \geq 3 \) and let \( r \leq n - 2 \). Then
\[
\text{rank} \langle \langle U(n, r) \rangle \rangle = \text{nilrank} \langle \langle U(n, r) \rangle \rangle = \begin{cases} \binom{\gamma}{r} + 1 & \text{if } r \geq 3, \\ \binom{\gamma}{r} & \text{if } r = 2, \\ n - 1 & \text{if } r = 1. \end{cases}
\]

Proof. From the fact that \( P_r \) (as an inverse semigroup) has rank \( \binom{\gamma}{r} + 1 \) it follows that rank \( \langle \langle U(n, r) \rangle \rangle \geq \binom{\gamma}{r} + 1 \). To complete the proof we must find a generating set of \( \langle \langle U(n, r) \rangle \rangle \) consisting of \( \binom{\gamma}{r} + 1 \) nilpotents.

Let \( A_1, A_2, \ldots, A_m \) be a list of the subsets of \( X_n \) of cardinality \( r \). Thus \( m = \binom{\gamma}{r} \). Let \( H_{A_i, A_j} \) denote the \( \mathcal{H} \)-class in \( J_r \) consisting of all the elements whose domain is \( A_i \) and image \( A_j \). Suppose that \( A_1 = \{1, 2, \ldots, r\} \). Then the \( \mathcal{H} \)-class \( H_{A_1, A_i} \) is the symmetric group on \( \{1, 2, \ldots, r\} \), and if \( r \geq 3 \) then it is generated by the elements \( \sigma \), \( \tau \), where
\[
\sigma = (1 2), \quad \tau = (1 2 \ldots r).
\]

We now show that each of \( \sigma \), \( \tau \) can be expressed as a product of nilpotents. For this purpose, we will suppose that \( A_2 = \{2, \ldots, r, r + 1\} \), \( A_3 = \{1, \ldots, r - 1, r + 1\} \) and \( A_4 = \{2, \ldots, r - 1, r + 1, r + 2\} \). The proof depends on whether \( r \) is odd or even. For \( r \) odd we have
\[
\sigma = \alpha_2^{-1} \beta \alpha_3 \quad \text{and} \quad \tau = \gamma_2^{-1} \alpha_2,
\]
where
\[
\begin{align*}
\alpha_2 &= \langle r + 1 r r - 1 \ldots 2 1 \rangle \in H_{A_2, A_1}, \\
\beta &= \langle r r - 2 r - 4 \ldots 3 r + 1 r - 1 \ldots 4 2 1 \rangle \in H_{A_2, A_3}, \\
\alpha_3 &= \langle r + 1 1 2 \ldots r \rangle \in H_{A_3, A_1}, \\
\gamma_2 &= \langle r + 1 r - 1 \ldots 2 r r - 2 \ldots 3 1 \rangle \in H_{A_2, A_1}.
\end{align*}
\]
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If for this case we now choose a nilpotent \( \alpha_i \in H_{A_i, A_1} \) for \( i = 4, \ldots, m \) in an arbitrary way, we see that

\[
\sigma, \tau, \alpha_2, \ldots, \alpha_m \in \langle \langle \alpha_2, \ldots, \alpha_m, \beta, \gamma_2 \rangle \rangle.
\]

By the remark before Theorem 3.3 in [3], the elements \( \sigma, \tau, \alpha_2, \ldots, \alpha_m \) generate \( P_r \). It follows that \( P_r \) and hence also \( U(n, r) \) is generated by the \( m + 1 \) nilpotents \( \alpha_2, \ldots, \alpha_m, \beta, \gamma_2 \) provided \( r \) is odd.

For \( r \) even we have

\[
\sigma = \alpha_3^{-1} \beta \alpha_4 \quad \text{and} \quad \tau = \gamma_4^{-1} \alpha_4,
\]

where

\[
\begin{align*}
\alpha_3 &= \| r + 1 \| 2 \ldots r - 2 r - 1 \| r \| \in H_{A_3, A_1}, \\
\beta &= \| r + 1 - 2 r + 3 \| 2 \ldots r - 5 \| 6 \ldots - 3 r - 4 r - 1 \| r - 2 \| \in H_{A_3, A_2}, \\
\alpha_4 &= \| r + 2 \| 2 4 \ldots r \| \cup \| r + 1 \| r - 1 \ldots 3 \| 1 \| \in H_{A_4, A_1}, \\
\gamma_4 &= \| r + 1 \| r - 2 \| r - 1 \| r - 4 \| 3 \| r - 6 \| 6 \ldots 3 \| 9 \| 6 \| 4 \| 5 \| 2 \| 3 \| r \| \cup \| r + 1 \| \in H_{A_4, A_1}.
\end{align*}
\]

In this case \( P_r \) and hence \( U(n, r) \) is generated by the \( m + 1 \) nilpotents \( \alpha_2, \ldots, \alpha_m, \beta, \gamma_4 \), where \( \alpha_i \in H_{A_i, A_1} \) are chosen arbitrarily for \( i = 2, 5, 6, \ldots, m \).

It now remains to show that the result is true for \( r = 2 \) and \( r = 1 \).

If \( r = 2 \), \( S_2 \) is cyclic and thus has only one generator. For this case we will suppose that \( A_1 = \{1, 2\} \) and \( A_m = \{n - 1, n\} \). The \( \mathcal{E} \)-class \( H_{A_1, A_1} \) is the symmetric group on \( A_1 \) and is generated by

\[
\sigma = (1 \ 2).
\]

Now,

\[
\sigma = \gamma_1^{-1} \alpha_m,
\]

where

\[
\begin{align*}
\alpha_m &= \| n - 1 \| 2 \| \cup \| n 1 \| \in H_{A_m, A_1}, \\
\gamma_m &= \| n - 1 \| 1 \| \cup \| n 2 \| \in H_{A_m, A_1}.
\end{align*}
\]

So, if we choose nilpotents \( \alpha_2, \ldots, \alpha_{m-1} \) as in the above cases, we see that \( \alpha_2, \ldots, \alpha_m, \gamma_m \) generate \( U(n, r) \). Thus \( U(n, r) \) has rank \( 1 + m - 1 = m \).

If \( r = 1 \), the symmetric group \( S_1 \) has rank 0, and it is easy to verify that the following \( n - 1 \) nilpotents generate \( U(n, r) \):

\[
\| 2 \ 1 \|, \| 3 \ 1 \|, \| 4 \ 1 \|, \ldots, \| n \ 1 \|.
\]

3. Partial transformations. The semigroup \( V(n, r) \) has \( r + 1 \) \( \mathcal{J} \)-classes, namely \( J_r, J_{r+1}, \ldots, J_0 \) (where \( J_0 \) consists of the empty map). For each \( t \) such that \( 1 \leq t \leq r \) we have

\[
J_t = \bigcup_{k=t}^{n-1} [k, t].
\]

The number of \( \mathcal{L} \)-classes in the \( \mathcal{J} \)-class \( J_t \) of \( V(n, r) \) is the number of image sets in \( X_n \) of cardinality \( r \), namely \( \binom{n}{r} \), and the number of \( \mathcal{R} \)-classes in \( J_t \) is the number of equivalence relations \( \rho \) on each of the subsets \( A \) of cardinality \( k \) (where \( n - 1 \geq k \geq r \)) for
which $|A/\rho| = r$, and this number is

$$
\sum_{k=r}^{n-1} \binom{n}{k} S(k, r) = \sum_{k=r}^{n} \binom{n}{k} S(k, r) - S(n, r) \\
= S(n + 1, r + 1) - S(n, r) \\
= (r + 1)S(n, r + 1).
$$

Like $U(n, r)$, the semigroup $V(n, r)$ is generated by the nilpotent elements in $J$, (see Lemma 2.3 in [1]). We also have from Lemma 3 in [6] that for $2 \leq r \leq n - 2$,

$$\text{rank}(V(n, r)) \equiv (r + 1)S(n, r + 1).$$

**Theorem 3.1.** For $n \geq 3$ and $2 \leq r \leq n - 2$, we have

$$\text{rank}(V(n, r)) = \text{nilrank}(V(n, r)) = (r + 1)S(n, r + 1).$$

The proof depends on the following lemma.

**Lemma 3.2.** Suppose that we can arrange the subsets $A_1, \ldots, A_m$ (where $m = \binom{n}{r}$ and $2 \leq r \leq n - 2$) of $X_n$ of cardinality $r$ in such a way that $|A_i \cap A_{i-1}| = r - 1$ for $i = 1, \ldots, m - 1$ and $|A_m \cap A_1| = r - 1$. Then there exist nilpotents $\alpha_1, \ldots, \alpha_p$ (where $p = (r + 1)S(n, r + 1)$) such that $\{\alpha_1, \ldots, \alpha_p\}$ is a set of generators for $V(n, r)$.

**Proof.** Notice first that every element $\alpha \in [k, r], r < k \leq n - 1$, is expressible as a product of a nilpotent in its own $\mathcal{R}$-class and an element in $[r, r]$. For

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_{r-1} & A_r \\ a_2 & a_3 & \cdots & a_r & x \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \cdots & b_{r-1} & b_r \end{pmatrix},$$

where

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r \\ b_1 & \cdots & b_r \end{pmatrix},$$

$a_i \in A_i$ for all $i \in \{2, \ldots, r\}$ and $x \in X_n \setminus \text{dom } \alpha$.

In the arrangement of our subsets $A_1, \ldots, A_m$ we shall assume that $A_1 = \{n - r + 1, n - r + 2, \ldots, n\}, A_2 = \{n - r, \ldots, n - 1\}$ and $A_m = \{1, n - r + 2, \ldots, n\}$. We shall also represent any two adjacent subsets $A_i, A_{i+1}$ by the two subsets $\{x_1, \ldots, x_{r-1}, y_i\}$ and $\{x_1, \ldots, x_{r-1}, z_{i+1}\}$, where $z_{i+1} \neq y_i$, and $z_{i+1}, y_i \neq x_i$ for any $i$. Define $H_{A_i, A_j}$ to consist of all elements $\alpha \in [r, r]$ for which $\text{dom } \alpha = A_i$ and $\text{im } \alpha = A_j$. For $i = 1, \ldots, m$ define a mapping $\xi_i \in H_{A_i, A_m}$ as follows:

$$\xi_1 = \begin{pmatrix} n - r + 1 & n - r + 2 & \cdots & n \\ 1 & n - r + 2 & \cdots & n \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} n - r & n - r + 1 & n - r + 2 & \cdots & n - 1 \\ n - r + 2 & 1 & n - r + \cdots & n \end{pmatrix},$$

and for $i = 2, \ldots, m - 1$ if

$$\xi_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_{r-1} & y_i \\ t_1 & t_2 & \cdots & t_{r-1} & t_r \end{pmatrix}.$$
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define

\[ \xi_{i+1} = \begin{pmatrix} x_1 & x_2 & \ldots & x_{r-1} & z \\ t_2 & t_3 & \ldots & t_r & t_1 \end{pmatrix}. \]

Then it is easy to see that the mapping

\[ \phi : B(S_r, \{1, \ldots, m\}) \to Q_r \]

defined by \((i, \eta, j)\phi = \xi_i \eta \xi_j^{-1}\) is an isomorphism. Here \(S_r\) is the symmetric group on \(\{1, n-r+2, \ldots, n\}\), \(Q_r\) is the principal factor

\[ [r, r]/\bigcup_{l=0}^{r-1} [l, l] = [r, r] \cup \{0\}. \]

From Proposition 2.4 in [1], the set

\[ T = \{(1, g_1, 1), (1, g_2, 2), (2, e, 3), \ldots, (m-1, e, m), (m, e, 1)\}, \]

where \(g_1 = (1 n-r+2 \ldots n)\), \(g_2 = (1 n-r+2)\) and \(e\) is the identity permutation in \(S_r\), generates \(B(S_r, \{1, \ldots, m\})\). Thus \(T\phi\) generates \(Q_r\) and hence \([r, r]\). If we now define

\[ \alpha_1 = \xi_1 g_2 \xi_2^{-1}, \alpha_i = \xi_i \xi_{i+1}^{-1} \quad \text{for } i = 2, \ldots, m-1 \]

and

\[ \beta = \xi_m \xi_1^{-1}, \quad \delta = \xi_1 g_1 \xi_1^{-1}, \]

we obtain a generating set \(\{\beta, \delta, \alpha_1, \ldots, \alpha_{m-1}\}\) of \([r, r]\), where

\[ \alpha_1 = \|n n-1 \ldots n-r+1 n-r\|, \]
\[ \alpha_i = \|y_i x_{r-1} \ldots x_1 z_{i+1}\| \quad \text{for } i = 2, \ldots, m-1 \]

are all nilpotents. On the other hand,

\[ \delta = \begin{pmatrix} n-r+1 & n-r+2 & \ldots & n-1 & n \\ n-r+2 & n-r+3 & \ldots & n & n-r+1 \end{pmatrix} \]

is clearly non-nilpotent. However if \(r\) is odd we have

\[ \delta = \alpha_1 \lambda_1, \quad (3.3) \]

where

\[ \lambda_1 = \|n-r n-r+2 \ldots n-1 n-r+1 n-r+3 \ldots n-2 n\|. \]

If \(r\) is even, and is of the form \(4q + 2(q \geq 0)\), then

\[ \delta = \alpha_1 \eta_1 \eta_2, \quad (3.4) \]

where

\[ \eta_1 = \|n-r+1 n-r+5 \ldots n-1 n-r+3 n-r+7 \ldots n-3 n-r n-r+4 \ldots n-2 1\|
\]
\[ \cup \|n-r+2 n-r+6 \ldots n\| \]

and

\[ \eta_2 = \|n-r n-1 n-3 \ldots n-r+3 n-r+1\|\cup \|1 n n-2 n-4 \ldots n-r+2\|. \]

If \(r\) is even and of the form \(4q(q \geq 1)\) then

\[ \delta = \alpha_1 \psi_1 \psi_2, \quad (3.5) \]
where
\[ \psi_1 = \|n - r + 1 n - r n - r + 3 n - r + 2 n - r + 5 n - r + 4 \ldots n - 1 n - 2 1\| \]
and
\[ \psi_2 = \|n - r n - r + 3 n - r + 2 n - r + 5 \ldots n - 2 n - r + 1\| \cup \|1 n\|. \]

Next, \( \beta \) may or may not be nilpotent. However, as \( \beta \in [r, r] \), if \( \beta \) is non-nilpotent, then by Lemma 2.1 it is expressible as a product of two nilpotents in \([r, r]\), say
\[ \beta = \xi \xi_2. \]

It is clear that \( \beta \xi_1 \) and \( \beta \xi_2 \), that is \( R_{\xi_1} = A_m \) and \( L_{\xi_2} = A_1 \).

We now define \( \lambda, \eta, \eta_2, \psi, \psi_2 \) and \( \xi_2 \) as follows:
\[
\begin{align*}
\lambda & = \lambda_1 \cup (1, n), \\
\eta & = \eta_1 \cup (n, 1), \\
\eta_2 & = \eta_2 \cup (n - r + 1, n), \\
\psi & = \psi_1 \cup (n, n - 2), \\
\psi_2 & = \psi_2 \cup (n, n - r + 1).
\end{align*}
\]

Before we define \( \xi_2 \), we note that from Theorem 2.8 in [2], \( \xi_2 \) can be expressed as a disjoint union of \( k \) primitive nilpotents, say
\[ \xi_2 = \mu_1 \cup \mu_2 \cup \ldots \cup \mu_k. \]

If \( k \geq 2 \), then assume
\[ \mu_1 = \|x_1 \ldots x_r\| \quad \text{and} \quad \mu_2 = \|y_1 \ldots y_r\| \]
and define \( \xi_2 \) as
\[ x_\xi_2 = x_\xi \quad \text{if} \quad x \in \text{dom} \xi \]
and
\[ x_\xi_2 = y_r. \]

On the other hand if \( k = 1 \) then \(|\text{dom} \xi_2 \cup \text{im} \xi_2| = r + 1\), and since \( r \leq n - 2 \) we have \( X_n \setminus (\text{dom} \xi_2 \cup \text{im} \xi_2) \) to be non-empty. Then define \( \xi_2 \) as
\[ \xi_2 = \xi_2 \cup (x, n - r + 1), \]
where \( x \in X_n \setminus (\text{dom} \xi_2 \cup \text{im} \xi_2) \).

Note that \( \lambda, \eta, \eta_2, \psi, \psi_2 \) and \( \xi_2 \) are distinct, and belong to \([r + 1, r]\). If we now replace \( \lambda, \eta, \eta_2, \psi, \psi_2 \) and \( \xi_2 \) by \( \lambda', \eta', \eta_2', \psi', \psi_2' \) respectively in equations (3.3)–(3.5) then it is easy to see that the equations remain unaltered. Since \( \beta, \xi_1, \xi_2 \) are all one-one and of the same height, we must have
\[ \text{dom} \beta = \text{dom} \xi_1, \quad \text{im} \xi_1 = \text{dom} \xi_2, \]
and since \( x, x \notin \text{dom} \xi_2 = \text{im} \xi_1 \) we conclude that
\[ \xi_1 \xi_2 = \xi_1 \xi_2'. \]

Now, if \( \beta \) is nilpotent then \( V(n, r) \) is generated by
\[ \{\beta, \lambda_1, \alpha_1, \ldots, \alpha_{p-3}\}, \quad \{\beta, \eta_1, \eta_2, \alpha_1, \ldots, \alpha_{p-3}\} \]
or

\{β, ψ', ψ'', α_1, \ldots, α_{p-3}\}

according to whether \(r\) is odd, even and of the form \(4q + 2\) \((q \geq 0)\) or even and of the form \(4q\) \((q \geq 1)\), and \(α_m, \ldots, α_{p-k}\) \((k = 2, 3)\) are chosen arbitrarily to cover all the \(R\)-classes in \(J_r\).

If \(β\) is non-nilpotent, then \(V(n, r)\) is generated by

\{ξ_1, ξ_2, λ', α_1, \ldots, α_{p-3}\}, \quad \{ξ_1, ξ_2, η_1, η_2, α_1, \ldots, α_{p-4}\}

or

\{ξ_1, ξ_2, ψ', ψ'', α_1, \ldots, α_{p-4}\}

according to whether \(r\) is odd, even and of the form \(4q + 2\) \((q \geq 0)\) or even and of the form \(4q\) \((q \geq 1)\), and \(α_m, \ldots, α_{p-k}\) \((k = 3, 4)\) are chosen arbitrarily to cover all the \(R\)-classes in \(J_r\).

To conclude the proof of Theorem 3.1, it remains to prove that the listing of the subsets of \(X_n\) of cardinality \(r\) as postulated in the statement of Lemma 3.2 can actually be carried out. Let \(n \geq 4\) and \(2 \leq r \leq n - 2\), and consider the following proposition.

\((P(n, r))\): there is a way of listing the subsets of \(X_n\) of cardinality \(r\) as \(A_1, A_2, \ldots, A_m\) \((m = \binom{n}{r})\), \(A_1 = \{n - r + 1, \ldots, n\}\), \(A_2 = \{n - r, \ldots, n - 1\}\), \(A_m = \{1, n - r + 2, \ldots, n\}\) such that \(|A_i \cap A_{i+1}| = r - 1\) for \(i = 1, \ldots, m - 1\) and \(|A_m \cap A_1| = r - 1\).

We shall prove this by a double induction on \(n\) and \(r\), the key step being a kind of Pascal’s Triangle implication.

\[P(n, 2) \quad \text{and} \quad P(n - 1, r - 1) \Rightarrow P(n, r).\]

First, however, we anchor the induction with two lemmas.

**Lemma 3.7.** \(P(n, 2)\) holds for every \(n \geq 4\).

**Proof.** Consider the following arrangement of the subsets of \(X_n\) of cardinality 2.

\[
\begin{align*}
\{1, 2\}, & \quad \{1, 3\}, \ldots, \quad \{1, n - 1\}, \quad \{1, n\}, \\
\{2, 3\}, & \quad \ldots, \quad \{2, n - 1\}, \quad \{2, n\}, \\
& \quad \vdots \\
\{n - 2, n - 1\}, & \quad \{n - 2, n\}, \\
& \quad \{n - 1, n\}.
\end{align*}
\]

If we denote the first row by \(R_1\), second row by \(R_2\), etc., then we note that the first entry in \(R_i\) is \(\{i, i + 1\}\) and the last entry is \(\{i, n\}\). Thus the number of elements in \(R_i\) is \(n - i\), and the total number of subsets in all the rows is

\[\sum_{i=1}^{n-1} (n - i) = \frac{n}{2} (n - 1) = \binom{n}{2}.\]

Hence above is a complete list of the subsets of \(X_n\) of cardinality 2.

Note that for any two subsets \(A_i, A_j\) in \(R_i, A_j \cup A_r = \{i\}\), and the intersection of the last entry in \(R_{i+1}\) with the first entry in \(R_i\) is \(\{i + 1\}\). Hence the following arrangement satisfies \(P(n, 2)\):

\[R_{n-1}, R_{n-2}, \ldots, R_{i+1}, R_i, \ldots, R_2, R_1.\]
That is, the list begins with all the subsets in $R_{n-1}$, followed by the subsets in $R_{n-2}$, followed by the subsets in $R_{n-3}$, and so on, until $R_1$ is reached.

**Lemma 3.8.** $P(n, n-2)$ holds for every $n \geq 4$.

**Proof.** Note that $P(4, 2)$ follows from Lemma 3.7. So we will assume that $n \geq 5$. Let $R'_i$ be the list of the complements of the subsets in $R_i$ (defined in the proof of Lemma 3.7) arranged in the same order as in $R_i$. Let $(R'_i)^{-1}$ be $R'_i$ arranged in the reverse order. For example

$$R_{n-2} = \{n-2, n-1\}, \{n-2, n\},$$
$$R'_{n-2} = \{1, \ldots, n-3, n\}, \{1, \ldots, n-3, n-1\},$$
$$(R'_{n-2})^{-1} = \{1, \ldots, n-3, n-1\}, \{1, \ldots, n-3, n\}.$$

Let $T' = \{1, 3\}, \{1, 4\}, \ldots, \{1, n-1\}$ and $T'' = R'_n \setminus \{(1, 2)', \{1, n\}'}$.

It is clear that, for any two subsets $A'_i$, $A'_j$ in $R'_i$, we have $|A'_i \cap A'_j| = n - 3$, and the intersection of the last subset in $R'_{i+1}$ and the first subset in $R'_i$ also contains $n - 3$ elements. We also have $n - 3$ elements in the intersection of the last subset in $R'_i$ with the first subset in $(R'_2)^{-1}$, and the same number of elements in the intersection of the last subset in $T''$ with the subset in $R'_{n-1}$. We now have the following arrangement satisfying $P(n, n-2)$:

$$A'_1, A'_2, T', R'_{n-1}, R'_{n-2}, \ldots, R'_3, (R'_2)^{-1},$$
where $A'_1 = \{1, 2\}'$ and $A'_2 = \{1, n\}'$.

**Lemma 3.9.** Let $n \geq 6$ and $3 \leq r \leq n-3$. Then $P(n-1, r-1)$ and $P(n-1, r)$ together imply $P(n, r)$.

**Proof.** From the assumption $P(n-1, r)$ we have a list $A_1, \ldots, A_m$ (where $m = \binom{n-1}{r-1}$) of the subsets of $X_{n-1}$ with cardinality $r$ such that $|A_i \cap A_{i+1}| = r-1$ for $i = 1, \ldots, m-1$, and

$$A_1 = \{n-r, \ldots, n-1\}, A_2 = \{n-r-1, \ldots, n-2\}, A_m = \{1, n-r+1, \ldots, n-1\}.$$  

From the assumption $P(n-1, r-1)$, we have a list $B_1, \ldots, B_t$ (where $t = \binom{n-1}{r-2}$) of subsets of $X_{n-1}$ of cardinality $r-1$ such that $|B_i \cap B_{i+1}| = r-2$ for $i = 1, \ldots, r-1$, and

$$B_1 = \{n-r+1, \ldots, n-1\}, B_2 = \{n-r, \ldots, n-2\}, B_t = \{1, n-r+2, \ldots, n-1\}.$$  

Let $B'_i = B_i \cup \{n\}$. Then

$$A_1, \ldots, A_m, B'_1, \ldots, B'_t$$
is a complete list of the subsets of $X_n$ of cardinality $r$. (Notice that $t + m = \binom{n}{r}$.) Now, arrange the above subsets as follows:

$$B'_1, A_1, A_m, \ldots, A_2, B'_2, \ldots, B'_t.$$  

Then it is easy to verify that this arrangement satisfies $P(n, r)$. Hence the induction is complete and we may deduce that $P(n, r)$ is true for all $n \geq 4$ and all $r$ such that $2 \leq r \leq n-2$.  

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