A proof of the "Theorem of the Means"

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Numerous proofs have been given of this familiar theorem,¹ which states that if \( a_1, a_2, \ldots, a_n \) are positive, and not all equal, then

\[
a_1^n + a_2^n + \ldots + a_n^n > na_1 a_2 \ldots a_n.
\]

The following is an elementary proof by induction, which I

¹ See e.g. Hardy, Littlewood & Pólya Inequalities, where many references will be found.
have not seen used before. It is, of course, not claimed to be
novel, and not likely to be so.

We have, easily,

\[ n \left( a_1^n + a_2^n + \ldots + a_n^n \right) - \left( a_1^{n-1} + a_2^{n-1} + \ldots + a_n^{n-1} \right) \left( a_1 + a_2 + \ldots + a_n \right) \]

\[ = \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} (a_p^{n-1} - a_q^{n-1}) \left( a_p - a_q \right) > 0 \]

since any term \((a_p^{n-1} - a_q^{n-1}) \left( a_p - a_q \right)\) on the right which does not vanish, is positive, being composed of two factors with the same sign. There must be at least one positive, as the \(a\)'s are not all equal.

Hence,

\[ \frac{a_1^n + a_2^n + \ldots + a_n^n}{a_1^{n-1} + a_2^{n-1} + \ldots + a_n^{n-1}} > \frac{a_1 + a_2 + \ldots + a_n}{n}. \]

(1)

Let us suppose, now, that the theorem of the means holds good for \(n - 1\). Then, omitting \(a_1, a_2, \ldots a_n\) in turn, there are \(n\) inequalities of the form

\[ a_2^{n-1} + a_3^{n-1} + \ldots + a_n^{n-1} > (n - 1) a_2 a_3 \ldots a_n. \]

As \(n - 1\) of the \(a\)'s can be equal, there may be equality in one, but not more than one, of these. Adding them all, and dividing by \(n - 1\), we get

\[ a_1^{n-1} + a_2^{n-1} + \ldots + a_n^{n-1} > a_1 a_2 \ldots a_n \left( 1/a_1 + 1/a_2 + \ldots + 1/a_n \right). \]

(2)

Finally, multiplying (1) and (2),

\[ a_1^n + a_2^n + \ldots + a_n^n > \frac{a_1 a_2 \ldots a_n (a_1 + a_2 + \ldots + a_n) (1/a_1 + 1/a_2 + \ldots + 1/a_n)}{n} \]

\[ > na_1 a_2 \ldots a_n, \text{ since, as is easily shewn} \]

\[ (a_1 + a_2 + \ldots + a_n) (1/a_1 + 1/a_2 + \ldots + 1/a_n) > n^2. \]

The theorem thus holds for \(n\). Being true when \(n = 2\), it is therefore, by induction, true for all \(n\).