SYMPLECTIC COMPLEX BUNDLES
OVER REAL ALGEBRAIC FOUR-FOLDS

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Abstract

Let $X$ be a compact affine real algebraic variety of dimension 4. We compute the Witt group of symplectic bilinear forms over the ring of regular functions from $X$ to $\mathbb{C}$. The Witt group is expressed in terms of some subgroups of the cohomology groups $H^{2k}(X, \mathbb{Z})$ for $k = 1, 2$.


1. Introduction

Let $X$ be an affine real algebraic variety, that is, $X$ is biregularly isomorphic to an algebraic subset of $\mathbb{R}^n$ for some $n$ (for definitions and notions of real algebraic geometry we refer to [3]). Denote by $\mathcal{R}(X, \mathbb{C})$ the ring of regular $\mathbb{C}$-valued functions on $X$ (cf. [3, page 279]). Thus if $X$ is an algebraic subset of $\mathbb{R}^n$ and $X_\mathbb{C}$ is its Zariski closure in $\mathbb{C}^n$, then $\mathcal{R}(X, \mathbb{C})$ is canonically isomorphic to the localization of the affine ring $A(X_\mathbb{C})$ of $X_\mathbb{C}$ with respect to the multiplicatively closed subset

$$S = \{ f \in A(X_\mathbb{C}) | f(X) \subset \mathbb{C} \setminus \{ 0 \} \}.$$ 

In this note we study symplectic (that is, skew-symmetric) nonsingular bilinear forms over $\mathcal{R}(X, \mathbb{C})$. More precisely, let $W^{-1}(\mathcal{R}(X, \mathbb{C}))$ denote the Witt group of symplectic bilinear forms over $\mathcal{R}(X, \mathbb{C})$ (cf. Section 2 or [1, 2, 11]).
In [4, 6] (cf. also Section 2) we have defined the graded subring

\[ H^{\text{even}}_{\text{alg}}(X, \mathbb{Z}) = \bigoplus_{k \geq 0} H^{2k}_{\text{alg}}(X, \mathbb{Z}) \]

of the cohomology ring \( H^{\text{even}}(X, \mathbb{Z}) \). Assuming that \( X \) is compact, nonsingular, \( \dim X = 4 \), we compute the group \( W^{-1}(\mathcal{H}(X, \mathbb{C})) \otimes \mathbb{Z}/2 \) and, in some cases, also the group \( W^{-1}(\mathcal{H}(X, \mathbb{C})) \) in terms of the groups \( H^{2k}_{\text{alg}}(X, \mathbb{Z}) \), \( k = 1,2 \). Combining this result with [4], we obtain that for “most” algebraic hypersurfaces \( X \) of the real projective space \( \mathbb{R}P^5 \) of sufficiently high degree, the group \( W^{-1}(\mathcal{H}(X, \mathbb{C})) \) is zero (the precise meaning of “most” is explained in Section 2). We also give examples of “exceptional” algebraic hypersurfaces \( X \) in \( \mathbb{R}P^5 \) of arbitrarily high degree with \( W^{-1}(\mathcal{H}(X, \mathbb{C})) \neq 0 \).

Let us recall that the real projective space \( \mathbb{R}P^n \) with its usual structure of an abstract real algebraic variety is in fact an affine variety [3, Theorem 3.4.4]. Hence every algebraic subvariety of \( \mathbb{R}P^n \) is also affine.

2. Results

Let \( A \) be a commutative ring with an identity element. A symplectic space over \( A \) is a pair \((P, s)\), where \( P \) is a finitely generated projective \( A \)-module and \( s : P \times P \to A \) is a bilinear nonsingular symplectic form (recall that \( s \) is said to be nonsingular if the homomorphism \( P \to P^* = \text{Hom}(P, A) \), \( x \to s(x, \cdot) \) is bijective). Every finitely generated projective \( A \)-module \( Q \) gives rise to a symplectic space \( H(Q) = (Q \oplus Q^*, h) \), where \( h((x, x^*), (y, y^*)) = x^*(y) - y^*(x) \) for \( x, y \) in \( Q \) and \( x^*, y^* \) in \( Q^* \). An isometry of symplectic spaces is an isomorphism of the underlying modules preserving the forms. The orthogonal sum of two symplectic space \((P_1, s_1)\) and \((P_2, s_2)\), denoted by \((P_1 \oplus P_2, s)\), where \( s((x_1, x_2), (y_1, y_2)) = s_1(x_1, y_1) + s_2(x_2, y_2) \) for \( x_1, y_1 \) in \( P_1 \) and \( x_2, y_2 \) in \( P_2 \). Two symplectic spaces \((P_1, s_1)\) and \((P_2, s_2)\) are said to be equivalent if there exist finitely generated projective \( A \)-modules \( Q_1 \) and \( Q_2 \) such that the symplectic spaces \((P_1, s_1) \perp H(Q_1)\) and \((P_2, s_2) \perp H(Q_2)\) are isometric. The set \( W^{-1}(A) \) of equivalence classes of symplectic spaces over \( A \) forms an abelian group with operation induced by orthogonal sum (we shall use additive notation). The equivalence class of \((P, s)\) in \( W^{-1}(A) \) will be denoted by \([P, s]\). The group \( W^{-1}(A) \), called the Witt group of symplectic bilinear forms over \( A \), is an interesting invariant of \( A \) (cf. [1, 2, 11]).

Now we need to recall some notions introduced in [4, 6].
Let $V$ be a quasi-projective nonsingular $n$-dimensional complex algebraic variety. One defines the natural ring homomorphism

$$cl: A^*(V) \to H^*(V, \mathbb{Z}),$$

where $A^*(V) = \bigoplus_{k \geq 0} A^k(V)$ is the Chow ring of $V$ and $H^*(V, \mathbb{Z})$ is the Čech cohomology of $V$, as follows. Let $Y \subset V$ be a closed irreducible subvariety of dimension $k$ and let $\{Y\}$ be the elements of $A^{n-k}(V)$ represented by $Y$. Denote by $[Y]$ the fundamental class of $Y$ in the Borel-Moore homology group $\tilde{H}^{BM}_2(Y, \mathbb{Z})$ (cf. [5] or [7, Chapter 19]). Then $cl(\{Y\})$ is the element of $H^{2n-2k}(V, \mathbb{Z})$ which corresponds, via Poincaré duality, to the image of $[Y]$ in $\tilde{H}^{BM}_2(V, \mathbb{Z})$ under the homomorphism $\tilde{H}^{BM}_2(Y, \mathbb{Z}) \to H^{BM}_2(V, \mathbb{Z})$ induced by the inclusion $Y \subset V$. Extending by linearity, $cl$ defines a natural homomorphism $cl: A^*(V) \to H^*(V, \mathbb{Z})$. We set

$$H^{2k}_{\text{alg}}(V, \mathbb{Z}) = cl(A^k(V)).$$

Now let $X$ be an affine nonsingular real algebraic variety and suppose for a moment that $X$ is embedded in $\mathbb{R}P^n$ as a locally closed subvariety. We shall consider $\mathbb{R}P^n$ as a subset of the complex projective space $\mathbb{C}P^n$. Let $X_C$ be the Zariski (complex) closure of $X$ in $\mathbb{C}P^n$ and let $U$ be a Zariski neighborhood of $X$ in the set of nonsingular points of $X_C$. We set

$$H^{2k}_{\text{C-alg}}(X, \mathbb{Z}) = H^*(i_U)(H^{2k}_{\text{alg}}(U, \mathbb{Z})),
H^{\text{even}}_{\text{C-alg}}(X, \mathbb{Z}) = \bigoplus_{k \geq 0} H^{2k}_{\text{C-alg}}(X, \mathbb{Z}),$$

where $H^*(i_U)$ is the homomorphism induced by the inclusion mapping $i_U: X \to U$. One easily sees that $H^{\text{even}}_{\text{C-alg}}(X, \mathbb{Z})$ does not depend on the choice of $U$ (cf. [4] and [6]).

Given a continuous complex vector bundle $\xi$ on $X$, let $c_k(\xi)$ denote its $k$th Chern class (cf. [10]). We shall consider $\mathcal{R}(X, \mathbb{C})$ as a subring of the ring $\mathcal{C}(X, \mathbb{C})$ of continuous $\mathbb{C}$-valued functions on $X$ (note that $\mathcal{R}(X, \mathbb{C})$ is dense in $\mathcal{C}(X, \mathbb{C})$ in the $C^0$ topology). If $P$ is a finitely generated projective $\mathcal{R}(X, \mathbb{C})$-module, then $\mathcal{C}(X, \mathbb{C}) \otimes P$ is a finitely generated projective $\mathcal{C}(X, \mathbb{C})$-module. We shall denote by $\xi_P$ the continuous complex vector bundle on $X$ associated with $\mathcal{C}(X, \mathbb{C}) \otimes P$ in the usual way (cf. [12]).

**Lemma 1.** Let $X$ be an affine nonsingular real algebraic variety.

(i) If $P$ is a finitely generated projective $\mathcal{R}(X, \mathbb{C})$-module, then $c_k(\xi_P)$ belongs to $H^{2k}_{\text{C-alg}}(X, \mathbb{Z})$ for $k \geq 0$.

(ii) If $v$ is in $H^2_{\text{C-alg}}(X, \mathbb{Z})$, then there exists an invertible $\mathcal{R}(X, \mathbb{C})$-module $L$ with $c_1(\xi_L) = v$. 

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PROOF. Both (i) and (ii) are quite straightforward consequences of the definition of $H^k_{\text{C-alg}}(X,\mathbb{Z})$; (i) is proved in [4, Theorem 5.3] (cf. also [6]), while (ii) follows from [4, Proposition 5.1, Remark 5.4] (cf. also the proof of Lemma 2 below).

**Lemma 2.** Let $X$ be a compact affine nonsingular real algebraic variety of dimension 4.

(i) For every element $u$ in $H^4_{\text{C-alg}}(X,\mathbb{Z})$, there exists a symplectic space $(P, s)$ over $\mathcal{R}(X,\mathbb{C})$ with $c_2(\xi_P) = u$.

(ii) If $(P, s)$ is a symplectic space over $\mathcal{R}(X,\mathbb{C})$ and $c_2(\xi_P) = 0$, then $(P, s)$ is isometric to $H(\mathcal{R}(X,\mathbb{C})^n)$, where $2n = \text{rank } P$.

**Proof.** First observe that every finitely generated projective $\mathcal{R}(X,\mathbb{C})$-module $M$ with $\text{rank } M \geq 3$ has a unimodular element. Indeed, since $\dim X = 4$, the complex vector bundle $\xi_M$ admits a nowhere zero continuous section (cf. [9, Chapter 8, Proposition 1.1]). This implies, from [13, Theorem 2.2(a)], that $M$ has a unimodular element.

In the proof of (i) we may assume that $X$ is a locally closed subvariety of $\mathbb{R}P^n$. Let $U$ be a Zariski neighborhood of $X$ in the set of nonsingular points of the Zariski (complex) closure of $X$ in $\mathbb{C}P^n$. By definition of $H^4_{\text{C-alg}}(X,\mathbb{Z})$, there exists an element $v$ in $A^2(U)$ such that $H^*(i)(\text{cl}(v)) = u$, where $H^*(i): H^4(U,\mathbb{Z}) \to H^4(X,\mathbb{Z})$ is the homomorphism induced by the inclusion mapping $i: X \to U$. Clearly, we may assume that $U$ is an affine variety (cf. for example the proof of [4, Proposition 5.1]). Now it follows from [7, Example 15.3.6] that there exists an algebraic (complex) vector bundle $\eta$ on $U$ with $C_1(\eta) = 0$ and $C_2(\eta) = v$, where $C_k(\cdot)$ stands for the $k$th Chern class with values in the Chow ring. Since $\text{cl} \circ C_k = c_k$ (cf. [5, (4.13)]), where this relation is proved for $k = 1$; by a standard argument, $\text{cl} \circ C_k = c_k$ must be true for all $k$), we obtain $c_1(\eta|X) = 0$ and $c_2(\eta|X) = u$, where the restriction $\eta|X$ is considered as a continuous complex vector bundle on $X$. It easily follows (cf. [4, Proposition 5.1]) that $\eta|X$ is topologically isomorphic to a vector bundle of the form $\xi_Q$ for some finitely generated projective $\mathcal{R}(X,\mathbb{C})$-module $Q$. By the remark at the beginning of the proof, $Q = P \oplus F$, where $F$ is free and $\text{rank } P = 2$. In particular,

$$c_1(\xi_P) = c_1(\xi_Q) = 0, \quad c_2(\xi_P) = c_2(\xi_Q) = u.$$  

Let $L = \det P$. Since $c_1(\xi_L) = c_1(\xi_P) = 0$, the bundle $\xi_L$ is topologically trivial (cf. [9, Chapter 16, Theorem 3.4]) and, by virtue of [13, Theorem 2.2(a)], $L$ is free.
In order to finish the proof of (i) it suffices to show that there exists a
symplectic nonsingular bilinear form on $P$. This however is obvious because
$\det P$ is free and rank $P = 2$.

Now we turn to the proof of (ii). First suppose that rank $P > 2$. Then $P$ has
a unimodular element and, by [2, (4.11.2)], $(P, s)$ is isometric to a symplectic
space of the form $(Q, t) \perp H(\mathcal{R}(X, \mathbb{C}))$. Since, obviously, $c_2(\xi_Q) = 0$, using
induction with respect to rank $P$, one reduces the proof to the case rank $P = 2$. In that case, $c_2(\xi_P) = 0$ implies that $\xi_P$ has a nowhere zero continuous
section (cf. [10, page 171, Problem 14-C]). Thus, by [13, Theorem 2.2(a)], $P$
has a unimodular element and, finally, by [2, (4.11.2)], $(P, s)$ is isometric to
$H(\mathcal{R}(X, \mathbb{C}))$.

Let $X$ be an affine nonsingular real algebraic variety. Observe that
$$G(X) = \{ 2u + v^2 | u \in H^4_{c-\text{alg}}(X, \mathbb{Z}), v \in H^2_{c-\text{alg}}(X, \mathbb{Z}) \}$$
is a subgroup of $H^4_{c-\text{alg}}(X, \mathbb{Z})$. Indeed, if $u_i$ are in $H^4_{c-\text{alg}}(X, \mathbb{Z})$ and $v_i$ are in
$H^2_{c-\text{alg}}(X, \mathbb{Z})$ for $i = 1, 2$, then
$$(2u_1 + v_1^2) - (2u_2 + v_2^2) = 2(u_1 - u_2 + v_1v_2 - v_2^2) + (v_1 - v_2)^2$$
is in $G(X)$.

For every finitely generated projective $\mathcal{R}(X, \mathbb{C})$-module $Q$, we have
$$c_2(\xi_{Q \oplus Q^*}) = c_2(\xi_Q \oplus \xi_{Q^*})
= c_2(\xi_Q) + c_2(\xi_{Q^*}) + c_1(\xi_Q)c_1(\xi_{Q^*})
= c_2(\xi_Q) + c_2((\xi_Q)^*) + c_1(\xi_Q)c_1((\xi_Q)^*)
= 2c_2(\xi_Q) - c_1(\xi_Q)^2$$
and hence, by Lemma 1(i), $c_2(\xi_{Q \oplus Q^*})$ is in $G(X)$. It easily follows (again
from Lemma 1(i)) that
$$\varphi_X : W^{-1}(\mathcal{R}(X, \mathbb{C})) \to H^4_{c-\text{alg}}(X, \mathbb{Z})/G(X)$$
$$\varphi_X([P, s]) = c_2(\xi_P) + G(X)$$
is a well-defined group homomorphism.

**Theorem 3.** Let $X$ be a compact affine nonsingular real algebraic variety
of dimension 4. Then the homomorphism
$$\varphi_X : W^{-1}(\mathcal{R}(X, \mathbb{C})) \to H^4_{c-\text{alg}}(X, \mathbb{Z})/G(X)$$
is surjective and
$$\ker \varphi_X = 2W^{-1}(\mathcal{R}(X, \mathbb{C})).$$

In particular,
$$W^{-1}(\mathcal{R}(X, \mathbb{C}))/2W^{-1}(\mathcal{R}(X, \mathbb{C})) \cong W^{-1}(\mathcal{R}(X, \mathbb{C})) \otimes \mathbb{Z}/2$$
is canonically isomorphic to $H^4_{C\text{-alg}}(X, \mathbb{Z})/G(X)$. Moreover, if $2H^4_{C\text{-alg}}(X, \mathbb{Z}) = 0$, then $\varphi_X$ is bijective.

**Proof.** It follows from Lemma 2(i) that $\varphi_X$ is surjective.

Now we turn to the proof of $\ker \varphi_X = 2W^{-1}(\mathcal{R}(X, \mathbb{C}))$.

Let $[P, s]$ be in $W^{-1}(\mathcal{R}(X, \mathbb{C}))$. Then

$$\varphi_X(2[P, s]) = c_2(\xi_P \oplus \xi_P) + G(X)$$

$$= c_2(\xi_P \oplus \xi_P) + G(X)$$

$$= 2c_2(\xi_P) + c_1(\xi_P)^2 + G(X) = 0.$$  

This shows that $2W^{-1}(\mathcal{R}(X, \mathbb{C}))$ is contained in $\ker \varphi_X$.

Suppose that $[P, s]$ is in $\ker \varphi_X$. Then $c_2(\xi_P) = 2u + v^2$, where $u$ is in $H^4_{C\text{-alg}}(X, \mathbb{Z})$ and $v$ is in $H^2_{C\text{-alg}}(X, \mathbb{Z})$. By Lemma 2(i), there exists a symplectic space $(Q, t)$ over $\mathcal{R}(X, \mathbb{C})$ such that $c_2(\xi_Q) = -u$. Also, by Lemma 1(ii), one can find an invertible $\mathcal{R}(X, \mathbb{C})$-module $L$ with $c_1(\xi_L) = v$. Let

$$(P', s') = (P, s) \perp (Q, t) \perp (Q, t) \perp H(L).$$

Then one obtains

$$c_2(\xi_{P'}) = c_2(\xi_P) + 2c_2(\xi_Q) - c_1(\xi_L)^2$$

$$= (2u + v^2) - 2u - v^2 = 0.$$  

By Lemma 2(ii), $[P', s'] = 0$ and hence $[P, s] = -2[Q, t]$. Thus $[P, s]$ is in $2W^{-1}(\mathcal{R}(X, \mathbb{C}))$, which shows that $\ker \varphi_X$ is contained in $2W^{-1}(\mathcal{R}(X, \mathbb{C}))$.

To finish the proof of the theorem, we note that if $2H^4_{C\text{-alg}}(X, \mathbb{Z}) = 0$, then, by Lemma 2(ii), $2W^{-1}(\mathcal{R}(X, \mathbb{C})) = 0$ and hence $\varphi_X$ is an isomorphism.

Theorem 3 immediately implies the following

**Corollary 4.** Let $X$ be a compact affine nonsingular real algebraic variety of dimension 4. Assume that each connected component of $X$ is nonorientable as a $C^\infty$ manifold. Then the groups $W^{-1}(\mathcal{R}(X, \mathbb{C}))$ and $H^4_{C\text{-alg}}(X, \mathbb{Z})/G(X)$ are canonically isomorphic.

**Proof.** Let $M$ be a connected component of $X$. Since $M$ is nonorientable, $H^4(M, \mathbb{Z}) \cong \mathbb{Z}/2$ (cf. [8, (23.28), (22.28), (26.18)]). It follows that $2H^4(X, \mathbb{Z}) = 0$ and hence $2H^4_{C\text{-alg}}(X, \mathbb{Z}) = 0$. Now it suffices to apply Theorem 3.

Our next result says that for a “generic” hypersurface $X$ of $\mathbb{R}P^5$ of sufficiently high degree, one has $W^{-1}(\mathcal{R}(X, \mathbb{C})) = 0$.

More precisely, let $n$ and $k$ be positive integers. Denote by $P(n, k)$ the projective space associated with the vector space of all homogeneous polynomials in $\mathbb{R}[x_0, \ldots, x_n]$ of degree $k$. If an element $H$ in $P(n, k)$ is represented
by a polynomial $G$, then $V(H)$ will denote the subvariety of $\mathbb{RP}^n$ defined by $G$.

**Theorem 5.** There exists a nonnegative integer $k_0$ such that, for every integer $k$ greater than $k_0$, one can find a subset $\Sigma_k$ of $P(5,k)$ which is a countable union of proper Zariski closed algebraic subvarieties of $P(5,k)$ and has the property that for every $H$ in $P(5,k) \setminus \Sigma_k$, the set $V(H)$ is empty or $V(H)$ is nonsingular, $\dim V(H) = 4$, and $W^{-1}(\mathcal{R}(V(H),\mathbb{C})) = 0$.

**Proof.** Let $n$ be an integer, $n \geq 3$. It is proved in [4, Theorem 4.10] (cf. also [6]) that there exists a positive integer $k_0$ such that for every integer $k$ greater than $k_0$, one can find a subset $\Sigma_k$ of $P(n,k)$ which is a countable union of proper Zariski closed algebraic subvarieties of $P(n,k)$ and has the property that for every $H$ in $P(n,k) \setminus \Sigma_k$, the set $V(H)$ is empty or $V(H)$ is nonsingular, $\dim V(H) = n - 1$, and $H^4_{\text{alg}}(V(H),\mathbb{Z})$ is equal to the image of the homomorphism

$$H^{\text{even}}(\mathbb{RP}^n,\mathbb{Z}) \to H^{\text{even}}(V(H),\mathbb{Z})$$

induced by the inclusion $V(H) \subset \mathbb{RP}^n$.

Recall that $H^{2k}(\mathbb{RP}^n,\mathbb{Z}) \cong \mathbb{Z}/2$ for $0 < 2k \leq n$. Moreover, if $n \geq 4$, then the nonzero element $u$ of $H^4(\mathbb{RP}^n,\mathbb{Z})$ is of the form $u = v^2$, where $v$ is the nonzero element of $H^2(\mathbb{RP}^n,\mathbb{Z})$. Hence $2H^{2k}_{\text{alg}}(V(H),\mathbb{Z}) = 0$ for $0 < 2k \leq n$ and $H^4_{\text{alg}}(V(H),\mathbb{Z}) = G(V(H))$ for $H$ in $P(n,k) \setminus \Sigma_k$.

With $n = 5$, the conclusion follows from Theorem 3.

**Remark 6.** Theorem 5 cannot be much improved. More precisely, for every positive integer $k_0$ there exists an integer $k$ greater than $k_0$ and an element $H_{2k}$ in $P(5,2k)$ such that $V(H_{2k})$ is a nonsingular algebraic hypersurface of $\mathbb{RP}^5$ and $W^{-1}(\mathcal{R}(V(H_{2k}),\mathbb{C})) \neq 0$. Let $H_{2k}$ be the element of $P(5,2k)$ represented by the polynomial $x_0^{2k} - \sum_{i=1}^5 x_i^{2k}$. Clearly, $V(H_{2k})$ is a nonsingular algebraic hypersurface of $\mathbb{RP}^5$ diffeomorphic to the 4-dimensional sphere $S^4$. Moreover, by [4, Proposition 4.8],

$$H^4_{\text{alg}}(V(H_{2k}),\mathbb{Z}) = H^4(V(H_{2k}),\mathbb{Z}) \cong \mathbb{Z}.$$

Since $H^2(V(H_{2k}),\mathbb{Z}) \cong H^2(S^4,\mathbb{Z}) = 0$, one obtains

$$G(V(H_{2k})) = 2H^4_{\text{alg}}(V(H_{2k}),\mathbb{Z}).$$

Hence, by Theorem 3, $W^{-1}(\mathcal{R}(V(H_{2k}),\mathbb{C})) \otimes \mathbb{Z}/2$ is isomorphic to $\mathbb{Z}/2$, and $W^{-1}(\mathcal{R}(V(H_{2k}),\mathbb{C})) \neq 0$. 

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