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FINITE COVERINGS BY 2-ENGEL GROUPS

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Baer's characterisation of central-by-finite groups as groups possessing a finite covering by abelian subgroups is the starting point for this investigation. We characterise groups with a finite covering by 2-Engel subgroups as groups for which the subgroup of right 2-Engel elements has finite index; and the groups having a finite covering by normal 2-Engel subgroups are exactly the 3-Engel groups among those having a finite covering by 2-Engel subgroups. The second centre of a group having a finite covering by class two subgroups does not necessarily have finite index. However, a group has a finite covering by subgroups in a variety containing all cyclic groups if the margin of this variety in the group has finite index.

1. INTRODUCTION AND RESULTS

Bernhard Neumann [7],suggested the following problem:

Given a group G covered by finitely many subgroups $H_1, H_2, \ldots, H_n$ with intersection D. If $H_1, H_2, \ldots, H_n$ possess a certain property $\mathcal{E}$, what can be said about D in relation to G, or about G itself?

His attention was drawn to this problem by the characterisation of central-by-finite groups given by Reinhold Baer:

THEOREM A. ([7, 9, 4.16]) *A group is central-by-finite if and only if it has a finite covering consisting of abelian subgroups.*

In [7, Section 6], B.H. Neumann raises the question as to whether Baer's result can be extended to finite coverings by Engel-groups and k-Engel groups. To give an affirmative answer in the case of finite coverings by 2-Engel subgroups is the main topic of this paper.

We define $\varepsilon_k(x,y) = [x,_k y] = [[x,_{k-1} y], y]$ as the k-Engel word, where $\varepsilon_1(x,y) = [x,_1 y] = [x,y]$ is the commutator of x and y. An element a in a group G is a right k-Engel element if $[a,_k x] = 1$ for all x in G. A group is a k-Engel group if $[x,_k y] = 1$ for all $x, y \in G$. Let G be a group and let

$$L(G) = \{\, a \in G; \forall x \in G\ [a,_2 x] = 1 \,\}$$

be the set of right 2-Engel elements. In [4], W. Kappe has shown that $L(G)$ is a characteristic subgroup of G. For finite coverings by 2-Engel groups we can now formulate our result.

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THEOREM 1. *A group G is the union of finitely many 2-Engel subgroups if and only if $G/L(G)$ is finite.*

It remains an open question in which way Baer's characterisation can be extended to finite coverings by k-Engel groups, particularly in light of the fact that for $k \geqslant 3$ the set of right k-Engel elements does not necessarily form a subgroup (see [**5**]).

In [**1**], Theorem A was extended to the case of finite coverings by abelian normal subgroups as follows.

THEOREM B. ([**1**, Theorem 2]) *A group G is the union of finitely many abelian normal subgroups if and only if $G/Z(G)$ is finite and G is a 2-Engel group.*

A direct analogue for finite coverings by normal 2-Engel subgroups is the following theorem.

THEOREM 2. *A group G is the union of finitely many normal 2-Engel subgroups if and only if $G/L(G)$ is finite and G is a 3-Engel group.*

This theorem is based on Theorem 1, a characterisation of 3-Engel groups given in [**3**], and a corollary which is in fact another characterisation of 3-Engel groups. (See Theorem E and Corollary 6.) Theorem A brings to mind a familiar result which is essentially due to Schur [**10**]. For other sources we refer to [**9**, page 102].

THEOREM C. *If G is a group whose centre has index n, then G' is finite and $(G')^n = 1$.*

In light of this result and Theorem 1, we may ask whether the finiteness of $G/L(G)$ implies that the word subgroup $\varepsilon_2(G) = \langle [x,_2 y]\,;\ x, y \in G\rangle$ is finite. However, this is not the case. In the last section a group H is constructed with $H/L(H)$ finite but $\varepsilon_2(H)$ is not finite (Proposition 8, (iv) and (viii)).

In this context we may raise the question as to whether the k-th centre has finite index in a group if the group has a finite covering by subgroups of nilpotency class k. The question has a negative answer, as shown by the same counterexample. The group H has a finite covering by subgroups of class 2, but $Z_2(H)$ does not have finite index in H (Proposition 8, (v) and (vi)).

It is however true that the finite index of the k-th centre implies that the group has a finite covering by subgroups of class k. This is a consequence of our next result, relating the finite index of a suitable margin to the existence of a finite covering by subgroups in a variety $\mathcal{V}$.

The margin of a word was introduced by P. Hall in [**2**]. Let $\psi(x_1, \ldots, x_n)$ be a word in the variables $x_1, \ldots, x_n$. The i-th partial margin of ψ in a group G is defined as

$$\psi_i^*(G) = \{ a \in G; \psi(g_1, \ldots, ag_i, \ldots, g_n) = \psi(g_1, \ldots, g_i, \ldots, g_n) \text{ for } g_1, \ldots, g_n \in G \}.$$

The margin of ψ in G is defined as

$$\psi^*(G) = \bigcap_{i=1}^{n} \psi_i^*(G).$$

The partial margins, and thus the margin of a word in a group G, are characteristic subgroups in G.

Let V be a set of words and $\mathcal{V}$ the variety of groups defined by the set V. Consider

$$\psi_V^*(G) = \bigcap_{\psi \in V} \psi^*(G).$$

We say that $\psi_V^*(G)$ is the margin of the variety $\mathcal{V}$ in G, Obviously $\psi_V^*(G)$ is in $\mathcal{V}$.

The following result holds for a large class of words, the class of all commutator words (see [8, page 4]).

THEOREM 3. *Let V be a set of words and let G be a group. Assume that the variety $\mathcal{V}$ defined by V contains all cyclic groups. Then $G/\psi_V^*(G)$ finite implies that G has a finite covering by subgroups in $\mathcal{V}$.*

In general, the finiteness of $G/\psi_V^*(G)$ does not imply that G has a finite covering by subgroups in $\mathcal{V}$, as the following example shows. Let $\psi(x) = x^2$. It can be shown easily that for any group G

$$\psi^*(G) = \{ a \in Z(G);\ a^2 = 1 \}.$$

Consider $G = F \times E_2$, where F is a finite group of odd order greater than one, and E_2 an elementary abelian 2-group which is not necessarily finite. Now $\psi^*(G) = E_2$, and $G/E_2 = F$ is finite. But G is not the union of finitely many elementary abelian 2-groups. Observing that $\gamma_k(x_1, \ldots, x_k) = [x_1, \ldots, x_k]$, the lower central word of weight k, and the k-Engel word $\varepsilon_k(x, y)$ are commutator words, and that the respective margins in a group G are $Z_{k-1}(G)$ (see [2]) and $\varepsilon_k^*(G)$, we obtain the following corollary.

COROLLARY 4. *Let G be a group with $G/Z_k(G)$ or $G/\varepsilon_k^*(G)$ finite. Then G has a finite covering by subgroups of nilpotency class k, or by k-Engel subgroups, respectively.*

The converse of Theorem 3, that is, that the existence of a finite covering of a group by subgroups in a variety $\mathcal{V}$ which contains all cyclic groups implies the finiteness of the index of $\psi_V^*(G)$, is rarely true. As already mentioned, the counterexample H has a finite covering by class 2 subgroups, however $Z_2(H)$ does not have finite index. In addition, $\varepsilon_2^*(H)$, the 2-Engel margin of H, does not have finite index, but H has a finite covering by 2-Engel subgroups (Proposition 8, (iv) and (vi)).

2. Some Preliminary Steps

The following result of Neumann is basic in the investigation of finite coverings of groups by subgroups. It allows us to assume that all subgroups in a finite covering of a group have finite index.

THEOREM D. ([6, 4.4]) *Let* $G = \bigcup_{i=1}^{n} H_i g_i$ *where* $H_1, \ldots, H_n$ *are (not necessarily distinct) subgroups of* G. *Then if we omit from the union any coset* $H_i g_i$ *for which* $[G : H_i]$ *is infinite, the union of the remaining cosets is still all of* G.

The next lemma collects essential facts about 2-Engel elements needed throughout this paper.

LEMMA 5. *([4]) Let* $a \in L(G)$. *Then:*

(i) $[x,_2 a] = 1$ *for all* $x \in G$;

(ii) $[[x,y],[x,a]] = 1$ *for all* $x, y \in G$;

(iii) $[x, a^k] = [x^k, a] = [x, a]^k$ *for all* $x \in G$, $k \in \mathbb{Z}$.

Denote by $\langle x^G \rangle$ the normal closure of an element x in G, and let $c(G)$ denote the nilpotency class of a group G. The following characterisation of 3-Engel groups was given earlier by W. Kappe and the author.

THEOREM E. *([3]) The following conditions on a group* G *are equivalent:*

(i) $c(\langle x^G \rangle) \leqslant 2$ *for all* $x \in G$;

(ii) $\langle x^G \rangle$ *is a 2-Engel group for all* $x \in G$;

(iii) G *is a 3-Engel group.*

Here we prove the following corollary.

COROLLARY 6. *A group* G *is a 3-Engel group if and only if* $\langle x^G \rangle L(G)$ *is a 2-Engel group for all* $x \in G$.

PROOF: It suffices to show that for any 3-Engel group G the normal subgroup $\langle x^G \rangle L(G)$ is a 2-Engel group for all $x \in G$, that is $[ua,_2 vb] = 1$ for all $u, v \in \langle x^G \rangle$ and all $a, b \in L(G)$. The other direction follows trivially from Theorem E.

By [11], we have $L(G) = \varepsilon_{2,1}^*(G)$, where

$$\varepsilon_{2,1}^*(G) = \{ a \in G; [ax,_2 y] = [x,_2 y] \text{ for all } x, y \in G \},$$

the first partial 2-Engel margin. Thus for $a \in L(G)$ we observe that $[ua,_2 vb] = [u,_2 vb]$. The usual commutator expansion and observing that $c(\langle x^G \rangle) = 2$, as well as Lemma 5 (i), yield

$$\begin{aligned}[u,_2 vb] &= [u,b,vb]^{[u,v]^b}[[u,v]^b, vb] \\ &= [u,_2 b] \cdot [u,b,v]^b \cdot [[u,v]^b, b] \cdot [[u,v],v]^b \\ &= ([u,b,v][u,v,b])^b.\end{aligned}$$

Thus we arrive at

$$[ua,_2 vb] = ([u,b,v][u,v,b])^b. \tag{1}$$

Let $u = \prod_{i=1}^{n} x^{\delta_i g_i}$ and $v = \prod_{i=1}^{m} x^{\varepsilon_i h_i}$ with $\delta_i, \varepsilon_i = \pm 1$ and $g_i, h_i \in G$. Observing that $[u,b,v] \in Z(\langle x^G \rangle)$, we obtain by linear expansion that $[u,b,v]$ is a product of commutators of the form $[x^{\pm g}, b, x^{\pm h}]$ with $g, h \in G$. Without loss of generality it suffices to show that $[x^g, b, x] = 1$ for $x, g \in G$, $b \in L(G)$. But this is an immediate consequence of Lemma 5 (ii). It follows that each of the factors in the expansion of $[u,b,v]$ equals 1. A similar argument shows $[u,v,b] = 1$. By (1) we conclude that $[ua,_2 vb] = 1$. ∎

The following lemma is needed for the verification of certain properties of the counterexample in Proposition 8.

Lemma 7. *Let G be a group of class 3 and $a \in L(G)$. Then $a \in \varepsilon_2^*(G)$ if and only if $[a,x,y]^3 = 1$ for all $x, y \in G$.*

Proof: By linear expansion and observing Lemma 5 (i), we obtain

$$[x,_2 ay] = [x,y,a][x,a,y][x,_2 y]. \tag{2}$$

Using $1 = [a,_2 xy] = [a,x,y][a,y,x]$ together with the Jacobi identity yields

$$[x,y,a] = [a,x,y]^{-2}.$$

This together with (2) implies

$$[x,_2 ay] = [a,x,y]^{-3}[x,_2 y]. \tag{3}$$

If $[a,x,y]^3 = 1$, it follows that $[x,_2 ay] = [x,_2 y]$. By [11], we have already $[xa,_2 y] = [x,_2 y]$ for all $x, y \in G$, $a \in L(G)$. Hence $a \in \varepsilon_2^*(G)$. Conversely, if $a \in \varepsilon_2^*(G)$, we obtain $[a,x,y]^3 = 1$ by (3). ∎

3. Proofs of the Theorems

Proof of Theorem 1: Suppose $G = \bigcup_{i=1}^{n} H_i$ with H_i a 2-Engel group. By Theorem D, we can assume that $[G : H_i] < \infty$ for all H_i in the covering. Thus $[G : D] < \infty$ for $D = \bigcap_{i=1}^{n} H_i$. Let $h \in D$ and $g \in G$. Since $g \in H_i$ for some i and $D \subseteq H_i$, we have $[h, g, g] = 1$, as H_i is a 2-Engel group. Thus $D \subseteq L(G)$, and hence $G/L(G)$ is finite.

Conversely, assume that $G/L(G)$ is finite. Consider $[y, x, x]$ with $y = g^i a$, $x = g^j b$ for $g \in G$ and $a, b \in L(G)$, $i, j \in \mathbb{Z}$. Observing again that $L(G) = \varepsilon^*_{2,1}(G)$, we have

$$[y, x, x] = [g^i, x, x] = [g^i, b, x].$$

Further expansion and use of Lemma 5 (iii) and (i) yield

$$[[g^i, b], g^j b] = [g^i, b, b]\left([[g, b], g]^b\right)^{ij} = 1.$$

We conclude that $[y,_2 x] = 1$ for all $y, x \in \langle g \rangle L(G)$. Thus $\langle g \rangle L(G)$ is a 2-Engel group.

Choose a transversal $T = \{g_1, \ldots, g_n\}$ of $L(G)$ in G. Then $G = \bigcup_{i=1}^{n} \langle g_i \rangle L(G)$, since each $g \in G$ can be written as $g = g_i w$ for some $w \in L(G)$ and some $g_i \in T$. By the above, it follows that G has a covering by finitely many 2-Engel subgroups. ∎

Proof of Theorem 2: Let $G = \bigcup_{i=1}^{n} N_i$ with N_i a normal 2-Engel subgroup of G. It follows immediately by Theorem 1 that $G/L(G)$ is finite. For every $x \in G$ there exists an N_i in the normal covering of G with $\langle x^G \rangle \subseteq N_i$. Hence every normal closure $\langle x^G \rangle$ is 2-Engel as a subgroup of a 2-Engel group N_i. It follows by Theorem E that G is a 3-Engel group.

Conversely, assume that G is a 3-Engel group and that $G/L(G)$ is finite. Corollary 6 implies that $\langle x^G \rangle L(G)$ is a normal 2-Engel subgroup for every x in G. Choose a transversal $T = \{g_1, \ldots, g_n\}$ of $L(G)$ in G. Since each $g \in G$ can be written as $g_i w$ for some $w \in L(G)$ and some $g_i \in T$ we obtain that $G = \bigcup_{i=1}^{n} \langle g_i^G \rangle L(G)$. By the above, it follows that G has a covering by finitely many normal 2-Engel subgroups. ∎

Proof of Theorem 3: For $g \in G$ let $H_g = \langle g \rangle \psi^*_V(G)$. Every element in H_g can be written as $g^i a$ with $a \in \psi^*_V(G)$ and $i \in \mathbb{Z}$. Let $\psi(x_1, \ldots, x_n) \in V$. For $a_j \in \psi^*_V(G)$, $i_j \in \mathbb{Z}$, $j = 1, \ldots, n$, we have $\psi(g^{i_1} a_1, \ldots, g^{i_n} a_n) = \psi(g^{i_1}, \ldots, g^{i_n}) = 1$, since $\langle g \rangle \in \mathcal{V}$. Thus $H_g \in \mathcal{V}$ for all g in G.

Choose a transversal $T = \{g_1, \ldots, g_m\}$ of $\psi^*_V(G)$ in G. Then $G = \bigcup_{j=1}^{m} \langle g_j \rangle \psi^*_V(G)$, since each $g \in G$ can be written as $g_j a$ for some $a \in \psi^*_V(G)$ and some $g_j \in T$. By the above, each $\langle g_j \rangle \psi^*_V(G)$ is in $\mathcal{V}$. Thus G has a finite covering by subgroups in $\mathcal{V}$. ∎

4. A Counterexample

For every prime p a p-group H of class 3 is constructed whose relevant properties are stated in Proposition 8. The construction of H follows the usual practice. Starting from a group isomorphic to the commutator subgroup of H, we will reach H by three split extensions.

Construction of the Counterexample. Let E_p denote an elementary abelian p-group of countable rank, and T_9 the direct sum of a countable number of cyclic groups of order 9. Set $V = \langle v_1, v_2, \dots \rangle$, $W = \langle w_1, w_2, \dots \rangle$ and $Z = \langle z_1, z_2, \dots \rangle$ with

$$V \cong W \cong Z \cong \begin{cases} E_p & \text{for } p \neq 3, \\ T_9 & \text{for } p = 3. \end{cases}$$

Define $X = \langle u \rangle \times V \times W \times Z$ with $\langle u \rangle \cong C_p$ for $p \neq 3$, and $\langle u \rangle \cong C_9$ for $p = 3$.

Let $A = [X]\langle a \rangle$, the semidirect product of X with a cyclic group $\langle a \rangle$, where $\langle a \rangle \cong C_p$ for $p \neq 3$, and $\langle a \rangle \cong C_9$ for $p = 3$. The automorphism induced by a on X has order p, or 9, respectively. The action of a on the generators of X is given as follows:

$$(4) \qquad \begin{cases} [u, a] = 1, [w_i, a] = z_i^{-1}, & i = 1, 2, \dots \\ [v_i, a] = [z_i, a] = 1, & i = 1, 2, \dots. \end{cases}$$

The defining relations of A are those of X, (4), and $a^p = 1$ for $p \neq 3$, or $a^9 = 1$ for $p = 3$, respectively.

Similarly, let $B = [A]\langle b \rangle$ with

$$\langle b \rangle \cong \begin{cases} C_p & \text{for } p \neq 3 \\ C_9 & \text{for } p = 3, \end{cases}$$

where b induces an automorphism of order p or 9, respectively, on A. The action of b on the generators of A is given as follows:

$$(5) \qquad \begin{cases} [u, b] = 1, \quad [w_i, b] = [z_i, b] = 1, & i = 1, 2, \dots \\ [a, b] = u, \quad [v_i, b] = z_i, & i = 1, 2, \dots. \end{cases}$$

The defining relations of B are those of A, (5), and $b^p = 1$ for $p \neq 3$, or $b^9 = 1$ for $p = 3$, respectively.

For the last extension we set

$$C = \langle c_1, c_2, \dots \rangle \cong \begin{cases} E_p & \text{for } p \neq 3, \\ T_9 & \text{for } p = 3. \end{cases}$$

Let $H = [B] \cdot C$. The elements of C induce automorphisms of order p for $p \neq 3$, or order 9 for $p = 3$.

The action of the generators of C on the generators of B is given as follows:

$$\text{(6)} \qquad \begin{cases} [b, c_i] = w_i, \quad [a, c_i] = v_i, \quad [u, c_i] = z_i^2, & i = 1, 2, \dots, \\ [v_j, c_i] = [w_j, c_i] = [z_j, c_i] = 1, & i, j = 1, 2, \dots. \end{cases}$$

The defining relations of H are those of B, (6) and those of C. This concludes the construction of H.

For the notation in Proposition 8 we refer to that used in the construction of H.

PROPOSITION 8. *Let H be the group constructed above. Then:*

(i) $H = \langle a, b, c \rangle$ *and* $c(H) = 3$, *precisely.*

(ii) $\exp H = \begin{cases} p & \text{for } p \geqslant 5, \\ p^2 & \text{for } p = 2, 3; \end{cases}$

(iii) $L(H) = \begin{cases} C \cdot H' & \text{for } p \neq 3, \\ \langle a^3, b^3, C \cdot H' \rangle & \text{for } p = 3; \end{cases}$

(iv) $H/L(H)$ *is finite, and H is the union of finitely many (normal) 2-Engel subgroups;*

(v) *H is the union of finitely many (normal) class 2 subgroups;*

(vi) $H/\varepsilon_2^*(H)$ *and* $H/Z_2(H)$ *are not finite;*

(vii) $\varepsilon_2(H)$ *is not finite.*

PROOF: The verification of (i) and (ii) is straightforward and will be omitted here.

(iii) Let $h = a^\alpha b^\beta c \cdot x \in H$ with $c \in C$, $x \in X$, $\alpha, \beta \in \mathbb{Z}$. The relations of H together with $c(H) = 3$ imply that

$$[c_i, h, h] = ([c_i, a, b][c_i, b, a])^{\alpha\beta} = \left(z_i z_i^{-1}\right)^{-\alpha\beta} = 1.$$

Hence $C \subseteq L(H)$. Since $H' \subseteq L(H)$ because of $c(H) = 3$, it follows that $CH' \subseteq L(H)$.

To establish the rest of the claim, let $x = a^\alpha b^\beta c h'$ with $c \in C$, $h' \in H'$, and $\alpha, \beta \in \mathbb{Z}$. Using the relations of H, we obtain for $s, t \in \mathbb{Z}$

$$\text{(7)} \qquad [a^s b^t, x, x] = z^{3(\beta s - \alpha t)} \text{ for some } z \in Z.$$

Let $p = 3$. If $s = 3$ and $t = 0$, or $s = 0$ and $t = 3$, we have $[a^s b^t, x, x] = 1$ for all $x \in H$. Hence $a^3, b^3 \in L(H)$ in this case.

Conversely, for any p, let $0 \leqslant s \leqslant p-1$, $0 \leqslant t \leqslant p-1$, and set $c = c_1$. Then (7) becomes

$$[a^s b^t, x, x] = z_1^{3(\beta s - \alpha t)}.$$

It can be easily seen that for given s and t, not both equal to zero, there exist integers α, β such that $z_1^{3(\beta s - \alpha t)} \neq 1$, hence $a^s b^t \notin L(H)$ in this case. It follows that $L(H) \subseteq CH'$ for $p \neq 3$, and $L(H) \subseteq \langle a^3, b^3, CH' \rangle$ for $p = 3$.

(iv) The finiteness of $H/L(H)$ is an immediate consequence of (iii). The rest follows by Theorem 1, Theorem 2, and the fact that $c(H) = 3$.

(v) First let $p \neq 3$. The 2-Engel group $\langle g \rangle L(H)$, $g \in H$, is of nilpotency class 2. Thus the finite covering by 2-Engel groups resulting from Theorem 1 is a finite covering by (normal) subgroups of class 2.

This is not the case if $p = 3$, for example for $a, b^3, c_1 \in \langle a \rangle L(H)$ we have $[a, b^3, c_1] = z_1^6 \neq 1$. Thus $c(\langle a \rangle L(H)) > 2$. However, for $p = 3$ we will show that $H = \bigcup_{0 \leqslant s,t \leqslant 8} \langle a^s b^t \rangle C \cdot H'$ is a finite covering by (normal) subgroups of class 2.

Set $g = a^s b^t$, $0 \leqslant s, t \leqslant 8$, and let $y_1, y_2, y_3 \in CH'$. Using $CH' \subseteq L(H)$ and $c(H) = 3$, we obtain by linear expansion

$$(8) \qquad [g^i y_1, g^j y_2, g^k y_3] = [g, y_2, y_3]^i [y_1, g, y_3]^j.$$

Using the relations of H, we obtain $[y_1, g] = g_1$ and $[g, y_2] = g_2$ with $g_1, g_2 \in V \times W \times Z$. Hence $[g, y_2, y_3] = [y_1, g, y_3] = 1$. This together with (8) leads to $[g^i y_1, g^j y_2, g^k y_3] = 1$. Thus we may conclude that the above finite covering consists of subgroups of class 2.

(vi) We will show $c_i \not\equiv c_j \mod \varepsilon_2^*(H)$ for $i \neq j$. By Lemma 7, it suffices to establish $[c_i c_j^{-1}, a, b]^3 \neq 1$. Using the relations of H, we obtain

$$[c_i c_j^{-1}, a, b]^3 = [c_i, a, b]^3 [c_j, a, b]^{-3} = z_i^{-3} z_j^3 \neq 1.$$

Thus there exist infinitely many different cosets mod $\varepsilon_2^*(H)$ in H. Since $Z_2(H) \subseteq \varepsilon_2^*(H)$ by [11], we have that $H/Z_2(H)$ is not finite either.

(vii) We observe that $[a, bc_i, bc_i] = z_i^3 \neq 1$. Thus $Z^3 \subseteq \varepsilon_2(H)$. Since Z^3 is not finite, the claim follows. ∎

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