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DILATIONS OF INTERACTION GROUPS THAT EXTEND ACTIONS OF ORE SEMIGROUPS

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Abstract

We show that if a unital injective endomorphism of a C^*-algebra admits a transfer operator, then both of them are compressions of mutually inverse automorphisms of a bigger algebra. More generally, every interaction group – in the sense of Exel – extending an action of an Ore semigroup by injective unital endomorphisms of a C^*-algebra, admits a dilation to an action of the corresponding enveloping group on another unital C^*-algebra, of which the former is a C^*-subalgebra: the interaction group is obtained by composing the action with a conditional expectation. The dilation is essentially unique if a certain natural condition of minimality is imposed, and it is faithful if and only if the interaction group is also faithful.

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1. Introduction

The notions of interaction groups and their crossed products have been introduced and studied by Exel in [9], with the aim of dealing with irreversible dynamical systems. Exel's paper emerges as the culmination of previous work in the subject appearing in [6–8, 11]. Related work may be found as well in [3–5, 13, 14].

In [10], Exel and Renault studied a family of interaction groups that extend actions of some semigroups on unital commutative C^*-algebras. Such semigroups are known as Ore semigroups, and their definition can be found at the beginning of Section 3. The class of Ore semigroups contains all cancellative abelian monoids, like $\mathbb{N}^k$. On the other hand, in [4] De Castro considered an interaction group defined by an iterated function system associated to inverse branches of continuous functions. The original motivation of the present work was to show that, in both cases, these interaction groups can be seen as a kind of 'compression' of a classical dynamical system, in a sense we briefly describe below.

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Suppose that X is a compact Hausdorff space and $\theta : X \to X$ is a covering map, that is, a surjective local homeomorphism. For $A = C(X)$, let $\alpha : A \to A$ be the dual map of θ, that is, $\alpha(a) = a \circ \theta$, which is a unital injective endomorphism of A. If there exists a transfer operator [6] for α, that is, a positive linear map $\mathcal{L} : A \to A$ such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$, for all $a, b \in A$, then $V : \mathbb{Z} \to \mathsf{B}(A)$ (here $\mathsf{B}(A)$ is the algebra of bounded operators from A into itself) given by

$$V_n = \begin{cases} \alpha^n & n \geq 0, \\ \mathcal{L}^{-n} & n < 0, \end{cases}$$

is called an interaction group (Definition 2.1). This interaction group is clearly an extension of the action $\bar{\alpha} : \mathbb{N} \times A \to A$ given by $(n, a) \mapsto \alpha^n(a)$. Conversely, it can be shown that if $W : \mathbb{Z} \to \mathsf{B}(A)$ is an interaction group that extends $\bar{\alpha}$, then W_{-1} is a transfer operator for α, and W is retrieved from α and W_{-1} from the construction above. That is, interaction groups that extend $\bar{\alpha}$ are in a natural bijection with transfer operators for α. In the same way, interaction groups that extend the action of an Ore semigroup correspond to semigroups of transfer operators associated to the endomorphisms of the action. In the case of actions on commutative algebras the work in [10] shows that one can replace transfer operators by *cocycles*. We will show later that an interaction group like the one above can be written as the composition of an action β on a bigger algebra with a conditional expectation F: $V_n = F\beta_n|_A$, for all $n \in \mathbb{Z}$, a decomposition that reflects the combination of the deterministic and probabilistic elements included in the concept of interaction group. The main aim of this paper is to show that a similar dilation exists for any interaction group that extends an action of an Ore semigroup by injective unital endomorphisms.

On the other hand, it seems that interaction groups are closely related with partial actions. Propositions 2.6 and 3.6 below are instances of this relation. Moreover, under certain conditions one may construct interaction groups from actions of groups and conditional expectations, in a way that resembles the construction of partial actions by the restriction of global ones. In fact, suppose that A is a C^*-subalgebra of the unital C^*-algebra B, $F : B \to B$ is a conditional expectation with range A, and $\beta : G \times B \to B$ is an action of a group G on B. Let $F_t : B \to B$ be given by $F_t := \beta_t F \beta_{t^{-1}}$. Then F_t is a conditional expectation onto $\beta_t(A)$, for all $t \in G$. It is not hard to prove that if $FF_t = F_tF$, for all $t \in G$, then $V : G \to \mathsf{B}(A)$ such that $V_t(a) = F(\beta_t(a))$, for all $a \in A, t \in G$, is an interaction group (provided $F(\beta_t(1_A)) = 1_A$, for all $t \in G$, see Proposition 2.3 below).

In the same spirit as the work done in [1], although with different methods, we show in the present paper that any interaction group that extends an action of an Ore semigroup by unital injective endomorphisms (for instance, those studied in [10]) is of this form, that is, it can be obtained by composing an action with a conditional expectation. The existence of the action is due to Laca's theorem (see [12] and Theorem 3.15 below) on the dilation of actions of Ore semigroups. The conditional expectation is constructed as the limit of the directed system of transfer operators corresponding to the endomorphisms of the Ore semigroup action. When the

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interaction group is faithful and the dilation is minimal, it is possible to show that the left ideal associated to the conditional expectation is in fact a two-sided ideal invariant under the action (Corollary 3.9), and from this fact one can prove that the conditional expectation must be faithful.

In particular our results apply, of course, to interaction groups defined by an endomorphism and a transfer operator for it. If α is an injective unital endomorphism of a C^*-algebra A and $\mathcal{L}$ is a unital transfer operator for α, that is, if we have the necessary ingredients to perform Exel's crossed product by an endomorphism [6], then there exist a unital C^*-algebra B containing A as a unital subalgebra, an automorphism β of B, and a conditional expectation $F : B \to A$, such that $\alpha = F\beta|_A$ and $\mathcal{L} = F\beta^{-1}|_A$, and F is faithful if and only if $\mathcal{L}$ is faithful.

The main examples of interaction groups considered in the existing literature usually occur on a commutative C^*-algebra. In the commutative case one is able to do a deeper analysis by using Gelfand–Naimark duality. In particular, the dilations of the Exel–Renault interaction groups can be described very precisely. However, in order to keep the present work to a reasonable length, we will not consider the general commutative case in the present paper, but we restrict ourselves to the consideration of just one family of examples: the interaction groups associated to some iterated function systems considered in [4], associated to inverse branches of a continuous function.

The structure of the present paper is as follows. In the next section we show a general construction of interaction groups, which we hope will serve to describe any interaction group, and we study some relations between interaction groups and partial actions. Then in the third section we introduce the notion of dilation of an interaction group, and we prove our main result, Theorem 3.18. In the final section we see how this theorem applies to some $\mathbb{Z}$-interaction groups considered by De Castro in [4], related to iterated function systems.

2. Preliminaries

2.1. Interaction groups. We show here how to get interaction groups from suitable pairs of actions and conditional expectations. Recall that a partial representation of a group G on a Banach space A is a map $V : G \to \mathsf{B}(A)$, the Banach algebra of bounded linear operators on A, such that

$$\begin{cases} V_e = \mathrm{Id} & (e \text{ the unit of } G), \\ V_{s^{-1}} V_s V_t = V_{s^{-1}} V_{st} & \forall s, t \in G, \\ V_s V_t V_{t^{-1}} = V_{st} V_{t^{-1}} & \forall s, t \in G. \end{cases}$$

Definition 2.1 (cf. [9]). An interaction group is a triple (A, G, V) where A is a unital C^*-algebra, G is a group, and V is a map from G into $\mathsf{B}(A)$, which satisfies:

(1) V_t is a positive unital map, for all $t \in G$;

(2) V is a partial representation;

(3) $V_t(aa') = V_t(a)V_t(a')$ if either a or a' belongs to $V_{t^{-1}}(A)$.

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If the group G is understood we will put write (A, V) (or even V if A is understood as well) instead of (A, G, V). A morphism $(A, G, V) \xrightarrow{\psi} (A', G, V')$ is a unital homomorphism of C^*-algebras $\psi : A \to A'$ that intertwines the interaction groups: $\psi V_t = V'_t \psi$, for all $t \in G$.

Let us introduce and fix some notation before proceeding. We will denote by $\mathcal{T}_G$ the set of triples $\mathsf{T} = (B, \beta, F)$, where β is an action of the group G on the unital C^*-algebra B, and $F : B \to B$ is a conditional expectation, that is, a norm-one idempotent whose range is a C^*-subalgebra of B. Recall that a conditional expectation F is a positive $F(B)$-bimodule map. Each such a triple T has associated the following objects (here $A = F(B)$):

(1) $F_r := \beta_r F \beta_{r^{-1}}$, which is a conditional expectation with image $\beta_r(A)$;
(2) $K_{\mathsf{T}} = \overline{\operatorname{span}}\{\beta_r(A) : r \in G\}$, the closed linear β-orbit of A. Note that K_{T} is F_r invariant, for every $r \in G$, and $\tilde{F}_r(K_{\mathsf{T}}) = \beta_r(A)$.

In virtue of (2) above, we can define $\tilde{F}_r : K_{\mathsf{T}} \to K_{\mathsf{T}}$ such that $\tilde{F}_r(k) = F_r(k)$. If $r = e$ we will write $\tilde{F}$ instead of $\tilde{F}_e$. We also have the restricted action $\tilde{\beta} : G \times K_{\mathsf{T}} \to K_{\mathsf{T}}$ such that $\tilde{\beta}_r(k) = \beta_r(k)$, for all $r \in G, k \in K_{\mathsf{T}}$. Then we have the following proposition.

Proposition 2.2.

(1) $\tilde{F}_r^2 = \tilde{F}_r$, $\|\tilde{F}_r\| = 1$ *and* $\tilde{F}_r \geq 0$, *for all* $r \in G$.
(2) $F_r \beta_s = \beta_s F_{s^{-1}r}$, *for all* $r, s \in G$.
(3) $F_r F_s = \beta_s F_{s^{-1}r} F \beta_{s^{-1}}$, *or equivalently* $F_{st} F_s = \beta_s F_t F \beta_{s^{-1}}$, *for all* $r, s, t \in G$.
(4) *If* $t \in G$, *then* $F_t F = F F_t$ *if and only if* $F_{st} F_s = F_s F_{st}$, *for all* $s \in G$.
(5) $[F_r, F_s] = \beta_r [F, F_{r^{-1}s}] \beta_{r^{-1}}$, *for all* $r, s \in G$, *where* $[S, T] =: ST - TS$ *is the commutator of the operators* S *and* T.
(6) *The same relations hold for* $\tilde{F}_r$ *and* $\tilde{\beta}_r$ *instead of* F_r *and* β_r.

Proof. Since $\tilde{F}_r$ is a restriction of the conditional expectation F_r, it is clear that $\tilde{F}_r^2 = \tilde{F}_r$, $\|\tilde{F}_r\| \leq 1$ and $\tilde{F}_r \geq 0$. On the other hand, every nonzero idempotent element of a normed algebra, like $\tilde{F}_r$, has norm greater than or equal to 1. Therefore statement (1) is proved. As for (2), we just make the necessary computations:

$$F_r \beta_s = \beta_r F \beta_{r^{-1}} \beta_s = \beta_s \beta_{s^{-1}r} F \beta_{r^{-1}s} = \beta_s F_{s^{-1}r}.$$

Assertion (3) follows now from (2):

$$F_r F_s = (\beta_r F \beta_{r^{-1}s}) F \beta_{s^{-1}} = (\beta_r \beta_{r^{-1}s} F_{s^{-1}r}) F \beta_{s^{-1}} = (\beta_s F_{s^{-1}r}) F \beta_{s^{-1}}.$$

Statements (4) and (5) follow at once from (3). Statement (6) is just a consequence of (1)–(5) and the invariance of K_{T} under every F_r and every β_r. □

We show next that the formula $V_t := F\beta_t|_A$ $(= \tilde{F}\tilde{\beta}_t|_A)$ defines an interaction group $V : G \to B(A)$ provided the following two conditions hold:

$$\tilde{F}\tilde{\beta}_t(1_A) = 1_A, \quad \forall t \in G; \tag{2.1}$$

$$\tilde{F}_r \tilde{F}_s = \tilde{F}_s \tilde{F}_r, \quad \forall r, s \in G. \tag{2.2}$$

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Thus it will be convenient to consider the subset $\mathcal{D}_G$ of $\mathcal{T}_G$, whose elements are those triples $\mathsf{T} = (B, \beta, F)$ that satisfy conditions (2.1) and (2.2). Note that (2.1) is equivalent to $F\beta_t F(1_B) = F(1_B)$, for all $t \in G$.

Proposition 2.3. *Let $\mathsf{T} = (B, \beta, F) \in \mathcal{T}_G$, $A := F(B)$, K_T, $\tilde{F}_r$ and $\tilde{F}$ as above. Suppose that $\tilde{F}\tilde{F}_t = \tilde{F}_t\tilde{F}$, for all $t \in G$. Then the following statements hold:*

(1) *$\tilde{F}_r\tilde{F}_s = \tilde{F}_s\tilde{F}_r$, for all $r, s \in G$, and $\tilde{F}_r\tilde{F}_s$ is a norm-one idempotent with range $\beta_r(A) \cap \beta_s(A)$.*
(2) *If $V_t := F\beta_t|_A$, for all $t \in G$, then the map $V : G \to \mathsf{B}(A)$ given by $t \mapsto V_t$ is a partial representation and satisfies condition (3) of Definition 2.1. Moreover, the range of V_t is $A \cap \beta_t(A)$, for all $t \in G$.*
(3) *If V is the map defined in (2), then V is an interaction group if and only if $\mathsf{T} \in \mathcal{D}_G$.*

Proof. Since $\tilde{F}$ commutes with every $\tilde{F}_t$, claim (1) follows directly from (4) and (6) of Proposition 2.2.

As for (2), V is a partial representation: on the one hand, it is clear that $V_e = F\beta_e|_A = F\,\mathrm{Id}_A = \mathrm{Id}_A$; and on the other hand,

$$V_{s^{-1}}V_sV_t = \tilde{F}\tilde{F}_{s^{-1}}\tilde{F}\tilde{\beta}_t|_A = \tilde{F}\tilde{F}_{s^{-1}}\tilde{\beta}_t|_A = V_{s^{-1}}V_{st},$$
$$V_sV_tV_{t^{-1}} = \tilde{F}\tilde{\beta}_{st}\tilde{F}_{t^{-1}}\tilde{F}\tilde{\beta}_{t^{-1}}|_A = V_{st}\tilde{F}\tilde{F}_{t^{-1}}\tilde{\beta}_{t^{-1}}|_A = V_{st}V_{t^{-1}}\tilde{F}|_A = V_{st}V_{t^{-1}}.$$

Now if $x \in A$, $a = V_{t^{-1}}(x)$ and $a' \in A$, then

$$\begin{aligned} V_t(aa') &= V_t(V_{t^{-1}}(x)a') = \tilde{F}(\tilde{F}_t(x)\tilde{\beta}_t(a')) = \tilde{F}(\tilde{F}_t\tilde{F}(x)\tilde{\beta}_t(a')) \\ &= \tilde{F}(\tilde{F}\tilde{F}_t(x)\tilde{\beta}_t(a')) = \tilde{F}\tilde{F}_t(x)\tilde{F}(\tilde{\beta}_t(a')) = V_tV_{t^{-1}}(x)V_t(a') = V_t(a)V_t(a'). \end{aligned}$$

Since $V_t(a'a) = V_t(a^*a'^*)^*$, we have shown that V satisfies condition (3) of Definition 2.1. On the other hand $V_t(A) = \tilde{F}\tilde{\beta}_t(A) = \tilde{F}\tilde{F}_t(K_\mathsf{T}) = A \cap \tilde{\beta}_t(A)$, where the latter equality is given by (1).

Now, in view of the preceding parts, V is an interaction group if and only if $\tilde{F}\tilde{\beta}_t(1_A) = 1_A$, for all $t \in G$, that is, if and only if $\mathsf{T} \in \mathcal{D}_G$. □

Observe that, if $F_rF = FF_r$, for all $r, s \in G$, one can follow the proof of (1) above to show that also $F_rF_s = F_sF_r$ for all $r, s \in G$, and then F_rF_s is a conditional expectation with range $\beta_r(A) \cap \beta_s(A)$.

Definition 2.4. An interaction group $V : G \to \mathsf{B}(A)$ will be called faithful if every V_t is a faithful positive map, that is, $V_t(a^*a) = 0$ if and only if $a = 0$.

We record the next obvious fact for future reference.

Proposition 2.5. *Let $\mathsf{T} = (B, \beta, F) \in \mathcal{D}_G$ be such that F is faithful. If $A := F(B)$ and $V : G \to \mathsf{B}(A)$ is the interaction group defined in Proposition 2.3, that is, $V_t = F\beta_t|_A$, then V is a faithful interaction group.*

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2.2. The partial action of an interaction group. We will see now that every interaction group has naturally associated a partial action of the group on the same algebra. Recall that a partial action of a discrete group G on a set X is a pair $(\{X_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$ where, for every $t \in G$, X_t is a subset of X, $\gamma_t : X_{t^{-1}} \to X_t$ is a bijection, and γ_{st} extends $\gamma_s\gamma_t$, for all $s, t \in G$. It is also assumed that $\gamma_e = \mathrm{id}_X$. When X is a C^*-algebra, it is usually supposed that X_t is an ideal and that γ_t is an isomorphism of C^*-algebras. So we warn the reader that, for the partial actions we consider in this paper, the sets X_t will be unital C^*-subalgebras rather than ideals.

Proposition 2.6. *Suppose that $V : G \to \mathsf{B}(A)$ is an interaction group. For $t \in G$, let $A_t := V_t(A)$, and let $\gamma_t : A_{t^{-1}} \to A_t$ be such that $\gamma_t(a) = V_t(a)$. Then the following statements hold:*

(1) *Every A_t is a unital C^*-subalgebra of A (with the same unit), and γ_t is an isomorphism between $A_{t^{-1}}$ and A_t, for all $t \in G$.*
(2) *The map $E_t : A \to A$ given by $E_t := V_tV_{t^{-1}}$ is a conditional expectation onto A_t, for all $t \in G$, and $E_rE_s = E_sE_r$, for all $r, s \in G$. Moreover, the interaction group is faithful if and only if every E_t is faithful.*
(3) *The pair $\gamma := (\{A_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$ is a partial action of G on A.*

Proof. We already know by [9, 3.2] that A_t is a unital C^*-subalgebra of A with unit $V_t(1_A) = 1_A$, and that γ_t is an isomorphism, for all $t \in G$. The first assertion of (2) also follows from [9, 2.2 and 3.2]. The composition of faithful positive maps is again a faithful positive map, so every E_t is faithful if V is faithful. On the other hand, if E_t is faithful and $V_t(a^*a) = 0$, then $E_t(a^*a) = V_{t^{-1}}V_t(a^*a) = 0$, so $a = 0$; then V_t is faithful. To prove (3) note first that, since V is a partial representation, we have $\gamma_e = V_e = \mathrm{Id}$. Suppose now that c belongs to the domain of $\gamma_s\gamma_t$, that is, $c \in A_{t^{-1}}$ is such that $\gamma_t(c) \in A_{s^{-1}}$. Then $\gamma_s\gamma_t(c) \in A_s$ and $\gamma_s\gamma_t(c) = V_sV_t(V_{t^{-1}}(\gamma_t(c))) = V_{st}(V_{t^{-1}}(\gamma_t(c))) = V_{st}(c) \in A_{st}$. Then $\gamma_s\gamma_t(c) \in A_s \cap A_{st}$, and we may apply $\gamma_{t^{-1}s^{-1}}$ to $\gamma_s\gamma_t(c)$. Since V is a partial representation we obtain

$$\gamma_{t^{-1}s^{-1}}\gamma_s\gamma_t(c) = V_{t^{-1}s^{-1}}V_s\gamma_t(c) = V_{t^{-1}}V_{s^{-1}}V_s\gamma_t(c) = \gamma_{t^{-1}}\gamma_{s^{-1}}\gamma_s\gamma_t(c) = c,$$

whence $\gamma_{st}(c) = \gamma_s\gamma_t(c)$. This shows that γ_{st} extends $\gamma_s\gamma_t$, for all $s, t \in G$, and therefore γ is a partial action. □

Observe that if V is an interaction group of the type considered in (3) of Proposition 2.3, then $E_r = FF_r|_A = F_r|_A$, and $E_rE_s = F_rF_s|_A$ (in the notation of Propositions 2.3 and 2.6).

As mentioned previously, the usual notion of partial actions of groups on C^*-algebras requires that the domains of the partial automorphisms involved are ideals. In the commutative case, partial actions on a C^*-algebra correspond exactly to partial actions on the spectrum of the algebra, where the domains of the partial homeomorphisms are open subsets of the spectrum [2, Proposition 1.5]. Instead, partial actions on unital commutative C^*-algebras such as the ones considered in Proposition 2.6 lead to a different notion of partial action on a topological space.

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In fact, let $A = C(X)$ be a unital commutative C^*-algebra, and let $\gamma = (\{A_t\}, \{\gamma_t\})$ be a partial action of G on A, where each A_t is a unital subalgebra of A, with the same unit. Then the dual notion of the partial action γ should be expressed in terms of the spectra of the subalgebras A_t and the maps induced by γ between them. Although we will not give here the exact conditions that such a collection of spaces and maps must satisfy, it is clear that the result is not a partial action in the usual sense, as the spectrum of A_t is not a subspace but a quotient of X.

3. Dilations

3.1. Dilations of interaction groups. We introduce next the notion of dilation of an interaction group V, and we study its relation with the partial action associated with V.

Definition 3.1. Let $V : G \to \mathsf{B}(A)$ be an interaction group. A dilation of V is a pair (i, T), where $\mathsf{T} = (B, \beta, F) \in \mathcal{T}_G$ and $i : A \to B$ is a homomorphism of C^*-algebras such that $iV_t = F\beta_t i$, for all $t \in G$. If $B = C^*(\{\beta_t i(a) : a \in A, t \in G\})$, that is, B is the C^*-algebra generated by $\{\beta_t i(a) : a \in A, t \in G\}$, we say that the dilation is minimal. The dilation is called faithful if i is injective and F is faithful.

In view of the general construction made in the previous section, it seems natural to single out the dilations which will produce interaction groups via (2) of Proposition 2.3. More precisely, we give the following definition.

Definition 3.2. Let $K_{\mathsf{T},i} := \overline{\operatorname{span}}\{\tilde{\beta}_t(i(A)) : t \in G\}$, so $K_{\mathsf{T},i}$ is a closed β-invariant subspace of B contained in K_{T}. We say that the dilation is admissible if $\tilde{F}_r\tilde{F}_s = \tilde{F}_s\tilde{F}_r$ on $K_{\mathsf{T},i}$, for all $r, s \in G$.

As in Proposition 2.3, it is easy to see that the dilation $(i, (B, \beta, F))$ of V is admissible if and only if $\tilde{F}_r\tilde{F} = \tilde{F}\tilde{F}_r$ on $K_{\mathsf{T},i}$, for all $r \in G$. A computation of each side of the equality $\tilde{F}_r\tilde{F}\beta_s i = \tilde{F}\tilde{F}_r\beta_s i$ then shows that the latter condition is equivalent to

$$iV_rV_{r^{-1}s} = \beta_r iV_{r^{-1}}V_s, \quad \forall r, s \in G. \tag{3.1}$$

In particular, $iV_r = \beta_r iV_{r^{-1}}V_r$, for all $r \in G$.

Note that (i, T) is admissible whenever $\mathsf{T} \in \mathcal{D}_G$, and if V is the interaction group given by (2) of Proposition 2.3 and i is the natural inclusion, then (i, T) is admissible if and only if $\mathsf{T} \in \mathcal{D}_G$.

We are mainly interested in admissible dilations. It will follow from the following result that this is always the case provided the conditional expectation F is faithful.

Proposition 3.3. *Let $V : G \to \mathsf{B}(A)$ be an interaction group, and suppose that (i, T) is a dilation of V, where $\mathsf{T} = (B, \beta, F)$. Then we have $[\tilde{F}_r, \tilde{F}_s](K_{\mathsf{T},i}) \subseteq I_F$, for all $r, s, t \in G$ (recall that $I_F := \{b \in B : F(b^*b) = 0\}$).*

Proof. By (5) and (6) of Proposition 2.2, we have $[\tilde{F}_r, \tilde{F}_s] = \tilde{\beta}_r[\tilde{F}, \tilde{F}_{r^{-1}s}]\tilde{\beta}_{r^{-1}}$, for all $r, s \in G$. So it is enough to show that $[\tilde{F}, \tilde{F}_s](\tilde{\beta}_t(i(A))) \subseteq I_F$, for all $s, t \in G$. More explicitly, we must prove that $F\beta_sF\beta_{s^{-1}}(\beta_t i(a)) - \beta_sF\beta_{s^{-1}}F(\beta_t i(a)) \in I_F$,

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that is, $iV_sV_{s^{-1}t}(a) - \beta_s iV_{s^{-1}}V_t(a) \in I_F$, for all $s \in G$, $a \in A$. Since F is an $i(A)$-bimodule map which is the identity on $i(A)$, and since $F\beta_t i = iV_t$, we have that $F((iV_sV_{s^{-1}t}(a) - \beta_s iV_{s^{-1}}V_t(a))^*(iV_sV_{s^{-1}t}(a) - \beta_s iV_{s^{-1}}V_t(a)))$ is equal to

$$i[V_sV_{s^{-1}t}(a^*)(V_sV_{s^{-1}t}(a) - V_sV_{s^{-1}}V_t(a)) \\ - V_sV_{s^{-1}}V_t(a^*)V_sV_{s^{-1}t}(a) + V_s(V_{s^{-1}}V_t(a^*)V_{s^{-1}}V_t(a))],$$

which is zero because V is a partial representation and V_s is multiplicative on $V_{s^{-1}}(A)$. □

Corollary 3.4. *Any dilation of an interaction group is admissible if the conditional expectation F is faithful.*

Proposition 3.5. *If (i, T) is a faithful dilation of the interaction group V, then V is a faithful interaction group.*

Proof. Since the homomorphism i is injective, the interaction group V is isomorphic to the restriction to $i(A)$ of one of those interaction groups considered in Proposition 2.5 and, as such, it is faithful. □

Suppose that β is an action of G on the C^*-algebra B, and that A is a C^*-subalgebra of B. The restriction of β to A is the partial action $\beta|_A := (\{A'_t\}_{t\in G}, \{\gamma'_t\}_{t\in G})$, where $A'_t := A \cap \beta_t(A)$ and $\gamma'_t(a) := \beta_t(a)$, for all $a \in A'_{t^{-1}}$, $t \in G$. If the C^*-algebra generated by $\{\beta_t(a) : a \in A, t \in G\}$ is all of B, we say that β is a *minimal globalization* of γ'. In particular, if $(i, (B, \beta, F))$ is a dilation of an interaction group V, we have two partial actions related with V: the restriction of β to $i(A)$ as above, and also that associated to V in Proposition 2.6. We study next the relationship between them.

Proposition 3.6. *Suppose that $V : G \to \mathsf{B}(A)$ is an interaction group with dilation $(i, (B, \beta, F))$, where A is a C^*-subalgebra of B and $i : A \to B$ is the natural inclusion. Let γ be the partial action of G on A given by Proposition 2.6, and let $\gamma' := \beta|_A$. Then $A_t \supseteq A'_t := A \cap \beta_t(A)$ and $\gamma_t(a) = \gamma'_t(a)$, for all $t \in G$, $a \in A'_{t^{-1}}$. If the dilation is admissible then $\gamma = \beta|_A$. In particular, if the dilation is faithful then γ is the restriction of β to A.*

Proof. If $a \in A$, then $a \in A'_{t^{-1}} \iff \beta_t(a) \in A \iff \beta_t(a) = F\beta_t(a) \iff \beta_t(a) = V_t(a)$. Then if $a \in A'_{t^{-1}}$ we have $\gamma'_t(a) = \beta_t(a) = V_t(a) \in A_t$, which shows that $A'_t \subseteq A_t$ and $\gamma'_t = V_t|_{A'_{t^{-1}}} = \gamma_t|_{A'_{t^{-1}}}$. On the other hand, if $\tilde{F}\tilde{F}_t = \tilde{F}_t\tilde{F}$, by (1) of Proposition 2.3 we have

$$A_t = V_t(A) = V_tV_{t^{-1}}(A) = \tilde{F}\tilde{F}_t(A) \subseteq A \cap \beta_t(A) \subseteq A'_t,$$

whence $A_t = A'_t$, and $\gamma_t = \gamma'_t$. The last two assertions follow respectively from Proposition 3.3 and Corollary 3.4. □

Corollary 3.7. *Suppose that $V : G \to \mathsf{B}(A)$ is an interaction group with admissible dilation $(i, (B, \beta, F))$, where $i : A \to B$ is an embedding (that is, i is injective). Then the restriction of β to $C := C^*(\{\beta_t i(a) : t \in G, a \in A\})$ is isomorphic to a minimal globalization of the partial action γ of G on A given by Proposition 2.6. In particular, if the dilation is minimal then β is isomorphic to a minimal globalization of γ.*

In the proof of our main result, Theorem 3.18, we will construct a minimal dilation $\mathsf{T} = (B, \beta, F)$ of a given interaction group V that extends an action of an Ore semigroup. This particular dilation is such that i is injective and

$$[F, F_t] = 0, \quad \forall t. \tag{3.2}$$

It turns out that, if in addition V is faithful, then there exists a faithful dilation of V. Since the latter fact is just a consequence of (3.2) and the injectivity of i, and not of the particular structure of the group or the nature of V, we will prove it in a more general context than that of Theorem 3.18. This is precisely the aim of the remaining results of the present subsection.

The key fact is that, in the above situation, the left ideal I_F is actually a β-invariant two-sided ideal of B. We begin by proving this fact.

Proposition 3.8. *Let (i, T) be a dilation of the interaction group $V : G \to \mathsf{B}(A)$, where $\mathsf{T} = (B, \beta, F)$.*

(1) *If V is faithful, i is injective and $[F, F_t] = 0$ for all $t \in G$, then I_F is β-invariant.*
(2) *If I_F is β-invariant and (i, T) is minimal, then I_F is a two-sided ideal of B.*

Proof. Let $b \in I_F$ and suppose that $F(b^*b) = 0$. If $[F, F_{t^{-1}}] = 0$, we have $iFF_{t^{-1}}(b^*b) = iF_{t^{-1}}F(b^*b) = 0$, that is, $V_{t^{-1}}F(\beta_t(b)^*\beta_t(b)) = 0$. Thus, if V is faithful, we have $F(\beta_t(b)^*\beta_t(b)) = 0$, which shows that I_F is β-invariant. To prove the second part we have to show only that I_F is a right ideal, for it is well known that I_F is always a left ideal in B. To do so, it is enough to show that $b\beta_t(i(a)) \in I_F$, for all $b \in I_F$, $a \in A$ and $t \in G$, because of the minimality of the dilation. Since we are assuming that I_F is β-invariant, the above is equivalent to showing that $\beta_{t^{-1}}(b\beta_t(i(a))) \in I_F$. But $F((\beta_{t^{-1}}(b)i(a))^*(\beta_{t^{-1}}(b)i(a))) = F(i(a)^*\beta_{t^{-1}}(b)^*\beta_{t^{-1}}(b)i(a))$ and, since F is a conditional expectation onto $F(B) \supseteq i(A)$, the latter expression is equal to $i(a)^*F(\beta_{t^{-1}}(b^*b))i(a)$, which is zero because I_F is β invariant. □

So combining both parts of Proposition 3.8, we get the following corollary.

Corollary 3.9. *Suppose that (i, T) is a minimal dilation of a faithful interaction group $V : G \to \mathsf{B}(A)$, where $\mathsf{T} = (B, \beta, F)$. If i is injective and $[F, F_t] = 0$ for all $t \in G$, then I_F is a β-invariant two-sided ideal of B.*

In particular, we have the following result.

Corollary 3.10. *Let $\mathsf{T} = (B, \beta, F) \in \mathcal{D}_G$ be such that $B = C^*(\{\beta_t(F(B)) : t \in G\})$ and $[F, F_t] = 0$ for all $t \in G$. If F is faithful, then I_F is a β-invariant two-sided ideal of B.*

Proof. Just apply Corollary 3.9 to the interaction group associated to T via (2) of Proposition 2.3. □

Our next goal is to show that if a faithful interaction group has a dilation in which i is injective and (3.2) holds, then the interaction group also has a faithful dilation. This will be accomplished in Corollary 3.13.

Let (i, T) be a dilation of the interaction group $V : G \to \mathsf{B}(A)$, where $\mathsf{T} = (B, \beta, F)$. Suppose that I_F is two-sided ideal of B, as happens to be the case when B is commutative, or under the circumstances of the second part of Proposition 3.8. Let $q : B \to B/I_F =: \bar{B}_F$ be the quotient map, and $\bar{i} = qi$. Since $I_F \subseteq \ker F$, there exists a unique linear map $\bar{F} : \bar{B}_F \to \bar{B}_F$ such that $\bar{F}q = qF$, and we have $\|\bar{F}\| = \|F\| = 1$. Moreover, $\bar{F}$ is clearly positive, and it is idempotent: $\bar{F}^2 q = qF^2 = qF = \bar{F}q$. Finally, $\bar{F}(\bar{B}_F) = \bar{F}q(B) = q(F(B))$, which is a C^*-subalgebra of $\bar{B}_F$, so $\bar{F}$ is a conditional expectation. We have the following proposition.

Proposition 3.11. *Suppose that I_F is a β-invariant two-sided ideal of B, and let $\bar{\beta}$ be the action induced by β on $\bar{B}_F$, so $\bar{\beta}_t q = q\beta_t$, for all $t \in G$. If $\bar{\mathsf{T}} := (\bar{B}_F, \bar{\beta}, \bar{F})$, then $(\bar{i}, \bar{\mathsf{T}})$ is a dilation of V, with $\bar{F}$ faithful. If i is injective, then so is $\bar{i}$, and $(\bar{i}, \bar{\mathsf{T}})$ is a faithful dilation of V.*

Proof. If $\bar{F}(q(b)^* q(b)) = 0$, then $q(F(b^*b)) = 0$, thus $F(b^*b) \in I_F \cap i(A) = 0$; hence $F(b^*b) = 0$, that is, $q(b) = 0$. So $\bar{F}$ is faithful. On the other hand,

$$\bar{i}V_t = qiV_t = qF\beta_t i = \bar{F}q\beta_t i = \bar{F}\bar{\beta}_t qi = \bar{F}\bar{\beta}_t \bar{i}.$$

Suppose that i is injective. Since $I_F \cap i(A) = 0$, then $q|_{i(A)}$ is injective. It follows that $\bar{i}$ is injective as well; hence $(\bar{i}, \bar{T})$ is a faithful dilation of V. □

Corollary 3.12. *If an interaction group has a dilation for which i is injective and I_F is β-invariant, then it also has a faithful dilation.*

Corollary 3.13. *If a faithful interaction group has a dilation in which i is injective and $[F, F_t] = 0$ for all $t \in G$, then it also has a minimal faithful dilation with the same poperties (in particular, it is admissible).*

Proof. We may suppose the dilation is minimal, so by Corollary 3.9 we are in the conditions of Proposition 3.11, which provides a faithful dilation $(\bar{B}_F, \bar{\beta}, \bar{F})$, which is minimal by construction. Since $\bar{F}_t = \bar{\beta}_t \bar{F} \bar{\beta}_{t^{-1}}$ we have $\bar{F}_t q = F_t q$. Thus $[\bar{F}, \bar{F}_t]q = q[F, F_t] = 0$, whence $[\bar{F}, \bar{F}_t] = 0$. □

3.2. A case of existence of dilation. We will prove next our main result: any interaction group that extends an action of an Ore semigroup by injective endomorphisms has an essentially unique minimal admissible dilation. We first review some facts about Ore monoids and also Laca's extension-dilation theorem about actions of these semigroups.

A cancellative monoid P is called an Ore semigroup if $Pr \cap Ps \neq \emptyset$, for all $r, s \in P$. It follows by induction that P is an Ore semigroup if and only if $P_{t_1} \cap \cdots \cap P_{t_n} \neq \emptyset$, for all $t_1, \ldots, t_n \in P$. Then P is partially ordered by the relation $r \leq s \iff s \in Pr$ (equivalently, $r \leq s \iff Pr \supseteq Ps$), and it is even directed by that relation.

Any cancellative abelian monoid P is an Ore semigroup. In fact, such a monoid embeds in its Grothendieck group G, and every element $t \in G$ can be written as $t = v^{-1}u$, with $u, v \in P$. Therefore, if $r, s \in P$, writing $rs^{-1} = u^{-1}v$, with $u, v \in P$, gives $t := ur = vs \in Pr \cap Ps$, so P is an Ore semigroup (and $P \ni t \geq r, s$). More generally, we

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have the following theorem [12, Theorem 1.1.2], which shows that there is a functor from the category of Ore semigroups into the category of groups.

THEOREM 3.14 (Ore, Dubreil). *A semigroup P can be embedded in a group G with $P^{-1}P = G$ if and only if it is an Ore semigroup. In this case the group G is determined up to canonical isomorphism and every semigroup homomorphism ϕ from P into a group H extends uniquely to a group homomorphism $\varphi : G \to H$.*

If P is an Ore semigroup we say that the group G in Theorem 3.14 is the enveloping group of P.

A key ingredient in our process of dilating the interaction groups under consideration is Laca's theorem [12, 2.1.1]. For the convenience of the reader we recall it next.

THEOREM 3.15 (Laca, [12]). *Assume that P is an Ore semigroup with enveloping group $G = P^{-1}P$ and let α be an action of P by unital injective endomorphisms of a unital C^*-algebra A. Then there exists a C^*-dynamical system (B, G, β), unique up to isomorphism, consisting of an action β of G by automorphisms of a C^*-algebra B and an embedding $i : A \to B$ such that:*

(1) *β dilates α, that is, $\beta_t i = i\alpha_t$, for t in P; and*
(2) *(B, G, β) is minimal, that is, $\bigcup_{t\in P} \beta_t^{-1}(i(A))$ is dense in B.*

Note that i is unital:

$$\beta_{t^{-1}} i(a)i(1_A) = \beta_{t^{-1}}(i(a)\beta_t(i(1_A))) = \beta_{t^{-1}}(i(a\alpha_t(1_A))) = \beta_{t^{-1}} i(a), \quad \forall t \in P,$$

so taking adjoints and recalling that $\{\beta_{t^{-1}}(i(a)) : t \in P, a \in A\}$ is dense in B, we see that $i(1_A) = 1_B$.

From now on G will denote the enveloping group of the Ore semigroup P.

Suppose now that α is an action of the Ore semigroup P by unital injective endomorphisms of the unital C^*-algebra A, and that $V : G \to \mathsf{B}(A)$ is an interaction group such that $V|_P = \alpha$. To prove our main result it will be useful to establish first some easy relations between α, V and their possible dilations.

LEMMA 3.16. *In the conditions above, we have:*

(1) *$V_{r^{-1}} V_r = \mathrm{id}_A$, that is, $E_{r^{-1}} = \mathrm{id}_A$, for all $r \in P$;*
(2) *$V_{r^{-1}t} = V_{r^{-1}} V_t$, for all $r \in P$, $t \in G$;*
(3) *$V_{tr} = V_t V_r$, for all $r \in P$, $t \in G$.*

PROOF. To prove (1), note that $\alpha_r = V_r = V_r V_{r^{-1}} V_r = \alpha_r V_{r^{-1}} \alpha_r$; thus $V_{r^{-1}} \alpha_r = \mathrm{Id}_A$, because α_r is injective. Since V is an interaction group, in particular a partial representation, from (1) we obtain (2): $V_{r^{-1}t} = V_{r^{-1}} V_r V_{r^{-1}t} = V_{r^{-1}} V_t$. Similarly, we obtains (3) by multiplying V_{tr} by $V_{r^{-1}} V_r$ on the right. □

LEMMA 3.17. *If $(i, (B, \beta, F))$ is an admissible dilation of V, then:*

(1) *$\beta_r i = iV_r = i\alpha_r$, for all $r \in P$;*
(2) *$\beta_t i V_{t^{-1}} = iV_t V_{t^{-1}}$, for all $t \in G$.*

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Proof. By (3.1) we have $iV_r = \beta_r i V_{r^{-1}} V_r$, for all $r \in G$. Thus our first claim follows from (1) of the previous lemma. The second assertion holds in general for admissible dilations, for it is a particular case of (3.1). □

One can define morphisms between elements of the set $\mathcal{T}_G$, in such a way that $\mathcal{T}_G$ with these morphisms is a category. If $T = (B, \beta, F)$, $T' = (B', \beta', F') \in \mathcal{T}_G$, by a morphism $\phi : T \to T'$ we mean a unital homomorphism of C^*-algebras $\phi : B \to B'$ that intertwines the actions and the conditional expectations: $\phi F = F'\phi$ and $\phi\beta_t = \beta'_t\phi$, for all $t \in G$. Of course $\mathcal{D}_G$ with such morphisms is a full subcategory of $\mathcal{T}_G$.

Theorem 3.18. *Let α be an action of the Ore semigroup P by unital injective endomorphisms of the unital C^*-algebra A, and suppose that $V : G \to \mathsf{B}(A)$ is an interaction group such that $V|_P = \alpha$. Then V has a minimal admissible dilation (i, T), where $\mathsf{T} = (B, \beta, F)$ and $i : A \to B$ is a unital embedding, which has the following universal property. If $(i', (B', \beta', F'))$ is another admissible dilation of V, then there exists a unique morphism $\phi : (B, \beta, F) \to (B', \beta', F')$ such that $\phi i = i'$. Therefore the dilation (i, T) is unique up to isomorphism in the class of minimal and admissible dilations. Moreover, the dilation is faithful if and only if V is a faithful interaction group.*

Proof. Let $i : (A, \alpha) \to (B, \beta)$ be the minimal dilation of (A, α) provided by Laca's theorem. We suppose, as we can do, that i is the natural inclusion, so $A \subseteq B$. We proceed next to define a conditional expectation $F : B \to A$. To this end note first that if $r, s \in P$, with $r \le s$, and $a_r, a_s \in A$ are such that $\beta_{r^{-1}}(a_r) = \beta_{s^{-1}}(a_s)$, then $\beta_{sr^{-1}}(a_r) = a_s$, so $\alpha_{sr^{-1}}(a_r) = a_s$ by Theorem 3.15. Therefore

$$V_{s^{-1}}(a_s) = V_{s^{-1}}\alpha_{sr^{-1}}(a_r) = V_{s^{-1}}V_{sr^{-1}}(a_r) = V_{s^{-1}}\alpha_s V_{r^{-1}}(a_r) = V_{r^{-1}}(a_r).$$

Thus we may define $F_0 : \bigcup_{t \in P} \beta_{t^{-1}}(A) \to B$ such that $F_0(b) = V_{t^{-1}}(\beta_t(b))$, for all $b \in \beta_{t^{-1}}(A)$. Since $\|F_0(b)\| = \|V_{t^{-1}}(\beta_t(b))\| \le \|b\|$, F_0 extends uniquely to a bounded operator $F : B \to A$, which is easily seen to be positive and to satisfy $F^2 = F$ and $F(B) = A$. Then F is a conditional expectation with range A. We claim that (B, β, F) is a minimal admissible dilation of V. In fact, if $t \in G$ and $r, s \in P$ are such that $t = r^{-1}s$, then

$$F\beta_t|_A = F\beta_{r^{-1}}\beta_{rt}|_A = F\beta_{r^{-1}}\alpha_s = V_{r^{-1}}\alpha_s = V_{r^{-1}}V_r V_{r^{-1}s} = V_{r^{-1}s} = V_t.$$

Since $\bigcup_{t \in P} \beta_{t^{-1}}(A)$ is dense in B we have that (B, β, F) is minimal, and to see that it is also admissible it is enough to check that $FF_t\beta_{r^{-1}}|_A = F_t F\beta_{r^{-1}}|_A$, for all $t \in G, r \in P$. On the one hand, we have

$$FF_t\beta_{r^{-1}}|_A = F\beta_t F\beta_{t^{-1}r^{-1}}|_A = V_t F V_{t^{-1}r^{-1}} = E_t V_{r^{-1}}. \tag{3.3}$$

On the other hand, if $t \in G$, we have $F_t F = \beta_t F\beta_{t^{-1}} F = \beta_t V_{t^{-1}} F = V_t V_{t^{-1}} F = E_t F$, where the third equality follows from (ii) of Lemma 3.17. Then

$$F_t F\beta_{r^{-1}}|_A = E_t F\beta_{r^{-1}}|_A = E_t V_{r^{-1}}. \tag{3.4}$$

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From (3.3) and (3.4) we conclude that (B,β,F) is admissible. We see next that (B,β,F) has the claimed universal property. Then suppose that $(i',(B',\beta',F'))$ is another admissible dilation of V. By (1) of Lemma 3.17 we have that $\beta'_r i' = i'\alpha_r$, for all $r \in P$, for all $r \in P$, and then by the universal property of the pair (B,β) there exists a unique homomorphism $\phi : B \to B'$ such that $\phi i = i'$ and $\beta'_t\phi = \phi\beta_t$ for all $t \in G$. In particular, $\phi\beta_{r^{-1}} i = \beta'_{r^{-1}}\phi i = \beta'_{r^{-1}} i'$, for all $r \in P$. Thus

$$F'\phi\beta_{r^{-1}} i = F'\beta'_{r^{-1}} i' = i' V_{r^{-1}} = \phi i V_{r^{-1}} = \phi F\beta_{r^{-1}} i, \quad \forall r \in P.$$

The equality $\phi F = F'\phi$ follows now from the density of $\bigcup_{r\in P}\beta_{r^{-1}} i(A)$ in B and the continuity of the maps involved.

We consider next the question of the faithfulness of F. Suppose first that F is faithful. Then the dilation is faithful, and therefore V is a faithful interaction group by Proposition 3.5. Suppose conversely that V is faithful, and observe that the dilation obtained satisfies $K_{\mathsf{T}} = B$, so $F_r = \tilde{F}_r$, for all $r \in G$. Then V has an admissible minimal faithful dilation (B',β',F') by Corollary 3.13. By the universal property of the dilation $(i,(B,\beta,F))$, there exists a unique morphism $\phi : (B,\beta,F) \to (B',\beta',F')$ such that $\phi i = i'$. But the map ϕ must be an isomorphism by Theorem 3.15. Therefore, since $F = \phi^{-1}F'\phi$, we conclude that F is faithful. □

Corollary 3.19. *Let α be an action of the Ore semigroup P by unital injective endomorphisms of the unital commutative C^*-algebra A. Then the following assertions are equivalent:*

(1) *there exists a (faithful) interaction group $V : G \to \mathsf{B}(A)$ such that $V|_P = \alpha$;*
(2) *if $\beta : G \times B \to B$ is the minimal dilation of α (provided by Theorem 3.15), there exists a (faithful) conditional expectation $F : B \to B$ such that $(B,\beta,F) \in \mathcal{D}_G$.*

Proof. Just apply Theorem 3.18 to see that (1)→(2), and Propositions 2.3 (3) and 2.5 for the converse implication. □

Remark 3.20. Suppose that V and V' are interaction groups that extend actions by injective unital endomorphisms of the Ore semigroup P. Suppose as well that $\psi : (A,V) \to (A',V')$ is a morphism of interaction groups, and let (i,T) and (i',T') be the corresponding minimal admissible dilations of V and V'. Then $(i'\psi,T')$ is an admissible dilation of V, so there exists a unique morphism $\phi : T \to T'$ such that $\phi i = i'\psi$. In this way we obtain a functor from the category of interaction groups that extend actions by injective unital endomorphisms of the Ore semigroup P into the category $\mathcal{D}_G$, where G is the enveloping group of P.

When an interaction group extends an action of an Ore semigroup it is possible to determine if it is faithful by examining just the conditional expectations corresponding to the elements of the semigroup.

Proposition 3.21. *Let $\alpha : P \to \mathsf{B}(A)$ be an action of the Ore semigroup P by injective unital endomorphisms of the unital C^*-algebra A. If $V : G \to \mathsf{B}(A)$ is an interaction group that extends the action α, then V is faithful if and only if E_t is faithful, for every $t \in P$.*

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Proof. As shown in Proposition 2.6, V is faithful if and only if E_t is faithful, for all $t \in G$. Suppose now that E_t is faithful, for all $t \in P$. If $r \in P$, then $V_r = \alpha_r$, which is faithful because it is injective, and, since $E_r = V_r V_{r^{-1}}$ is faithful, so must $V_{r^{-1}}$ also be faithful. Finally, if $t \in G$ is any element, write it as $t = r^{-1}s$, with $r, s \in P$. Then, by Lemma 3.16, $V_t = V_{r^{-1}s} = V_{r^{-1}} V_s$, which is faithful for is the composition of faithful maps. □

Proposition 3.22. *Under the assumptions of Proposition 3.21, and if $r_1, \ldots, r_n$ are generators of P, then the following assertions are equivalent:*

(1) *V is faithful;*
(2) *$V_{r_i^{-1}}$ is faithful, for all $i = 1, \ldots, n$;*
(3) *E_{r_i} is faithful, for all $i = 1, \ldots, n$.*

Proof. It is easy to check the path of implications (1) → (2) → (3) → (2). To see that (2) → (1), note first that $V_r = \alpha_r$ is faithful for all $r \in P$. If $r = r_{i_1} \cdots r_{i_k}$, we have $V_{r^{-1}} = V_{r_{i_k}^{-1}} \cdots V_{r_{i_1}^{-1}}$ by (2) of Lemma 3.16, which shows that $V_{r^{-1}}$ is faithful. Finally, if $t \in G$ is arbitrary, write it as $t = r^{-1}s$. A new invocation of the second part of Lemma 3.16 shows that V_t is faithful. □

A particular relevant case is when the interaction group $V : \mathbb{Z} \to \mathsf{B}(A)$ extends an action of $\mathbb{N}$ by injective endomorphisms.

Corollary 3.23. *Let $\alpha : A \to A$ be a unital injective endomorphism of the unital C^*-algebra A, and suppose that $\mathcal{L} \in \mathsf{B}(A)$ is a transfer operator for α. Consider the interaction group $V : \mathbb{Z} \to \mathsf{B}(A)$ such that $V_n = \alpha^n$ if $n \geq 0$, $V_n = \mathcal{L}^{-n}$ otherwise. Then the following are equivalent: (a) V is faithful; (b) $\mathcal{L}$ is faithful; (c) $E := \alpha\mathcal{L}$ is faithful.*

We end the section with a result concerning minimal globalizations.

Proposition 3.24. *Let V be an interaction group as in Theorem 3.18, and let γ be the partial action associated to V via Proposition 2.6. Then γ has a minimal globalization, which is unique up to isomorphism.*

Proof. It follows from Theorem 3.18 and Corollary 3.7 that the action β provided by Theorem 3.18 is a minimal globalization of γ. Suppose now that $\beta' : G \times B' \to B'$ is another minimal globalization of γ, where B' is a C^*-algebra which contains A. To show that β and β' are isomorphic, it is enough to show that β' satisfies properties (1) and (2) of Theorem 3.15. By definition β' satisfies the first property, so let us see that it also satisfies the second. Note that if $t = r^{-1}s \in G$, with $r, s \in P$, then $\beta'_t(A) = \beta'_{r^{-1}}\alpha_s(A) \subseteq \beta'_{r^{-1}}(A)$, which shows that $\bigcup_{t\in G}\beta'_t(A) \subseteq \bigcup_{r\in P}\beta'_{r^{-1}}(A)$. On the other hand, suppose $r, s \in P$, with $r \leq s$. Then, since $sr^{-1} \in P$, we have $A \supseteq \alpha_{sr^{-1}}(A) = \beta'_s\beta'_{r^{-1}}(A)$, so $\beta'_{s^{-1}}(A) \supseteq \beta'_{r^{-1}}(A)$. This shows that $\bigcup_{r\in P}\beta'_{r^{-1}}(A)$ is a $*$-subalgebra of B' which contains $\bigcup_{t\in G}\beta'_t(A)$. This implies that B' is the closure of $\bigcup_{r\in P}\beta'_{r^{-1}}(A)$, as we wanted to prove. □

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4. Iterated function systems

We end the paper with an example originating in the so-called iterated function systems associated to inverse branches of continuous functions (see [4]). More precisely, suppose that $\gamma : X \to X$ is a continuous function on the compact Hausdorff space X, for which there exist continuous functions $\gamma_1, \ldots, \gamma_d$ on X such that $\gamma \circ \gamma_i = \mathrm{id}_X$ and $X = \bigcup_{i=1}^{d} \gamma_i(X)$. Let $A := C(X)$, and $\alpha, \alpha_i \in \mathsf{B}(E)$ be the endomorphisms induced by γ and γ_i respectively. Note that $\alpha_i \alpha = \mathrm{id}_A$, so α is injective, each α_i is surjective, and α_i is a transfer operator for α, for all $i = 1, \ldots, d$. Since convex combinations of transfer operators for α are transfer operators for α as well, it follows that $\mathcal{L} := (1/d) \sum_{i=1}^{d} \alpha_i$ is also a transfer operator for α. Thus we can consider the corresponding interaction group $V^\gamma : \mathbb{Z} \to \mathsf{B}(A)$, which is an extension of the action $\mathbb{N} \times A \to A$: $(n, a) \mapsto \alpha^n(a)$.

Proposition 4.1. *The interaction group V^γ has a faithful dilation.*

Proof. By Theorem 3.18 V^γ has a minimal admissible dilation, which is faithful if and only if V^γ is faithful. Now, by Corollary 3.23, V^γ is faithful if and only if $\mathcal{L}$ is faithful. So suppose that $a \in A$ is such that $\mathcal{L}(a^*a) = 0$. Then $\alpha_i(a^*a) = 0$ for all $i = 1, \ldots, d$, that is, $a^*a \circ \gamma_i = 0$. Since $X = \bigcup_{i=1}^{d} \gamma_i(X)$, we conclude that $a^*a = 0$, so $a = 0$. Then $\mathcal{L}$ is faithful. □

Since $x = \gamma(\gamma_i(x))$, for all $x \in X$, $i = 1, \ldots, d$, we have $\gamma^{-1}(x) \supseteq \{\gamma_1(x), \ldots, \gamma_d(x)\}$, for all $x \in X$. The converse inclusion follows from $X = \bigcup_{i=1}^{d} \gamma_i(X)$: if $x = \gamma(y)$, there exist $z \in X$ and some $i \in \{1, \ldots, d\}$ such that $y = \gamma_i(z)$, and therefore $x = \gamma(\gamma_i(z)) = z$, that is, $y = \gamma_i(x)$. Then $\gamma^{-1}(x) = \{\gamma_1(x), \ldots, \gamma_d(x)\}$ is a nonempty finite set of at most d elements, for each $x \in X$. We have

$$\mathcal{L}(b)(x) = \sum_{i=1}^{d} \frac{1}{d} b(\gamma_i(x)) = \sum_{y \in \gamma^{-1}(x)} \frac{n(y, x)}{d} b(y) = \sum_{y \in X} \frac{n(y, x)}{d} b(y),$$

where $n(y, x)$ is the number of elements of the set $\{j : \gamma_j(y) = x\}$.

A particular important case is when the iterated function system satisfies the strong separation condition, that is, when X is the disjoint union of the $\gamma_i(X)$, because then the iterated function system is isomorphic to the one we define next (see [4]). We will also describe very explicitly its dilation. Let $\Omega := \{1, \ldots, d\}^{\mathbb{N}}$, and let $\sigma : \Omega \to \Omega$ be given by $\sigma(x)(j) = x(j + 1)$, for all $x \in \Omega$, $j \geq 0$. For every $i \in \{1, \ldots, d\}$, let $\sigma_i : \Omega \to \Omega$ be such that

$$\sigma_i(x)(j) = \begin{cases} i & \text{if } j = 0, \\ x(j-1) & \text{if } j \geq 1. \end{cases}$$

Then we have $\sigma \circ \sigma_i = \mathrm{id}_\Omega$, for all $i = 1, \ldots, d$, and $\Omega = \biguplus_{i=1}^{d} \sigma_i(\Omega)$ (disjoint union). It is clear that $\sigma_i(\Omega)$ is open, and that σ is a local homeomorphism. Let $A := C(\Omega)$, and $\alpha, \alpha_i : A \to A$ be the dual maps of σ and σ_i respectively. Finally, let $\mathcal{L} := (1/d) \sum_{i=1}^{d} \alpha_i$, which is a transfer operator for α, and call V^σ the corresponding interaction group. We proceed now to describe its dilation. Let $\tilde{\Omega} := \{1, \ldots, d\}^{\mathbb{Z}}$, and let $\tilde{\sigma} : \tilde{\Omega} \to \tilde{\Omega}$

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be the Bernoulli shift, that is, $\tilde{\sigma}(y)(j) = y(j+1)$, for all $y \in \tilde{\Omega}$, $j \in \mathbb{Z}$. We have a natural inclusion $\iota : A \hookrightarrow B := C(\tilde{\Omega})$ given by the dual map of $\pi : \tilde{\Omega} \to \Omega$, defined as $\pi(y)(j) = y(j)$, for all $y \in \tilde{\Omega}$, $j \geq 0$. Note that $\pi \circ \tilde{\sigma}^n = \sigma^n \circ \pi$, for all $n \in \mathbb{N}$. Let $\beta : \mathbb{Z} \times B \to B$ be the action induced by $\tilde{\sigma}$. It is easy to see that (B, β) is the dilation of (A, α) announced by Theorem 3.15. It remains to find the conditional expectation $F : B \to \iota(A)$ such that (B, β, F) is the dilation of V^σ. For each $i = 1, \ldots, d$, let $\tau_i : \Omega \to \tilde{\Omega}$ be given by

$$\tau_i(y)(j) = \begin{cases} i & \text{if } j < 0, \\ y(j) & \text{if } j \geq 0. \end{cases}$$

Then $\pi \circ \tau_i = \mathrm{id}_\Omega$, so $\rho\iota = \mathrm{id}_A$, where ρ_i is the dual map of τ_i. Define $F_i : B \to \iota(A)$ as $F_i = \iota\rho_i$ (the dual map of $\tau_i \circ \pi$). Then F_i is a homomorphism of C^*-algebras, and also a conditional expectation onto $\iota(A)$. Moreover, $F_i \beta_{-1} \iota(a) = \iota(\alpha_i(a))$, for all $a \in A$ and $i = 1, \ldots, d$. So if we define $F := (1/d) \sum_{i=1}^{d} F_i$, we have that F is a conditional expectation onto $\iota(A)$, and $F\beta_{-1}\iota = \iota\mathcal{L}$.

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