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INTEGER POINTS OF ANALYTIC FUNCTIONS IN A HALF-PLANE

ALASTAIR N. FLETCHER AND J. K. LANGLEY

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK (jkl@maths.nott.ac.uk)

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Abstract It is shown that if f is an analytic function of sufficiently small exponential type in the right half-plane, which takes integer values on a subset of the positive integers having positive lower density, then f is a polynomial.

Keywords: analytic functions; integer values; half-plane

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1. Introduction

A classical theorem of Pólya (see [**13**] and [**19**, p. 55]) shows that 2^z is the slowest growing transcendental entire function which takes integer values at the non-negative integers. That is, let f be entire and take integer values on $\mathbb{N} \cup \{0\}$. Pólya shows that if

$$\limsup_{r\to\infty} \frac{M(r,f)}{2^r} < 1, \quad \text{where } M(r,f) = \max\{|f(z)| : |z| = r\},$$

then f is a polynomial and, further, that if

$$M(r,f) = O(r^N 2^r) \tag{1.1}$$

as $r \to \infty$ for some $N > 0$, then there exist polynomials P_1 and P_2 such that $f(z) \equiv P_1(z)2^z + P_2(z)$. Further results on integer-valued entire functions may be found in [**1**–**3**, **5**, **10**, **11**, **14**–**18**].

This paper is concerned with similar results for analytic functions in a half-plane. It was proved in [**7**, Lemma 5] that if f is analytic of polynomial growth in the right half-plane and takes integer values at the positive integers, then f is a polynomial. This result has several applications to value distribution theory and differential equations [**7**–**9**]. In [**12**], an analogue of Pólya's result for a half-plane is given. That is, let f be analytic in the closed right half-plane $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant 0\}$ with maximum modulus

$$M_\Omega(r,f) = \max\{|f(z)| : z \in \Omega,\ |z| \leqslant r\}, \tag{1.2}$$

and assume that $f(n)$ is an integer for all sufficiently large positive integers n. If f satisfies (1.1) as $r \to \infty$ for some $N > 0$, with $M(r, f)$ replaced by $M_\Omega(r, f)$, then again there exist polynomials P_1 and P_2 with $f(z) \equiv P_1(z)2^z + P_2(z)$. Furthermore, if f takes integer values at all the non-negative integers and

$$\limsup_{|z|\to\infty, z\in\Omega} \frac{|f(z)|}{2^{|z|}} < 1,$$

then f is a polynomial.

We remark that a result was proved in [**20**] for functions holomorphic on the product Ω^n of n half-planes and taking integer values on $\mathbb{N}^n$. This result contains [**7**, Lemma 5], but not the theorem from [**12**]. We are very grateful to the referee for drawing our attention to this reference and to others such as [**1**, **2**, **21**, **22**].

In order to state our result, some terminology will be required. Let f be analytic in Ω, and let $0 \leqslant \lambda < \infty$. Then f is of exponential type λ in Ω if

$$\limsup_{r\to\infty} \frac{\log^+ M_\Omega(r, f)}{r} = \lambda, \tag{1.3}$$

where $\log^+ x = \max\{0, \log x\}$ and $M_\Omega(r, f)$ is as in (1.2). This is of course in direct analogy with the definition of exponential type for entire functions. The main result to be proved is the following half-plane analogue of a theorem of Waldschmidt for entire functions [**15**].

Theorem 1.1. *Let d, J, λ satisfy*

$$0 < d < 1, \quad J \in \mathbb{N}, \quad \lambda > 0, \quad 16\left(\frac{1 + \log(1 + J/2)}{J}\right) + 8(J-1)\lambda < d^2. \tag{1.4}$$

Let $E \subset \mathbb{N}$ have lower linear density

$$\underline{D}(E) = \liminf_{n\to\infty} \frac{|E \cap \{1, \dots, n\}|}{n} > d,$$

where $|X|$ denotes the number of elements of the set X. Let the function f be analytic of exponential type less than λ in the closed right half-plane Ω, and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then f is a polynomial.

2. Lemmas used in the proof of Theorem 1.1

2.1. Linear forms

The following lemma is a slight modification of [**5**, Lemma I, p. 11]: a proof is given for completeness.

Lemma 2.1. *Let $A \geqslant 1$ and $N \geqslant 2$ be integers. Suppose that $L_1, \dots, L_m$ are linear forms in the n variables $x_1, \dots, x_n$, with real coefficients $a_{j,k}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$, that is,*

$$L_j = a_{j,1}x_1 + \dots + a_{j,n}x_n.$$

Suppose further that $n > m$ and

$$\max_{j,k} |a_{j,k}| \leqslant A.$$

Then there exist integers $x_1, \dots, x_n$, not all zero, such that

$$|L_j| \leqslant \frac{1}{N}$$

for $j = 1, \dots, m$, and

$$|x_k| \leqslant 2(2nAN)^{m/(n-m)}$$

for $k = 1, \dots, n$.

Proof. Define X by

$$X = [(2nAN)^{m/(n-m)}],$$

where $[x]$ denotes the greatest integer not exceeding x. An n-tuple of integers $(x_1, \dots, x_n)$, in which each x_k has absolute value no greater than X, gives rise to a point $(L_1, \dots, L_m)$ lying in the closed m-dimensional cube of centre $(0, \dots, 0)$ and side length $2nAX$. Divide this cube into $(2nAXN)^m$ closed subcubes, each of side length $1/N$. The number of distinct n-tuples $(x_1, \dots, x_n)$ is evidently

$$(2X+1)^n \geqslant (2(2nAN)^{m/(n-m)} - 1)^n > (2nAN)^{nm/(n-m)} \geqslant (2nAXN)^m,$$

since if this is not the case, we get

$$(2nAN)^n < (2nAXN)^{n-m} \leqslant (2nAN)^{n-m}(2nAN)^m,$$

which is impossible. Hence, there are distinct n-tuples giving rise to points $(L_1', \dots, L_m')$ and $(L_1'', \dots, L_m'')$ lying in the same subcube. But then we may write

$$\left| \sum_{k=1}^{n} a_{j,k}(x_k' - x_k'') \right| = |L_j' - L_j''| \leqslant \frac{1}{N}$$

for $j = 1, \dots, m$, where

$$|x_k' - x_k''| \leqslant 2X \leqslant 2(2nAN)^{m/(n-m)}$$

and $x_k = x_k' - x_k'' \neq 0$ for at least one k. This completes the proof. □

2.2. An application of the maximum principle

Lemma 2.2. *Let d, M, L, K satisfy*

$$0 < d < 1, \quad M > 0, \quad 1 < K < L < \infty, \quad ML^2K < d^2(L-K). \tag{2.1}$$

Let $G \subseteq \mathbb{N}$ and let F be analytic in the closed right half-plane Ω such that $F(z) \in \mathbb{Z}$ for all $z \in G$. Let $s > 0$ be such that $M_\Omega(Ls, F) \leqslant \mathrm{e}^{MLs}$ and F has $m \geqslant ds$ distinct zeros in $G \cap [1, s]$. Then $F(z) = 0$ for all $z \in G \cap [s, Ks]$.

Proof. Let $x_1, \dots, x_m$ be distinct zeros of F in $G \cap [1, s]$. For $0 < x \leqslant s$ let

$$p(z) = p(z, x) = \frac{z - x}{z + x}.$$

Then p satisfies

$$|p(z)| \begin{cases} = 1, & z \in \mathrm{i}\mathbb{R}, \\ \geqslant \dfrac{Ls - x}{Ls + x}, & |z| = Ls, \\ \leqslant \dfrac{Ks - x}{Ks + x}, & z \in [s, Ks] \subseteq \mathbb{R}, \end{cases}$$

the last estimate following from monotonicity. Next, let

$$g(x) = \log \left[\left(\frac{Ls + x}{Ls - x} \right) \left(\frac{Ks - x}{Ks + x} \right) \right].$$

Then, for $0 \leqslant x \leqslant s$,

$$\begin{aligned} g'(x) &= \frac{1}{Ls + x} + \frac{1}{Ls - x} - \frac{1}{Ks + x} - \frac{1}{Ks - x} \\ &= \frac{2Ls}{L^2 s^2 - x^2} - \frac{2Ks}{K^2 s^2 - x^2} \\ &= \frac{2s^3 KL(K - L) + 2x^2 s(K - L)}{(L^2 s^2 - x^2)(K^2 s^2 - x^2)} \\ &\leqslant \frac{2(K - L)}{LKs} \end{aligned}$$

and hence

$$g(x) \leqslant \frac{2x(K - L)}{LKs}.$$

The function

$$F_1(z) = F(z) \prod_{j=1}^{m} \frac{1}{p(z, x_j)}$$

is analytic in Ω and satisfies

$$|F_1(z)| \leqslant M_\Omega(Ls, F) \prod_{j=1}^{m} \frac{Ls + x_j}{Ls - x_j}$$

on the boundary of the region given by $z \in \Omega$, $|z| \leqslant Ls$, and this estimate also holds for $z \in [s, Ks]$, by the maximum principle. For $z \in [s, Ks]$ it therefore follows that

$$\begin{aligned} |F(z)| &\leqslant M_\Omega(Ls, F) \prod_{j=1}^{m} \left[\left(\frac{Ls + x_j}{Ls - x_j} \right) \left(\frac{Ks - x_j}{Ks + x_j} \right) \right] \\ &= M_\Omega(Ls, F) \exp\left(\sum_{j=1}^{m} g(x_j) \right) \end{aligned}$$

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$$\begin{aligned}
&\leqslant M_\Omega(Ls,F)\exp\left(\frac{2(K-L)}{LKs}\sum_{j=1}^{m}x_j\right)\\
&\leqslant M_\Omega(Ls,F)\exp\left(\frac{2(K-L)}{LKs}\cdot\frac{m(m+1)}{2}\right)\\
&\leqslant \exp\left(MLs+\frac{d^2(K-L)s}{LK}\right)\\
&< 1,
\end{aligned}$$

using (2.1) and the fact that the x_j are distinct positive integers, which proves the lemma. □

In order to apply Lemma 2.2, it is necessary for a given d to choose M, L and K with (2.1) in mind. Evidently, if

$$ML^2 < d^2(L-1),$$

then K may be chosen with $K-1$ small and positive so that (2.1) is satisfied. Since elementary calculus gives

$$q(L) = \frac{L-1}{L^2} \leqslant q(2) = \tfrac{1}{4}$$

for $1 < L < \infty$, the appropriate condition is $4M < d^2$.

Lemma 2.3. *Let $0 < d < 1$ and $0 < 4M < d^2$. Let $G \subseteq \mathbb{N}$ and let F be analytic in Ω such that $F(z) \in \mathbb{Z}$ for all $z \in G$. Let $S > 0$ be such that*

$$Q(r) = |G \cap [0,r]| \geqslant dr \quad \text{and} \quad M_\Omega(r,F) \leqslant \mathrm{e}^{Mr}$$

for all $r \geqslant S$, and assume that $F(z) = 0$ for all z in $G \cap [1,S]$. Then $F(z) = 0$ for all $z \in G$.

Proof. Choose $L = 2$ and $K \in (1,2)$ such that (2.1) is satisfied. Then F has at least dS distinct zeros in $G \cap [1,S]$. Applying Lemma 2.2 with $s = S$ then shows that $F(z) = 0$ for all $z \in G \cap [S,KS]$, from which it follows at once that F has at least $Q(KS) \geqslant dKS$ distinct zeros in $[1,KS]$. Hence, Lemma 2.2 may again be applied, this time with $s = KS$. Repetition of this argument proves Lemma 2.3. □

2.3. The Nevanlinna characteristic in a half-plane

This section provides a brief overview of a half-plane characteristic analogous to the Nevanlinna characteristic in the plane, the details of which may be found in [**6**, p. 38]. Let f be meromorphic in the closed upper half-plane

$$\overline{\mathbb{H}} = \{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant 0\}$$

with poles at $\rho_n \mathrm{e}^{\mathrm{i}\psi_n}$, where $\rho_n \geqslant 0$ and $0 \leqslant \psi_n \leqslant \pi$. The counting function of the poles is

$$c(r,f) = \sum_{1<\rho_n\leqslant r} \sin\psi_n,$$

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and the integrated counting function takes the form

$$C(r,f) = 2\int_1^r c(t,f)\left(\frac{1}{t^2}+\frac{1}{r^2}\right)\mathrm{d}t = 2\sum_{1<\rho_n\leqslant r}\left(\frac{1}{\rho_n}-\frac{\rho_n}{r^2}\right)\sin\psi_n.$$

The analogue of the Nevanlinna proximity function consists of the following two functions:

$$A(r,f) = \frac{1}{\pi}\int_1^r\left(\frac{1}{t^2}-\frac{1}{r^2}\right)[\log^+|f(t)|+\log^+|f(-t)|]\,\mathrm{d}t,$$
$$B(r,f) = \frac{2}{\pi r}\int_0^\pi \log^+|f(re^{\mathrm{i}\phi})|\sin\phi\,\mathrm{d}\phi.$$

The half-plane characteristic is then given by

$$S(r,f) = A(r,f)+B(r,f)+C(r,f)$$

and satisfies, for non-constant f and $a\in\mathbb{C}$,

$$S\left(r,\frac{1}{f-a}\right) = S(r,f)+O(1) \tag{2.2}$$

as $r\to\infty$.

The following lemma uses the half-plane characteristic and is in the spirit of Carlson's theorem [**4**]. For generalizations in other directions see [**21**, **22**].

Lemma 2.4. *Let $E\subseteq \mathrm{i}\mathbb{N}=\{\mathrm{i},2\mathrm{i},\dots\}$ have lower density D. Let f be analytic in $\overline{\mathbb{H}}$, of exponential type $\lambda<\pi D$, with $f(z)=0$ for all $z\in E$. Then $f\equiv 0$.*

Here the lower density of E and exponential type relative to the upper half-plane are defined in straightforward analogy with § 1.

Proof. Assume that f is not identically zero. As $r\to\infty$,

$$B(r,f)\leqslant \frac{2}{\pi r}\int_0^\pi(\lambda+o(1))r\sin\phi\,\mathrm{d}\phi = O(1)$$

and

$$A(r,f)\leqslant \frac{1}{\pi}\int_1^r\left(\frac{1}{t^2}-\frac{1}{r^2}\right)2(\lambda+o(1))t\,\mathrm{d}t+O(1)\leqslant \frac{2(\lambda+o(1))}{\pi}\log r.$$

Since f has no poles in $\overline{\mathbb{H}}$, applying (2.2) with $a=0$ now gives

$$S\left(r,\frac{1}{f}\right)\leqslant A(r,f)+B(r,f)+O(1)\leqslant \frac{2(\lambda+o(1))}{\pi}\log r. \tag{2.3}$$

But since the lower density of E is D we have

$$c\left(r,\frac{1}{f}\right)\geqslant \sum_{n\in\mathbb{N}\cap(1,r],\ \mathrm{i}n\in E} 1 \geqslant (D-o(1))r$$

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as $r \to \infty$. Integrating this yields

$$S\left(r, \frac{1}{f}\right) \geqslant C\left(r, \frac{1}{f}\right) \geqslant 2 \int_1^r (D - o(1))t\left(\frac{1}{t^2} + \frac{1}{r^2}\right) \mathrm{d}t \geqslant 2(D - o(1)) \log r$$

as $r \to \infty$, which, on combination with (2.3), gives $\lambda \geqslant \pi D$, a contradiction. Therefore, f must be identically zero. □

2.4. A class of polynomials

The following lemma summarizes some basic facts from [**15**] concerning a class of polynomials which are key to the proof of Theorem 1.1.

Lemma 2.5. *Define polynomials* $p_0, p_1, \dots$ *by*

$$p_0(z) = 1, \quad p_1(z) = z, \quad p_h(z) = \frac{z(z-1)\cdots(z-h+1)}{h!}, \quad h = 2, 3, \dots. \tag{2.4}$$

Then $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$*, and for* $R > 0$ *and* $H \in \mathbb{N}$ *we have*

$$|p_h(z)| \leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H \quad \text{for } |z| \leqslant R,\ h = 0, \dots, H. \tag{2.5}$$

Proof. It is easy to check that $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$. To prove (2.5) we write, following [**15**],

$$|p_h(z)| \leqslant \frac{(R+H)^h}{h!} \leqslant \frac{H^h}{h!}\left(\frac{R}{H} + 1\right)^H \leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H.$$

□

2.5. Algebraic functions mapping integers to integers

Proposition 2.6. *Let the algebraic function* f *be analytic in* Ω *and let it satisfy* $f(E) \subseteq \mathbb{Z}$ *for some set* $E \subseteq \mathbb{N}$ *of positive lower density. Then* f *is a polynomial.*

To prove Proposition 2.6, let E and f be as in the hypotheses, and assume that the lower density of E exceeds $D > 0$. We assert that f maps the positive real axis into $\mathbb{R}$. To see this, observe that the functions $\overline{f(\bar{z})}$ and $f(z) - \overline{f(\bar{z})}$ are algebraic because f is algebraic. Since $f(z) \in \mathbb{R}$ for $z \in E$ and since an algebraic function having a sequence of zeros tending to infinity must vanish identically, the assertion follows.

Again since f is algebraic, there exists a positive integer m such that, for all sufficiently large r,

$$M_\Omega(r, f) \leqslant r^m. \tag{2.6}$$

Let n and N be integers with n/m and N/n large, and in particular with

$$DN \geqslant n + 1. \tag{2.7}$$

Lemma 2.7. *There exist arbitrarily large* $r \in \mathbb{N}$ *such that*

$$|E \cap \{r, r+1, \dots, r+N-1\}| \geqslant n + 1. \tag{2.8}$$

Proof. Assume that there exists $p_0 \in \mathbb{N}$ such that, for every $p \geqslant p_0$,

$$|E \cap \{Np, \dots, N(p+1)-1\}| \leqslant n.$$

Since the lower density of E exceeds D, for large p, this gives

$$DNp \leqslant |E \cap \{1, \dots, Np\}| \leqslant (p-p_0)n + O(1) \leqslant (n+o(1))p,$$

which contradicts (2.7). □

Let ε be small and positive and choose a large positive integer r satisfying (2.8). Let $\Gamma = \Gamma_r$ be the circle of centre r, radius εr, described once anticlockwise. Choose distinct

$$a_0, \dots, a_n \in E \cap \{r, r+1, \dots, r+N-1\}. \tag{2.9}$$

Then $a_0, \dots, a_n$ lie inside Γ, since r is large.

For $k = 0, \dots, n$ it follows from Cauchy's integral formula and the identity

$$\frac{1}{t-z} = \frac{1}{t-a_0} + \frac{z-a_0}{(t-a_0)(t-a_1)} + \dots + \frac{(z-a_0)\cdots(z-a_k)}{(t-a_0)\cdots(t-a_k)(t-z)},$$

which is easily proved by induction, that

$$f(z) = P_k(z) + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(z-a_0)\cdots(z-a_k)f(t)\,\mathrm{d}t}{(t-a_0)\cdots(t-a_k)(t-z)} \tag{2.10}$$

for z inside Γ, where

$$P_k(z) = A_0 + A_1(z-a_0) + \dots + A_k(z-a_0)\cdots(z-a_{k-1}) \tag{2.11}$$

is given by

$$A_j = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(t)\,\mathrm{d}t}{(t-a_0)\cdots(t-a_j)}. \tag{2.12}$$

Thus, $P_k(z)$ is the interpolating polynomial of degree at most k which equals $f(z)$ at the $k+1$ points $a_0, \dots, a_k$ [**5**, p. 103].

Next, let

$$Q = \prod_{0 \leqslant j < k \leqslant n} |a_k - a_j| \leqslant C = N^{(n+1)^2}, \tag{2.13}$$

and observe that C is independent of r. Since $f(a_j) \in \mathbb{Z}$ for $j = 0, \dots, n$, it follows from (2.12) and the residue theorem that

$$QA_j \in \mathbb{Z} \quad \text{for } j = 0, \dots, n. \tag{2.14}$$

On the other hand, since r is large, (2.9) gives

$$|t - a_j| \geqslant \tfrac{1}{2}\varepsilon r$$

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for $t \in \Gamma$. Thus, combining (2.6), (2.12) and (2.13) yields, for $m < j \leqslant n$, again since r is large,

$$|QA_j| \leqslant \frac{C(\varepsilon r)(2r)^m}{(\varepsilon r/2)^{j+1}} < \frac{1}{2},$$

which in conjunction with (2.14) gives $A_j = 0$.

Recalling (2.10) and the definition (2.11) of P_k, it now follows that $P_m = P_n$ and that $f - P_m$ has $n+1$ zeros $a_0, \dots, a_n$ in the interval $[r, r+N-1]$. Hence, $f^{(n)} = (f - P_m)^{(n)}$ has a zero in the same interval, using Rolle's theorem. Since r may be chosen arbitrarily large, the algebraic function $f^{(n)}$ must vanish identically, and f is a polynomial. This proves Proposition 2.6.

3. Proof of Theorem 1.1

Let $E \subseteq \mathbb{N}$ and let d, J, λ, f be as in the hypotheses. Label the elements of E as $1 \leqslant \alpha_1 < \alpha_2 < \cdots$. Let R be a large positive integer such that

$$H = \frac{n}{J} \in \mathbb{N}, \quad \text{where } m = |E \cap [1, R]| \text{ and } n = 2m. \tag{3.1}$$

Define the functions

$$q_{\mu,\nu}(z) = p_\mu(z) f(z)^\nu, \quad \mu = 0, 1, \dots, H-1, \ \nu = 0, 1, \dots, J-1, \tag{3.2}$$

where p_μ is defined as in (2.4). This gives $HJ = n$ functions, which we label $g_1, \dots, g_n$, where

$$g_k(z) = p_{\mu(k)}(z) f(z)^{\nu(k)}.$$

In order to prove Theorem 1.1, it suffices to show that the functions $g_1, \dots, g_n$ are linearly dependent over $\mathbb{C}$. Once such a relation

$$\sum_{k=1}^{n} B_k g_k(z) \equiv 0$$

is established with the B_k constants, not all zero, then it cannot be the case that there is an integer q such that $B_k \neq 0$ implies $\nu(k) = q$, because p_h has degree h in (2.4). Hence, it follows that f is algebraic, and Proposition 2.6 shows that f is a polynomial.

In order to prove that the g_k are linearly dependent, observe first that

$$a_{j,k} = g_k(\alpha_j) \in \mathbb{Z},$$

using Lemma 2.5. Moreover, we have, for $j = 1, \dots, m$,

$$\begin{aligned} |a_{j,k}| &\leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H (1 + M_\Omega(R, f))^{J-1} \\ &\leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H \mathrm{e}^{(J-1)\lambda R} \\ &= J(R) \leqslant A = [J(R)] + 1, \end{aligned} \tag{3.3}$$

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by (2.5), (3.2) and the fact that R is large. Applying Lemma 2.1 with $N = 2$ yields integers $A_1, \dots, A_n$, not all zero, such that

$$\sum_{k=1}^{n} A_k g_k(\alpha_j) = 0 \tag{3.4}$$

for $j = 1, \dots, m$, and

$$|A_k| \leqslant 8nA, \tag{3.5}$$

since $n = 2m$. Set

$$F(z) = \sum_{k=1}^{n} A_k g_k(z). \tag{3.6}$$

Lemma 3.1. *Choose a real number M with*

$$4\left(\frac{1 + \log(1 + J/2)}{J}\right) + 2(J-1)\lambda < M < \tfrac{1}{4}d^2, \tag{3.7}$$

using (1.4). Provided that R is chosen large enough, we have

$$|E \cap [1, r]| \geqslant dr \quad \text{and} \quad \log^+ M_\Omega(r, F) \leqslant Mr \quad \text{for } r \geqslant R. \tag{3.8}$$

Proof. The first inequality of (3.8) holds provided, R is chosen large enough, since E has lower density greater than d. Let c denote positive constants which do not depend on r or R. Then we have

$$\begin{aligned} M_\Omega(r, F) &\leqslant 8n^2 A \mathrm{e}^H \left(\frac{r}{H} + 1\right)^H (1 + M_\Omega(r, f))^{J-1} \\ &\leqslant cr^2 \mathrm{e}^{2H} \left(\frac{r}{H} + 1\right)^{2H} \mathrm{e}^{2(J-1)\lambda r} \end{aligned}$$

using (2.5), (3.3), (3.5) and the fact that R is large. Now (3.1) gives

$$\frac{r}{H} \geqslant \frac{R}{H} = \frac{RJ}{n} \geqslant \frac{J}{2}.$$

Since the function

$$\frac{1 + \log(x+1)}{x}$$

is decreasing for $x > 0$ this yields, for $r \geqslant R$,

$$\begin{aligned} \log^+ M_\Omega(r, F) &\leqslant 2r\left(\frac{H}{r}\right)\left(1 + \log\left(\frac{r}{H} + 1\right)\right) + 2(J-1)\lambda r + O(\log r) \\ &\leqslant 4r\left(\frac{1 + \log(1 + J/2)}{J}\right) + 2(J-1)\lambda r + O(\log r) \\ &< Mr, \end{aligned}$$

provided R is chosen large enough. □

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The function F satisfies $F(z) \in \mathbb{Z}$ for all $z \in E$, and $F(z) = 0$ for all $z \in E \cap [1, R]$ by (3.4) and (3.6). It then follows from (3.7), (3.8) and Lemma 2.3, with $S = R$ and $G = E$, that $F(z) = 0$ for all $z \in E$. But (3.7) also gives

$$4M < d^2 < d, \quad M < \pi d,$$

and so (3.8) and Lemma 2.4, applied to the function $F(-\mathrm{i}z)$, show that $F(z)$ vanishes identically, which completes the proof of Theorem 1.1.

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