SOME REVERSES OF THE JENSEN INEQUALITY
WITH APPLICATIONS

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Abstract

Two new reverses of the celebrated Jensen’s inequality for convex functions in the general setting of the Lebesgue integral, with applications to means, Hölder’s inequality and f-divergence measures in information theory, are given.

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1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set $\Omega$, a $\sigma$-algebra $\mathcal{A}$ of parts of $\Omega$ and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a $\mu$-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu$-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is } \mu\text{-measurable and } \int_{\Omega} w(x) |f(x)| \, d\mu(x) < \infty \right\}.$$ 

For simplicity of notation, we write everywhere in the following $\int_{\Omega} w \, d\mu$ instead of $\int_{\Omega} w(x) \, d\mu(x)$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu$-measurable functions, $\int_{\Omega} w \, d\mu = 1$ and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T_w(f, g) := \int_{\Omega} wfg \, d\mu - \int_{\Omega} w f \, d\mu \int_{\Omega} w g \, d\mu.$$ 

The following result is known in the literature as the Grüss inequality:

$$|T_w(f, g)| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta),$$

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provided

\[-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty\]

for \(\mu\)-a.e. \(x \in \Omega\).

The constant \(\frac{1}{4}\) is sharp in the sense that it cannot be replaced by a smaller constant.

If we assume that \(-\infty < \gamma \leq f(x) \leq \Gamma < \infty\) for \(\mu\)-a.e. \(x \in \Omega\), then, by the Grüss inequality for \(g = f\) and by Schwarz’s integral inequality,

\[
\int_\Omega w \left| f - \int_\Omega w f \, d\mu \right| \, d\mu \leq \left( \int_\Omega w f^2 \, d\mu - \left( \int_\Omega w f \, d\mu \right)^2 \right)^{1/2} \leq \frac{1}{2} (\Gamma - \gamma). \tag{1.1}
\]

To provide a reverse of the celebrated Jensen’s integral inequality for convex functions, in 2002, the author \cite{12} obtained the following result.

\textbf{Theorem 1.1.} Let \(\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}\) be a differentiable convex function on \((m, M)\) and \(f : \Omega \to [m, M]\) such that \(\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)\), where \(w \geq 0\) \(\mu\)-a.e. on \(\Omega\) with \(\int_\Omega w \, d\mu = 1\). Then we have the inequality

\[
0 \leq \int_\Omega w(\Phi \circ f) \, d\mu - \Phi\left( \int_\Omega w f \, d\mu \right) \leq \int_\Omega w(\Phi' \circ f) f \, d\mu - \int_\Omega w(\Phi' \circ f) \, d\mu \int_\Omega w f \, d\mu \tag{1.2}
\]

\[
\leq \frac{1}{2} (\Phi' (M) - \Phi' (m)) \int_\Omega \left| f - \int_\Omega w f \, d\mu \right| \, d\mu.
\]

For a generalisation of the first inequality in (1.2) without the differentiability assumption and the derivative \(\Phi'\) replaced with a selection \(\varphi\) from the subdifferential \(\partial \Phi\), see Niculescu \cite{27}.

If \(\mu(\Omega) < \infty\) and \(\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L(\Omega, \mu)\), then we have the inequality

\[
0 \leq \frac{1}{\mu(\Omega)} \int_\Omega (\Phi \circ f) \, d\mu - \Phi\left( \frac{1}{\mu(\Omega)} \int_\Omega f \, d\mu \right) \leq \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) f \, d\mu - \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) \, d\mu \cdot \frac{1}{\mu(\Omega)} \int_\Omega f \, d\mu \leq \frac{1}{2} (\Phi' (M) - \Phi' (m)) \frac{1}{\mu(\Omega)} \int_\Omega \left| f - \frac{1}{\mu(\Omega)} \int_\Omega f \, d\mu \right| \, d\mu.
\]

The following discrete inequality is of interest as well.

\textbf{Corollary 1.2.} Let \(\Phi : [m, M] \to \mathbb{R}\) be a differentiable convex function on \((m, M)\). If \(x_i \in [m, M]\) and \(w_i \geq 0\) \((i = 1, \ldots, n)\) with \(W_n := \sum_{i=1}^n w_i = 1\), then we have the...
Some reverses of the Jensen inequality

counterpart of Jensen’s weighted discrete inequality:

\[ 0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \]
\[ \leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \]
\[ \leq \frac{1}{2} (\Phi'(M) - \Phi'(m)) \left( \sum_{i=1}^{n} w_i x_i - \sum_{j=1}^{n} w_j x_j \right). \]

(1.3)

**Remark 1.3.** The inequality between the first and the second terms in (1.3) was proved in 1994 by Dragomir and Ionescu [15].

Using the results (1.2) and (1.1), we can state the following string of reverse inequalities:

\[ 0 \leq \int_{\Omega} w(\Phi \circ f) \, d\mu - \Phi\left(\int_{\Omega} w f \, d\mu\right) \]
\[ \leq \int_{\Omega} w(\Phi' \circ f) f \, d\mu - \int_{\Omega} w(\Phi' \circ f) \, d\mu \int_{\Omega} w f \, d\mu \]
\[ \leq \frac{1}{2} (\Phi'(M) - \Phi'(m)) \int_{\Omega} w \left| f - \int_{\Omega} w f \, d\mu \right| \, d\mu \]
\[ \leq \frac{1}{2} (\Phi'(M) - \Phi'(m)) \left( \int_{\Omega} w f^2 \, d\mu - \left( \int_{\Omega} w f \, d\mu \right)^2 \right)^{1/2} \]
\[ \leq \frac{1}{4} (\Phi'(M) - \Phi'(m))(M - m), \]

provided that \( \Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable convex function on \( (m, M) \) and \( f : \Omega \rightarrow [m, M] \) such that \( \Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L^2_{w}(\Omega, \mu) \), where \( w \geq 0 \) \( \mu \)-a.e. on \( \Omega \) with \( \int_{\Omega} w \, d\mu = 1 \).

**Remark 1.4.** The inequality between the first, second and last terms from (1.4) was proved in the general case of positive linear functionals in 2001 by the author [11].

Motivated by the above results, we establish in the current paper two new reverses of Jensen’s integral inequality for a convex function. Some natural applications for inequalities between means, reverses of Hölder’s inequality and for the \( f \)-divergence measure that play an important role in information theory are given as well.

## 2. Reverse inequalities

The following reverse of Jensen’s inequality holds.

**Theorem 2.1.** Let \( \Phi : I \rightarrow \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and let \( m, M \in \mathbb{R} \), \( m < M \), with \( [m, M] \subset \bar{I} \) (where \( \bar{I} \) is the interior of \( I \)).
If \( f : \Omega \to \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds
\[
-\infty < m \leq f(x) \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega
\]
and is such that \( f, \Phi \circ f \in L_\mu(\Omega, \mu) \), where \( w \geq 0 \) \( \mu \)-a.e. on \( \Omega \) with \( \int_\Omega w \, d\mu = 1 \), then
\[
0 \leq \int_\Omega w(\Phi \circ f) \, d\mu - \Phi(f_{\Omega,w})
\leq \frac{(M - f_{\Omega,w})(f_{\Omega,w} - m)}{M - m} \sup_{t \in (m, M)} \Psi(\Phi(t); m, M)
\leq \frac{1}{4} (M - m)(\Phi^\prime_-(M) - \Phi^\prime_+(m)),
\]
where \( f_{\Omega,w} := \int_\Omega w(x)f(x) \, d\mu(x) \in [m, M] \) and \( \Psi(\cdot; m, M) : (m, M) \to \mathbb{R} \) is defined by
\[
\Psi(\Phi(t); m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.
\]
We also have the inequality
\[
0 \leq \int_\Omega w(\Phi \circ f) \, d\mu - \Phi(f_{\Omega,w}) \leq \frac{1}{4} (M - m)\Psi(f_{\Omega,w}; m, M)
\leq \frac{1}{4} (M - m)(\Phi^\prime_-(M) - \Phi^\prime_+(m)),
\]
provided that \( f_{\Omega,w} \in (m, M) \).

**Proof.** By the convexity of \( \Phi \),
\[
\int_\Omega w(x)\Phi(f(x)) \, d\mu(x) - \Phi(f_{\Omega,w})
\leq \int_\Omega \left( \int_\Omega \frac{m(M - f(x)) + M(f(x) - m)}{M - m} \, d\mu(x) \right) \Phi(f(x)) - \Phi(f_{\Omega,w})
\leq \int_\Omega \left( \Phi\left( \int_\Omega \frac{m(M - f(x)) + M(f(x) - m)}{M - m} \, d\mu(x) \right) \right) \Phi(f(x)) - \Phi(f_{\Omega,w})
\leq \frac{(M - f(x))\Phi(m) + (f(x) - m)\Phi(M)}{M - m} w(x) \, d\mu(x)
\leq \Phi\left( \int_\Omega \frac{m(M - f_{\Omega,w}) + M(f_{\Omega,w} - m)}{M - m} \, d\mu(x) \right)
\leq \frac{(M - f_{\Omega,w})\Phi(m) + (f_{\Omega,w} - m)\Phi(M)}{M - m} := B.
\]
By denoting
\[
\Delta_{\Phi}(t; m, M) := \frac{(t-m)\Phi(M) + (M-t)\Phi(m)}{M-m} - \Phi(t), \quad t \in [m, M],
\]
we have
\[
\Delta_{\Phi}(t; m, M) = \frac{(t-m)\Phi(M) + (M-t)\Phi(m) - (M-m)\Phi(t)}{M-m} \\
= \frac{(t-m)(\Phi(M) - \Phi(t)) - (M-t)(\Phi(t) - \Phi(m))}{M-m} \\
= \frac{(M-t)(t-m)}{M-m} \Psi_{\Phi}(t; m, M)
\]
for any \(t \in (m, M)\).

Therefore we have the equality
\[
B = \frac{(M - \bar{f}_{\Omega,w})(\bar{f}_{\Omega,w} - m)}{M-m} \Psi_{\Phi}(\bar{f}_{\Omega,w}; m, M), \quad (2.4)
\]
provided that \(\bar{f}_{\Omega,w} \in (m, M)\).

For \(\bar{f}_{\Omega,w} = m\) or \(\bar{f}_{\Omega,w} = M\) the inequality (2.1) is obvious. If \(\bar{f}_{\Omega,w} \in (m, M)\), then
\[
\Psi_{\Phi}(\bar{f}_{\Omega,w}; m, M) \leq \sup_{\Omega \in (m, M)} \Psi_{\Phi}(t; m, M)
\]
\[
= \sup_{\Omega \in (m, M)} \left( \frac{\Phi(M) - \Phi(t)}{M-t} - \frac{\Phi(t) - \Phi(m)}{t-m} \right)
\]
\[
\leq \sup_{\Omega \in (m, M)} \left( \frac{\Phi(M) - \Phi(t)}{M-t} \right) + \sup_{\Omega \in (m, M)} \left( -\frac{\Phi(t) - \Phi(m)}{t-m} \right)
\]
\[
= \sup_{\Omega \in (m, M)} \left( \frac{\Phi(M) - \Phi(t)}{M-t} \right) - \inf_{\Omega \in (m, M)} \left( \frac{\Phi(t) - \Phi(m)}{t-m} \right)
\]
\[
= \Phi_{-}'(M) - \Phi_{-}'(m),
\]
which by (2.3) and (2.4) produces the desired result (2.1).

Since, obviously,
\[
\frac{(M - \bar{f}_{\Omega,w})(\bar{f}_{\Omega,w} - m)}{M-m} \leq \frac{1}{4} (M-m),
\]
then by (2.3) and (2.4) we deduce the first inequality (2.2). The second part is clear. \(\square\)

**Corollary 2.2.** Let \(\Phi : I \to \mathbb{R}\) be a continuous convex function on the interval of real numbers \(I\) and \(m, M \in \mathbb{R}, \ m < M, \) with \([m, M] \subset I\). If \(x_i \in [m, M]\) and \(p_i \geq 0\)
for \( i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i = 1 \), then we have the inequalities
\[
0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \\
\leq \frac{(M - \bar{x}_p)(\bar{x}_p - m)}{M - m} \sup_{t \in (m, M)} \Psi_\Phi(t; m, M) \\
\leq (M - \bar{x}_p)(\bar{x}_p - m) \frac{\Phi'(M) - \Phi'_+(m)}{M - m} \\
\leq \frac{1}{4}(M - m)(\Phi'_-(M) - \Phi'_+(m)),
\]
and
\[
0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \leq \frac{1}{4}(M - m)\Psi_\Phi(\bar{x}_p; m, M) \\
\leq \frac{1}{4}(M - m)(\Phi'_-(M) - \Phi'_+(m)),
\]
where \( \bar{x}_p := \sum_{i=1}^{n} p_ix_i \in (m, M) \).

**Remark 2.3.** Define the weighted arithmetic mean of the positive \( n \)-tuple \( x = (x_1, \ldots, x_n) \) with the nonnegative weights \( w = (w_1, \ldots, w_n) \) by
\[
A_n(w, x) := \frac{1}{W_n} \sum_{i=1}^{n} w_ix_i,
\]
where \( W_n := \sum_{i=1}^{n} w_i > 0 \), and the weighted geometric mean of the same \( n \)-tuple by
\[
G_n(w, x) := \left( \prod_{i=1}^{n} x_i^{w_i} \right)^{1/W_n}.
\]
It is well known that the following arithmetic mean–geometric mean inequality holds true:
\[
A_n(w, x) \geq G_n(w, x).
\]

Applying the inequality between the first and third terms in (2.5) for the convex function \( \Phi(t) = -\log t, t > 0 \),
\[
1 \leq \frac{A_n(w, x)}{G_n(w, x)} \leq \exp\left( \frac{1}{Mm} (M - A_n(w, x))(A_n(w, x) - m) \right) \\
\leq \exp\left( \frac{1}{4} \frac{(M - m)^2}{mM} \right),
\]
provided that \( 0 < m \leq x_i \leq M < \infty \) for \( i \in \{1, \ldots, n\} \).
Also, if we apply the inequality (2.6) for the same function \( \Phi \) we obtain

\[
1 \leq \frac{A_n(w, x)}{G_n(w, x)}
\]

\[
\leq \left( \frac{M}{A_n(w, x)} \right)^{M - A_n(w, x) - m} \left( \frac{m}{A_n(w, x)} \right)^{A_n(w, x) - m} - (M - m)/4
\]

\[
\leq \exp \left( \frac{1}{4} \frac{(M - m)^2}{mm} \right).
\]

The following result also holds.

**Theorem 2.4.** With the assumptions of Theorem 2.1, we have the inequalities

\[
0 \leq \int_{\Omega} w(\Phi \circ f) \, d\mu(x) - \Phi(\bar{f}_{\Omega_w})
\]

\[
\leq 2 \max \left\{ \frac{M - \bar{f}_{\Omega_w}}{M - m}, \frac{\bar{f}_{\Omega_w} - m}{M - m} \right\} \left( \frac{\Phi(m) + \Phi(M)}{2} - \Phi \left( \frac{m + M}{2} \right) \right)
\]

\[
\leq \frac{1}{2} \max \{M - \bar{f}_{\Omega_w}, \bar{f}_{\Omega_w} - m\} (\Phi' (M) - \Phi' (m)).
\]

**Proof.** We first recall the following result obtained by the author in [14] that provides a refinement and a reverse for the weighted Jensen’s discrete inequality:

\[
\begin{align*}
&n \min_{i \in \{1, \ldots, n\}} \{ p_i \} \left[ \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \right] \\
&\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \\
&\leq n \max_{i \in \{1, \ldots, n\}} \{ p_i \} \left[ \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \right].
\end{align*}
\]

where \( \Phi : C \to \mathbb{R} \) is a convex function defined on the convex subset \( C \) of the linear space \( X \), \( \{ x_i \}_{i \in \{1, \ldots, n\}} \subset C \) are vectors and \( \{ p_i \}_{i \in \{1, \ldots, n\}} \) are nonnegative numbers with \( P_n := \sum_{i=1}^{n} p_i > 0 \).

For \( n = 2 \) we deduce from (2.8) that

\[
2 \min \{ t, 1 - t \} \left( \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x + y}{2} \right) \right)
\]

\[
\leq t \Phi(x) + (1 - t) \Phi(y) - \Phi(tx + (1 - t)y)
\]

\[
\leq 2 \max \{ t, 1 - t \} \left( \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x + y}{2} \right) \right)
\]

for any \( x, y \in C \) and \( t \in [0, 1] \).
If we use the second inequality in (2.9) for the convex function \( \Phi : I \to \mathbb{R} \) and \( m, M \in \mathbb{R} \), \( m < M \), with \([m, M] \subset I\), we have for \( t = (M - \bar{f}_{\Omega,w})/(M - m) \) that
\[
\frac{(M - \bar{f}_{\Omega,w})\Phi(m) + (\bar{f}_{\Omega,w} - m)\Phi(M)}{M - m} - \Phi\left(\frac{m(M - \bar{f}_{\Omega,w}) + M(\bar{f}_{\Omega,w} - m)}{M - m}\right) \\
\leq 2 \max\left\{\frac{M - \bar{f}_{\Omega,w}}{M - m}, \frac{\bar{f}_{\Omega,w} - m}{M - m}\right\}\left(\Phi(m) + \Phi(M) - \Phi\left(\frac{m + M}{2}\right)\right).
\]
(2.10)

Using (2.3) and (2.10) we deduce the first inequality in (2.7).

Since
\[
\Phi\left(\frac{m + M}{2}\right) - \Phi\left(\frac{m + M}{2}\right) = \frac{1}{4} \left(\Phi(M) - \Phi\left(\frac{m + M}{2}\right) - \Phi\left(\frac{m + M}{2}\right) - \Phi(m)\right)
\]
and, by the gradient inequality,
\[
\frac{\Phi(M) - \Phi\left(\frac{m + M}{2}\right)}{M - \frac{m + M}{2}} \leq \Phi'(M)
\]
and
\[
\frac{\Phi\left(\frac{m + M}{2}\right) - \Phi(m)}{\frac{m + M}{2} - m} \geq \Phi'(m),
\]
then
\[
\frac{\Phi(m) + \Phi(M)}{M - m} - \frac{\Phi\left(\frac{m + M}{2}\right)}{\frac{m + M}{2}} \leq \frac{1}{4} (\Phi'(M) - \Phi'(m)).
\]
(2.11)

Making use of (2.10) and (2.11), we deduce the last part of (2.7). \( \square \)

**Corollary 2.5.** With the assumptions in Corollary 2.2, we have the inequalities

\[
0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}) \\
\leq 2 \max\left\{\frac{M - \bar{x}}{M - m}, \frac{\bar{x} - m}{M - m}\right\}\left(\Phi(m) + \Phi(M) - \Phi\left(\frac{m + M}{2}\right)\right) \\
\leq \frac{1}{2} \max\{M - \bar{x}, \bar{x} - m\}(\Phi'(M) - \Phi'(m)).
\]

**Remark 2.6.** Since, obviously,
\[
\frac{M - \bar{f}_{\Omega,w}}{M - m}, \frac{\bar{f}_{\Omega,w} - m}{M - m} \leq 1,
\]
we obtain from the first inequality in (2.7) the simpler but coarser inequality
\[
0 \leq \int_{\Omega} w(\Phi \circ f)\,d\mu(x) - \Phi(\bar{f}_{\Omega,w}) \leq 2\left(\Phi(m) + \Phi(M) - \Phi\left(\frac{m + M}{2}\right)\right).
\]
The discrete version of this result, namely
\[ 0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \leq 2 \left( \frac{\Phi(m) + \Phi(M)}{2} - \Phi\left( \frac{m + M}{2} \right) \right), \]
was obtained in 2008 by Simic [34].

**Remark 2.7.** With the assumptions in Remark 2.3 we have the following reverse of the arithmetic mean–geometric mean inequality
\[ 1 \leq \frac{A_n(w, x)}{G_n(w, x)} \leq \left( \frac{A(m, M)}{G(m, M)} \right)^{2 \max((M-A_n(w,x))/(M-m),(A_n(w,x)-m)/(M-m))}, \tag{2.12} \]
where \( A(m, M) \) is the arithmetic mean and \( G(m, M) \) is the geometric mean of the positive numbers \( m \) and \( M \).

**3. Applications for the Hölder inequality**

It is well known that if \( f \in L_p(\Omega, \mu) \), \( p > 1 \), where the Lebesgue space \( L_p(\Omega, \mu) \) is defined by
\[
L_p(\Omega, \mu) := \{ f : \Omega \to \mathbb{R} \mid f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(x)|^p \, d\mu(x) < \infty \},
\]
and \( g \in L_q(\Omega, \mu) \) with \( 1/p + 1/q = 1 \) then \( fg \in L(\Omega, \mu) = L_1(\Omega, \mu) \) and the Hölder inequality holds true:
\[
\int_{\Omega} |fg| \, d\mu \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q}.
\]

Assume that \( p > 1 \). If \( h : \Omega \to \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds
\[
0 < m \leq |h(x)| \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega
\]
and is such that \( h, |h|^p \in L_w(\Omega, \mu) \), for a \( \mu \)-measurable function \( w : \Omega \to \mathbb{R} \), with \( w(x) \geq 0 \) for \( \mu\text{-a.e. } x \in \Omega \) and \( \int_{\Omega} w \, d\mu > 0 \), then, from (2.1),
\[
0 \leq \frac{\int_{\Omega} |h|^p w \, d\mu}{\int_{\Omega} w \, d\mu} - \left( \frac{\int_{\Omega} |h| w \, d\mu}{\int_{\Omega} w \, d\mu} \right)^p \leq \frac{(M - \overline{|h|}_{\Omega,w})(\overline{|h|}_{\Omega,w} - m)}{M - m} B_p(m, M) \tag{3.1}
\]
\[
\leq \frac{p M^{p-1} - m^{p-1}}{M - m} (M - \overline{|h|}_{\Omega,w})(\overline{|h|}_{\Omega,w} - m)
\]
\[
\leq \frac{1}{4} p(M - m)(M^{p-1} - m^{p-1}).
\]
where \( \overline{|h|}_{p,q} := \int_{\Omega} |h| f d\mu / \int_{\Omega} w d\mu \in [m, M] \), \( \Psi_p(\cdot; m, M) : (m, M) \to \mathbb{R} \) is defined by

\[
\Psi_p(t; m, M) = \frac{M^p - t^p}{M - t} - \frac{t^p - m^p}{t - m},
\]

and

\[
B_p(m, M) := \sup_{t \in (m, M)} \Psi_p(t; m, M).
\]

From (2.2) we also have the inequality

\[
0 \leq \frac{\int_{\Omega} |h|^p w d\mu}{\int_{\Omega} w d\mu} - \left( \frac{\int_{\Omega} |h| w d\mu}{\int_{\Omega} w d\mu} \right)^p \leq \frac{1}{4} (M - m) \Psi_p(\overline{|h|}_{p,q}; m, M) \leq \frac{1}{4} p (M - m) (M^{p-1} - m^{p-1}).
\]

(3.2)

**Proposition 3.1.** If \( f \in L_p(\Omega, \mu) \), \( g \in L_q(\Omega, \mu) \) with \( p > 1 \), \( 1/p + 1/q = 1 \), and there exist constants \( \gamma, \Gamma > 0 \) such that

\[
\gamma \leq \frac{|f|}{|g|^{q-1}} \leq \Gamma \mu-a.e \text{ on } \Omega,
\]

then

\[
0 \leq \frac{\int_{\Omega} |f|^p g \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \left( \frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^p \leq \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \left( \Gamma - \frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right) \leq \frac{1}{4} p (\Gamma - \gamma) (\Gamma^{p-1} - \gamma^{p-1}),
\]

(3.3)

and

\[
0 \leq \frac{\int_{\Omega} |f|^p g \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \left( \frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^p \leq \frac{1}{4} (\Gamma - \gamma) \Psi_p \left( \frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^q \, d\mu}; \gamma, \Gamma \right) \leq \frac{1}{4} p (\Gamma - \gamma) (\Gamma^{p-1} - \gamma^{p-1}),
\]

(3.4)

where \( B_p(\cdot, \cdot) \) and \( \Psi_p(\cdot; \cdot, \cdot) \) are defined above.

**Proof.** The inequalities (3.3) and (3.4) follow from (3.1) and (3.2) by choosing

\[
h = \frac{|f|}{|g|^{q-1}} \quad \text{and} \quad w = |g|^q.
\]

The details are omitted. \(\square\)
Remark 3.2. We observe that for \( p = q = 2 \) we have \( \Psi_2(t; \gamma, \Gamma) = \Gamma - \gamma = B_2(\gamma, \Gamma) \) and then from the first inequality in (3.3) we get the following reverse of the Cauchy–Bunyakovsky–Schwarz inequality:

\[
\int_\Omega |g|^2 \, d\mu \int_\Omega |f|^2 \, d\mu - \left( \int_\Omega |fg| \, d\mu \right)^2 \\
\leq \left( \Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu} \right) \left( \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu} - \gamma \right) \left( \int_\Omega |g|^2 \, d\mu \right)^2,
\]

provided that \( f, g \in L_2(\Omega, \mu) \), and there exist constants \( \gamma, \Gamma > 0 \) such that

\[
\gamma \leq \frac{|f|}{|g|} \leq \Gamma \text{-a.e on } \Omega.
\]

Corollary 3.3. With the assumptions of Proposition 3.1 we have the following additive reverses of the Hölder inequality:

\[
0 \leq \left( \int_\Omega |f|^p \, d\mu \right)^{1/p} \left( \int_\Omega |g|^q \, d\mu \right)^{1/q} - \int_\Omega |fg| \, d\mu \\
\leq \left( \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \right)^{1/p} \left( \Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^p \, d\mu} \right) \left( \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^p \, d\mu} - \gamma \right)^{1/p} \int_\Omega |g|^q \, d\mu \\
\leq p^{1/p} \left( \frac{\Gamma^{-1} - \gamma^{p^{-1}}}{\Gamma - \gamma} \right)^{1/p} \left( \Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^p \, d\mu} \right) \left( \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^p \, d\mu} - \gamma \right)^{1/p} \int_\Omega |g|^q \, d\mu
\]

(3.5)

\[
\leq \frac{1}{4^{1/p}} p^{1/p} (\Gamma - \gamma)^{1/p} (\Gamma^{-1} - \gamma^{p^{-1}})^{1/p} \int_\Omega |g|^q \, d\mu
\]

and

\[
0 \leq \left( \int_\Omega |f|^p \, d\mu \right)^{1/p} \left( \int_\Omega |g|^q \, d\mu \right)^{1/q} - \int_\Omega |fg| \, d\mu \\
\leq \frac{1}{4^{1/p}} (\Gamma - \gamma)^{1/p} \Psi_p \left( \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^p \, d\mu}; m, M \right) \int_\Omega |g|^q \, d\mu
\]

(3.6)

\[
\leq \frac{1}{4^{1/p}} p^{1/p} (\Gamma - \gamma)^{1/p} (\Gamma^{-1} - \gamma^{p^{-1}})^{1/p} \int_\Omega |g|^q \, d\mu,
\]

where \( p > 1 \) and \( 1/p + 1/q = 1 \).
PROOF. By multiplying in (3.3) with \((\int_{\Omega} |g|^q \, d\mu)^p\),
\[
\int_{\Omega} |f|^p \, d\mu \left( \int_{\Omega} |g|^q \, d\mu \right)^{p-1} - \left( \int_{\Omega} |fg| \, d\mu \right)^p
\leq \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right) \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\leq p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right) \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
which is equivalent to
\[
\int_{\Omega} |f|^p \, d\mu \left( \int_{\Omega} |g|^q \, d\mu \right)^{p-1}
\leq \left( \int_{\Omega} |fg| \, d\mu \right)^p + \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right) \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\leq \left( \int_{\Omega} |fg| \, d\mu \right)^p + p \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right) \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\leq \left( \int_{\Omega} |fg| \, d\mu \right)^p + \frac{1}{4} p(\Gamma - \gamma)(\Gamma^{p-1} - \gamma^{p-1}) \left( \int_{\Omega} |g|^q \, d\mu \right)^p.
\]
Raising to the power \(1/p\) with \(p > 1\) and employing the elementary inequality that for \(p > 1\) and \(\alpha, \beta > 0\),
\[(\alpha + \beta)^{1/p} \leq \alpha^{1/p} + \beta^{1/p},\]
we have from the first part of (3.7) that
\[
\left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1-1/p}
\leq \int_{\Omega} |fg| \, d\mu + \left( \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \right)^{1/p} \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^{1/p} \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/p}
\]
\[
\leq \int_{\Omega} |fg| \, d\mu + \left( \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \right)^{1/p} \left( \frac{\int_{\Omega} |g|^q \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/p}
\]
and since \(1 - 1/p = 1/q\) we get from (3.8) the first inequality in (3.5). The rest is obvious.

The inequality (3.6) can be proved in a similar manner; the details are omitted. □
If \( h : \Omega \to \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds

\[
0 < m \leq |h(x)| \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega
\]

and is such that \( h, |h|^p \in L_w(\Omega, \mu) \), for a \( \mu \)-measurable function \( w : \Omega \to \mathbb{R} \), with \( w(x) \geq 0 \) for \( \mu \)-a.e. \( x \in \Omega \) and \( \int \Omega w \, d\mu > 0 \), then from (2.7) we also have the inequality

\[
0 \leq \frac{\int \Omega |h|^p w \, d\mu}{\int \Omega w \, d\mu} - \left( \frac{\int \Omega |h| w \, d\mu}{\int \Omega w \, d\mu} \right)^p
\]

\[
\leq 2 \left( \frac{m^p + M^p}{2} - \left( \frac{m + M}{2} \right)^p \right) \max \left\{ \frac{M - |h|_{\Omega,w}}{M - m}, \frac{|h|_{\Omega,w} - m}{M - m} \right\}
\]

(3.9)

Finally, the following additive reverse of the Hölder inequality can also be stated.

**Proposition 3.4.** With the assumptions of Proposition 3.1 we have

\[
0 \leq \frac{\int \Omega |f|^p d\mu}{\int \Omega |g|^q d\mu} - \left( \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} \right)^p
\]

\[
\leq 2 \cdot \frac{\gamma^p + \Gamma^p}{\Gamma - \gamma} \max \left\{ \Gamma - \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu}, \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} - \gamma \right\}
\]

\[
\leq \frac{1}{2} p (\Gamma^{p-1} - \gamma^{p-1}) \max \left\{ \Gamma - \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu}, \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} - \gamma \right\}.
\]

Finally, the following additive reverse of the Hölder inequality can also be stated.

**Corollary 3.5.** With the assumptions of Proposition 3.1,

\[
0 \leq \left( \int \Omega |f|^p d\mu \right)^{1/p} \left( \int \Omega |g|^q d\mu \right)^{1/q} - \int \Omega |fg| d\mu
\]

\[
\leq 2^{1/p} \cdot \left( \frac{\gamma^p + \Gamma^p}{\Gamma - \gamma} \right)^{1/p} \times \max \left\{ \left( \Gamma - \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} \right)^{1/p}, \left( \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} - \gamma \right)^{1/p} \right\} \int \Omega |g|^q d\mu
\]

\[
\leq \frac{1}{2^{1/p}} p^{1/p} \max \left\{ \left( \Gamma - \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} \right)^{1/p}, \left( \frac{\int \Omega |fg| d\mu}{\int \Omega |g|^q d\mu} - \gamma \right)^{1/p} \right\}
\]

\[
\times (\Gamma^{p-1} - \gamma^{p-1})^{1/p} \int \Omega |g|^q d\mu.
\]
Remark 3.6. As a simpler but coarser inequality we have the following result:

\[
0 \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q} - \int_{\Omega} |fg| \, d\mu \\
\leq 2^{1/p} \cdot \left( \frac{\gamma^p + \Gamma^p}{2} - \left( \frac{\gamma + \Gamma}{2} \right)^p \right)^{1/p} \int_{\Omega} |g|^q \, d\mu,
\]

where \( f \) and \( g \) are as above.

4. Applications for \( f \)-divergence

One of the important issues in many applications of probability theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [19], Kullback and Leibler [24], Rényi [30], Havrda and Charvat [17], Kapur [22], Sharma and Mittal [32], Burbea and Rao [4], Rao [29], Lin [25], Csiszár [7], Ali and Silvey [1], Vajda [39], Shioya and Da-Te [33] and others (see, for example, [26], and the references therein).

These measures have been applied in a variety of fields such as: anthropology [29], genetics [26], finance, economics and political science [31, 36, 37], biology [28], the analysis of contingency tables [16], approximation of probability distributions [6, 23], signal processing [20, 21] and pattern recognition [2, 5]. A number of these measures of distance are specific cases of Csiszár \( f \)-divergence and so further exploration of this concept will have a flow-on effect to other measures of distance and to areas in which they are applied.

Assume that a set \( \Omega \) and the \( \sigma \)-finite measure \( \mu \) are given. Consider the set of all probability densities on \( \mu \) to be \( \mathcal{P} := \{ p \mid p : \Omega \to \mathbb{R}, p(x) \geq 0, \int_{\Omega} p(x) \, d\mu(x) = 1 \} \). The Kullback–Leibler divergence [24] is well known among the information divergences. It is defined as

\[
D_{KL}(p, q) := \int_{\Omega} p(x) \log \left( \frac{p(x)}{q(x)} \right) \, d\mu(x), \quad p, q \in \mathcal{P}, \tag{4.1}
\]

where \( \log \) is to base \( e \).

In information theory and statistics, various divergences are applied in addition to the Kullback–Leibler divergence. These are, for example, the variation distance \( D_v \), Hellinger distance \( D_H \) [18], \( \chi^2 \)-divergence \( D_{\chi^2} \), \( \alpha \)-divergence \( D_{\alpha} \), Bhattacharyya distance \( D_B \) [3], harmonic distance \( D_{Ha} \), Jeffreys distance \( D_J \) [19], triangular discrimination \( D_\Delta \) [38]. They are defined as follows:

\[
D_v(p, q) := \int_{\Omega} |p(x) - q(x)| \, d\mu(x), \quad p, q \in \mathcal{P}; \tag{4.2}
\]

\[
D_H(p, q) := \int_{\Omega} |\sqrt{p(x)} - \sqrt{q(x)}| \, d\mu(x), \quad p, q \in \mathcal{P}; \tag{4.3}
\]
Some reverses of the Jensen inequality

$$D_{\chi^2}(p, q) := \int_{\Omega} p(x) \left( \frac{q(x)}{p(x)} \right)^2 - 1 \, d\mu(x), \quad p, q \in \mathcal{P};$$

$$D_{\alpha}(p, q) := \frac{4}{1 - \alpha^2} \left[ 1 - \int_{\Omega} (p(x))^{(1-\alpha)/2} (q(x))^{(1+\alpha)/2} \, d\mu(x) \right], \quad p, q \in \mathcal{P};$$

$$D_B(p, q) := \int_{\Omega} \sqrt{p(x)q(x)} \, d\mu(x), \quad p, q \in \mathcal{P};$$

$$D_{H\alpha}(p, q) := \int_{\Omega} \frac{2p(x)q(x)}{p(x) + q(x)} \, d\mu(x), \quad p, q \in \mathcal{P};$$

$$D_f(p, q) := \int_{\Omega} (p(x) - q(x)) \log \left( \frac{p(x)}{q(x)} \right) \, d\mu(x), \quad p, q \in \mathcal{P};$$

$$D_\Delta(p, q) := \int_{\Omega} \frac{(p(x) - q(x))^2}{p(x) + q(x)} \, d\mu(x), \quad p, q \in \mathcal{P}. \quad (4.9)$$

For other divergence measures, see Kapur [22] or the book online by Taneja [35].

Csiszár $f$-divergence is defined as follows [8]:

$$I_f(p, q) := \int_{\Omega} p(x) f \left( \frac{q(x)}{p(x)} \right) \, d\mu(x), \quad p, q \in \mathcal{P},$$

where $f$ is convex on $(0, \infty)$. It is assumed that $f$ is strictly convex and satisfies the condition that $f(1) = 0$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (4.1)–(4.9) are particular instances of Csiszár $f$-divergence. There are also many others which are not in this class (see, for example, [35]). For the basic properties of Csiszár $f$-divergence, see [8, 9] and [39].

The following result holds.

**Proposition 4.1.** Suppose that $f : (0, \infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exist constants $0 < r < 1 < R < \infty$ such that

$$r \leq \frac{q(x)}{p(x)} \leq R \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$ 

Then we have the inequalities

$$I_f(p, q) \leq \frac{(R - 1)(1 - r)}{R - r} \sup_{x \in (r, R)} \Psi_f(t; r, R)$$

$$\leq (R - 1)(1 - r) \frac{f'_+(R) - f'_+(r)}{R - r}$$

$$\leq \frac{1}{4} (R - r)(f'_-(R) - f'_+(r)), \quad (4.10)$$

where $\Psi_f(\cdot; r, R) : (r, R) \to \mathbb{R}$ is defined by

$$\Psi_f(t; r, R) = \frac{f(R) - f(t)}{R - t} - \frac{f(t) - f(r)}{t - r}.$$
We also have the inequality
\[
I_f(p, q) \leq \frac{1}{4}(R - r) \left( f(R)(1 - r) + f(r)(R - 1) \right) \frac{1}{(R - 1)(1 - r)} \\
\leq \frac{1}{4}(R - r)(f''(R) - f''(r)).
\] (4.11)

The proof follows by Theorem 2.1 by choosing \( w(x) = p(x) \), \( f(x) = q(x)/p(x) \), \( m = r \) and \( M = R \) and performing the required calculations. The details are omitted.

Using the same approach and Theorem 2.4 we can also state the following result.

**Proposition 4.2.** With the assumptions of Proposition 4.1,
\[
I_f(p, q) \leq 2 \max\left\{ \frac{R - 1}{R - r}, \frac{1 - r}{R - r} \right\} \left( \frac{f(r) + f(R)}{2} - f\left( \frac{r + R}{2} \right) \right) \\
\leq \frac{1}{2} \max\{R - 1, 1 - r\}(f''(R) - f''(r)).
\] (4.12)

The above results can be used to obtain various inequalities for divergence measures in information theory that are particular instances of \( f \)-divergence.

Consider the Kullback–Leibler divergence
\[
D_{KL}(p, q) := \int_{\Omega} p(x) \log \left( \frac{p(x)}{q(x)} \right) d\mu(x), \quad p, q \in \mathcal{P},
\]
which is an \( f \)-divergence for the convex function \( f : (0, \infty) \to \mathbb{R}, f(t) = -\log t \).

If \( p, q \in \mathcal{P} \) such that there exist constants \( 0 < r < 1 < R < \infty \) with
\[
r \leq \frac{q(x)}{p(x)} \leq R \quad \text{for } \mu\text{-a.e. } x \in \Omega,
\]
then we get from (4.10) that
\[
D_{KL}(p, q) \leq \frac{(R - 1)(1 - r)}{rR},
\]
from (4.11) that
\[
D_{KL}(p, q) \leq \frac{1}{4}(R - r) \log(R^{-1/(R-1)}r^{-1/(1-r)})
\]
and from (4.12) that
\[
D_{KL}(p, q) \leq 2 \max\left\{ \frac{R - 1}{R - r}, \frac{1 - r}{R - r} \right\} \log \left( \frac{A(r, R)}{G(r, R)} \right) \\
\leq \frac{1}{2} \max\{R - 1, 1 - r\} \left( \frac{R - r}{rR} \right),
\]
where \( A(r, R) \) is the arithmetic mean and \( G(r, R) \) is the geometric mean of the positive numbers \( r \) and \( R \).
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References


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