ON FLATNESS COVERS OF CYCLIC ACTS OVER MONOIDS

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Abstract. The covers of cyclic acts over monoids were investigated by Mahmoudi and Renshaw (M. Mahmoudi and J. Renshaw, On covers of cyclic acts over monoids, Semigroup Forum 77 (2008), 325–338) and the authors posed some open problems. In the present paper, we give answers to their problems 1 and 5, and we also give a sufficient and necessary condition that a cyclic act has a weakly pullback flat cover.

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1. Introduction. Throughout this paper, $S$ always stands for a monoid, and $N$ for the set of natural numbers.

Over the past several decades, the covers of modules have been investigated by many authors and ample results have been obtained (see [1, 3, 4, 12, 15]). Covers of acts over monoids are studied in [5, 7, 8]. Further investigations about this field were laid dormant until the recent appearance of [13].

Let us recall results and definitions that we shall use below. We refer the reader to [11] for a detailed account of these.

A monoid $S$ is said to be right reversible if for any $p, q \in S$ there exist $u, v \in S$ such that $up = vq$. A monoid $S$ is said to be weakly left collapsible if for any $p, q, r \in S$ with $pr = qr$ there exists $u \in S$ such that $up = uq$.

In [2], the acts, now called strongly flat, were introduced: A right $S$-act $A_S$ is strongly flat if the functor $A_S \otimes -$ preserves pullbacks and equalizers. In the same paper, strongly flat acts were characterized as the acts satisfying two interpolation conditions, later labelled as condition $(P)$ and condition $(E)$:

\begin{align*}
(P) & \quad (\forall a, a' \in A)(\forall s, t \in S)(as = a't \\
& \quad \quad \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u \land a' = a''v \land us = vt)), \\
(E) & \quad (\forall a \in A)(\forall s, s' \in S)(as = as' \\
& \quad \quad \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \land us = us')).
\end{align*}

In [14] Laan called a right $S$-act $A$ weakly pullback flat if it satisfies condition $(P)$ and the following condition $(E')$.

\begin{align*}
(E') & \quad (\forall a \in A)(\forall s, s', z \in S)(as = as' \land sz = s'z \\
& \quad \quad \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \land us = us')).
\end{align*}
DEFINITION 1.1 (Definition 2.1 in [13]). Let $S$ be a monoid and $A$ an $S$-act. An $S$-act $C$ together with an $S$-epimorphism $f: C \rightarrow A$ is a cover of $A$ if there is no proper subact $B$ of $C$ such that $f\mid B$ is onto. We shall usually refer to $C$ as the cover.

DEFINITION 1.2 (Definition 2.2 in [13]). Let $S$ be a monoid and $f: C \rightarrow A$ be an $S$-epimorphism. We call $f$ coessential if for each $S$-act $B$ and each $S$-map $g: B \rightarrow C$, if $fg$ is an epimorphism then $g$ is an epimorphism.

The purpose of the present paper is to answer the open problem 1 and open problem 5 in [13]. We also show that a cyclic $S$-act $S/\rho$ has a weakly pullback flat cover if and only if $[1]_{\rho}$ contains a right reversible and weakly left collapsible submonoid $R$ such that for all $u \in [1]_{\rho}$, $uS \cap R \neq \emptyset$.


PROBLEM. Is there a monoid $S$ and a cyclic $S$-act $A$ that do not have a ($P$)-cover?

In the following, we will give an affirmative answer to this question.

LEMMA 2.1 (Lemma 7 in [9]). Let $P \subseteq S$ be a right reversible submonoid and let $\rho$ be the relation on $S$ defined by

$$s\rho s' \iff (\exists p, q \in P)(ps = qs').$$

Then:

1. $\rho$ is a right congruence.
2. The right $S$-act $S/\rho$ satisfies condition ($P$).
3. If $P$ is weakly left collapsible, then $S/\rho$ is weakly pullback flat.

Let $X$ be a non-empty set, and let $X^+$ denote the free semigroup generated by $X$. If we adjoin an identity 1 to $X^+$, we obtain the free monoid on $X$ and we denote this by $X^*$. For each $w$ in $X$, by [6] the content $C(w)$ is defined as the (necessarily finite) set of elements of $X$ appearing in $w$. Let $R$ be a subsemigroup of a free semigroup $X^+$. We define the content $C(R)$ of $R$ as

$$C(R) = \bigcup_{w \in R} C(w).$$

LEMMA 2.2. Let $X$ be a non-empty set and let $R$ be a subsemigroup of $X^+$. Then $R$ is right reversible if and only if $|C(R)| = 1$.

Proof. For the sufficiency, since $|C(R)| = 1$, we suppose $C(R) = \{x\}$. For any $s, t \in R$, there exist $m, n \in N$ such that $s = x^m$ and $t = x^n$. It is clear that $x^m \cdot x^m = x^m \cdot x^n$. Now $R$ is right reversible.

For necessity if $|C(R)| > 1$, then these exist two distinct elements $x, x' \in X$ such that the elements $x_m \cdots x_{i+1} x_{i} x_{i-1} \cdots x_1$ and $x'_n \cdots x'_{j+1} x'_j x'_{j-1} \cdots x'_1$ belong to $R$, where $m, n \in N$ and $x_i, x'_j \in X$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Then by the property of the semigroup $X^+$, $R$ is not right reversible. This is a contradiction. Hence, $|C(R)| = 1$.

LEMMA 2.3 (Corollary 4.3 in [13]). The 1-element $S$-act $\Theta$ has a ($P$)-cover if and only if there exists a right reversible submonoid $R$ of $S$ such that for all $u \in S$, there exists $s \in S$ with $us \in R$.

EXAMPLE 2.4. Let $X$ be a set with more than 3 elements and $S = X^*$, the free monoid generated by $X$. Then the (cyclic) 1-element $S$-act $\Theta$ has no ($P$)-cover.
Proof. Suppose $R$ is a right reversible submonoid of $X^*$. By Lemma 2.2 $|C(R)| = 1$, and we suppose $C(R) = \{x\}$. There exists $u \in X^*$ such that for every $s \in X^*$, $us \notin R$. By Lemma 2.3 $\emptyset$ has no $(P)$-cover.


**PROBLEM.** Are strongly flat covers unique?

**LEMMA 3.1 (Lemma 2.1 in [10]).** Let $P \subseteq S$ be a left collapsible submonoid and let $\rho$ be the relation on $S$ defined by

$$s \rho t \iff (\exists p, q \in P)(ps = qt).$$

Then:

1. $\rho$ is a right congruence.
2. $S/\rho$ is strongly flat.

**LEMMA 3.2 (Theorem 3.2 in [13]).** Let $S$ be a monoid. Then the cyclic $S$-act $S/\rho$ has a strongly flat cover if and only if $[1]_\rho$ contains a left collapsible submonoid $R$ such that for all $u \in [1]_\rho$, $uS \cap [1]_\rho \neq \emptyset$.

**LEMMA 3.3 (Theorem 2.7 in [13]).** Let $S$ be a monoid and $S/\rho$ a cyclic $S$-act. The map $f : S/\sigma \rightarrow S/\rho$ given by $s \sigma \mapsto s\rho$ is a coessential epimorphism if and only if $\sigma \subseteq \rho$ and for all $u \in [1]_\rho$, $uS \cap [1]_\sigma \neq \emptyset$.

**EXAMPLE 3.4.** Let

$$S = \langle a, b, c | ab = ba = ac = ca = a, bc = c^2, cb = b^2, a^4 = a^5, b^5 = b^6, c^6 = c^7 \rangle \cup \{1\}.$$ Define an equivalence relation $\rho$ on $S$ by

$$s \rho t \iff (s, t \in \langle a \rangle) \text{ or } (s, t \in (\langle b \rangle \cup \langle c \rangle \cup \{1\}).$$

It is easy to verify that $\rho$ is a right congruence on $S$. Then $[1]_\rho = \langle b \rangle \cup \langle c \rangle \cup \{1\}$ and the strongly flat cover of $S/\rho$ is not unique.

**Proof.** By the definition of $\rho$, it is a proper right congruence of $S$. Denote $R_1 = \langle b \rangle \cup \{1\}$ and $R_2 = \langle c \rangle \cup \{1\}$, then $R_1$ and $R_2$ are both left collapsible submonoids of $[1]_\rho$.

Define a right congruence $\sigma_1$ on $S$ by

$$s \sigma_1 t \iff (\exists p, q \in R_1)(ps = qt).$$

Define a right congruence $\sigma_2$ on $S$ by

$$s \sigma_2 t \iff (\exists p, q \in R_2)(ps = qt).$$

Hence, for every $u \in [1]_\rho$, $uS \cap [1]_\sigma_i \neq \emptyset$ ($i = 1, 2$).

Then by Lemma 3.1, $S/\sigma_1$ and $S/\sigma_2$ are both strongly flat. But $\sigma_1 \neq \sigma_2$, since $(b, 1) \in \sigma_1$ but $(b, 1) \notin \sigma_2$, $(c, 1) \in \sigma_2$ but $(c, 1) \notin \sigma_1$.

By Lemmas 3.2 and 3.3, $S/\sigma_1$ and $S/\sigma_2$ are both strongly flat covers of $S/\rho$.

Now we have the following. □
PROPOSITION 3.5. Strongly flat covers of cyclic acts need not be unique.

This proposition gives a negative answer to the previous question.

REMARK 3.6. Let S be a monoid, and A be an S-act. As in [13], let X be a class of acts that is closed under isomorphism. By a X-precover of A we mean an S-map g : X → A from some X ∈ X such that for every S-map g′ : X′ → A, for X′ ∈ X, there exists an S-map f : X′ → X with g′ = gf.

\[ \begin{array}{ccc}
X' & \xrightarrow{g'} & A \\
\downarrow{f} & & \downarrow{\text{id}_A} \\
X & \xrightarrow{g} & A 
\end{array} \]

If in addition the precover satisfies the condition that each S-map f : X → X with gf = g is an isomorphism, then we shall call it a X-cover. It is clear that the X-cover is unique up to isomorphism. If SF is the class strongly flat acts then by Proposition 3.5 we now know the strongly flat covers and SF-covers do not coincide.

4. On weakly pullback flat covers of cyclic acts.

LEMMA 4.1 (Theorem 2.8 in [13]). Let S be a monoid and S/ρ a cyclic S-act. If R is a submonoid of [1]_ρ such that for all u ∈ [1]_ρ, uS ∩ R ≠ ∅, then there exists a right congruence σ on S such that R ⊆ [1]_σ and S/σ is a cover of S/ρ. Moreover, R = [1]_σ if and only if R is a left unitary submonoid of S.

LEMMA 4.2 (Lemma 9 in [9]). Let S be a monoid, σ a right congruence on S and let the cyclic S-act S/σ be weakly pullback flat. Then R = [1]_σ is a right reversible and weakly left collapsible submonoid of S.

THEOREM 4.3. Let S be a monoid. Then the cyclic S-act S/ρ has a weakly pullback flat cover if and only if [1]_ρ contains a right reversible and weakly left collapsible submonoid R such that for all u ∈ [1]_ρ, uS ∩ R ≠ ∅.

Proof. Suppose that S/ρ has a weakly pullback flat cover S/σ. Then by Lemma 3.3 we can assume that R = [1]_σ ⊆ [1]_ρ and that for all u ∈ [1]_ρ, uS ∩ R ≠ ∅. Moreover, R is right reversible and weakly left collapsible by Lemma 4.2.

Conversely, suppose that R is a right reversible and weakly left collapsible submonoid of [1]_ρ such that for all u ∈ [1]_ρ, uS ∩ R ≠ ∅. Define a right congruence σ on S by

\[ s \sigma t \iff (\exists p, q ∈ R)(ps = qt). \]

Then by Lemma 2.1 S/σ is weakly pullback flat. Finally, by Lemma 3.3, S/σ is a weakly pullback flat cover of S/ρ.

COROLLARY 4.4. The 1-element S-act Θ has a weakly pullback flat cover if and only if there exists a right reversible and weakly left collapsible submonoid R of S such that for all u ∈ S, there exists s ∈ S with us ∈ R.

Now by Example 2.4 we also have the following.

REMARK 4.5. There exists a monoid S and a cyclic S-act A which does not have a weakly pullback flat cover.
**Theorem 4.6.** If $S$ is a monoid then every cyclic $S$-act has a weakly pullback flat cover if and only if every left unitary submonoid $T$ of $S$ contains a right reversible and weakly left collapsible submonoid $R$ such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$.

Since commutative monoids are necessarily right reversible and weakly left collapsible, we have the following.

**Theorem 4.7.** Let $S$ be a commutative monoid. Then every cyclic $S$-act has a weakly pullback flat cover.

**Corollary 4.8 (Theorem 4.5 in [13]).** Let $S$ be a commutative monoid. Then every cyclic $S$-act has a $(P)$-cover.

**Corollary 4.9.** Let $S$ be a right cancellative monoid. The cyclic $S$-act $S/\rho$ has a weakly pullback flat cover if and only if $S/\rho$ has a $(P)$-cover.

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