ON SUBSEQUENTIAL LIMIT POINTS OF A SEQUENCE OF ITERATES. II

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Abstract

J. B. Diaz and F. T. Metcalf established some results concerning the structure of the set of cluster points of a sequence of iterates of a continuous self-map of a metric space. In this paper it is shown that their conclusions remain valid if the distance function in their inequality is replaced by a continuous function on the product space. Then this idea is extended to some other mappings and to uniform and general topological spaces.


Diaz and Metcalf [5, 6] have studied the structure of the set of subsequential limit points of a sequence of iterates of a continuous self-map $A$ of a metric space $(X, d)$ satisfying the condition $d(Ax, F(A)) < d(x, F(A))$, where $x \neq Ax$ and $F(A)$, the set of fixed points of $A$, is nonempty and compact. In this paper it is shown that the conclusions of Diaz and Metcalf [6] may still be derived after replacing $d$ by a continuous function $\varphi: X \times X \to R_0$, where $R_0$ is the subspace $[0, \infty)$ of the real line with usual topology. Then our analysis is extended to the mappings introduced by Dotson [7], Browder and Petryshyn [2, 3], Singh and Zorzitto [9] and Caristi [4]. Then we show that our results may be carried over to uniform spaces and, further, that some of our conclusions hold in Hausdorff topological spaces.

In what follows $\varphi(x, F(A))$ will be used to denote $\inf_{y \in F(A)}(x, y)$. The orbit of $x \in X$ generated by $A$ will be denoted by $O(x, A)$ and its closure by $\bar{O}(x, A)$. The set of subsequential limit points of the sequence $\{A^n x\}_{n=0}^\infty$ will be denoted by $L(x)$. 
We now establish the generalization of Theorem 2 of Diaz and Metcalf [6].

THEOREM 1. Let $A$ be a continuous self-map of a metric space $(X, d)$. Suppose that

(i) $F(A)$ is nonempty and compact.
(ii) there exists a non-negative continuous function $\varphi: X \times X \to \mathbb{R}_0$ such that $\varphi(Ay, F(A)) < \varphi(y, F(A))$ for $y \in X - F(A)$,
(iii) $\overline{O}(x, A)$ is compact for some $x \in X$.

Then $\mathcal{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathcal{L}(x)$ is a singleton or is uncountable. In the case $\mathcal{L}(x)$ is a singleton, $\lim_{n \to \infty} A^n x$ exists and belongs to $F(A)$. In the case $\mathcal{L}(x)$ is uncountable, it is contained in the boundary of $F(A)$.

PROOF. The compactness of $\overline{O}(x, A)$ implies nonemptiness of $\mathcal{L}(x)$. We now show that $\mathcal{L}(x) \subset F(A)$. If some iterate $A^k x \in F(A)$, we have $\mathcal{L}(x) = \{A^k x\} \subset F(A)$ and the theorem is proved. Therefore we assume that $A^k x \notin F(A)$ for $k = 0, 1, 2, \ldots$. Since for any fixed $y$, $z \to \varphi(z, y)$ is a continuous function from $X \to \mathbb{R}_0$, the function $z \to \varphi(z, F(A))$ is an upper semi-continuous function, being the infimum of a family of continuous functions. Since $A^k x \notin F(A)$ for all $k$, we have $\varphi(A^{k+1} x, F(A)) < \varphi(A^k x, F(A))$ for all $k$. Therefore $\{ \varphi(A^k x, F(A)) \}_{k=0}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers and so will converge to $r \geq 0$, say. Since $\mathcal{L}(x) \neq \emptyset$, for a $\xi \in \mathcal{L}(x)$ there exists a subsequence $\{A^n x\}_{n=1}^{\infty}$ with $A^n x \to \xi$ as $i \to \infty$. If $A^i \xi = \xi$, then we are through. Therefore we assume that $\xi \notin A^i \xi$. Then $r = \lim \varphi(A^{1+n} x, F(A)) = \lim \sup \varphi(A^{1+n} x, F(A)) = \varphi(\lim A^{1+n} x, F(A)) = \varphi(\xi, F(A))$. Now $y \to \varphi(A^n x, y)$, for fixed $A^n x$, is a continuous function: $X \to \mathbb{R}_0$ and so will attain its infimum on $F(A)$. Therefore there exists a $p_n \in F(A)$ such that $\varphi(A^n x, F(A)) = \varphi(A^n x, p_n)$. Corresponding to each $A^n x$ of the convergent subsequence $\{A^n x\}_{n=1}^{\infty}$ we have a $p_n \in F(A)$. Since $F(A)$ is compact, $\{p_n\}_{n=1}^{\infty}$ will have a convergent subsequence denoted by $\{p_{m_i}\}_{i=1}^{\infty}$ converging to $q$, say, in $F(A)$. Now $A^{m_i} x \to q$. From $\varphi(A^{n+1} x, p_{n+1}) = \varphi(A^{n+1} x, F(A)) < \varphi(A^n x, F(A)) = \varphi(A^n x, p_n)$, we have, for $m_i > n$, $\varphi(A^{m_i} x, p_{m_i}) < \varphi(A^n x, p_n)$. Letting $m_i \to \infty$, we have, since $\varphi$ is continuous, $\varphi(\xi, q) < \varphi(A^n x, p_n)$. Since $\varphi(A^n x, p_n) = \varphi(A^n x, F(A)) \to r \geq 0$, we have $\varphi(\xi, q) < r$. Thus $r < \varphi(A^i \xi, F(A)) < \varphi(\xi, F(A)) < \varphi(\xi, q) \leq r$. This is absurd. Therefore $\xi = A^i \xi$, that is, $\xi \in F(A)$. In other words $\mathcal{L}(x) \subset F(A)$. Further, $\mathcal{L}(x)$, being a closed subset of the compact set $F(A)$, is compact.

We now prove that $\mathcal{L}(x)$ is connected. Suppose the contrary. Then there exist two nonempty, disjoint, closed subsets $S_1, S_2$ of $\mathcal{L}(x)$ such that $\mathcal{L}(x) = S_1 \cup S_2$. 

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Since $S_1$ and $S_2$ are closed subsets of a compact set $\mathcal{L}(x)$, they are themselves compact. Hence $d(S_1, S_2) > 0$. Next, we show that $d(A^m x, F(A)) \to 0$ as $m \to \infty$. If not, there exists an $\epsilon > 0$ and a subsequence $\{A^m x\}_{m=1}^\infty$ such that $d(A^m x, F(A)) \geq \epsilon > 0$ for $i = 1, 2, 3, \ldots$. Since $\overline{\mathcal{O}(x, A)}$ is compact, $\{A^m x\}_{m=1}^\infty$ will have a subsequence $\{A^n x\}_{n=1}^\infty \to \xi \in F(A)$. Thus $d(A^n x, F(A)) \leq d(A^n x, \xi) \to 0$ as $i \to \infty$. This is a contradiction. Therefore we must have $d(A^m x, F(A)) \to 0$ as $m \to \infty$. We next prove that $\lim_{m \to \infty} d(A^m x, S_1 \cup S_2) = 0$. If it is not so, then there will exist an $\epsilon > 0$ and a subsequence $\{A^m x\}_{m=1}^\infty$ such that $d(A^m x, S_1 \cup S_2) \geq \epsilon > 0$ for $i = 1, 2, 3, \ldots$. Since $F(A)$ is compact, there exists a $q_m \in F(A)$ such that $d(A^m x, q_m) \to 0$. Hence $\epsilon \in \mathcal{L}(x) = S_1 \cup S_2$ and $d(A^m x, S_1 \cup S_2) \leq d(A^m x, \epsilon) \to 0$ as $i \to \infty$. This shows that $\lim_{m \to \infty} d(A^m x, S_1 \cup S_2) = 0$.

We further prove that $A$ is asymptotically regular. If not, there will exist an $\epsilon > 0$ and a subsequence $\{A^m x\}_{m=1}^\infty$ such that $d(A^m x, A^{m+1} x) \geq \epsilon > 0$. The corresponding sequence $\{A^m x\}_{m=1}^\infty$ will have a convergent subsequence $\{A^m x\}_{m=1}^\infty$ converging to $q \in F(A)$. As above $A^m x \to q \in F(A)$. Since $A$ is continuous at $q$, we have $A^{1+n} x \to A q = q$. Now $d(A^n x, A^{1+n} x) \leq d(A^n x, q) + d(A^{1+n} x, q) \to 0$ as $i \to \infty$. This is contrary to hypothesis. Hence we have proved that $d(A^m x, A^{m+1} x) \to 0$ as $i \to \infty$. Thus from the results found in this paragraph, we can find an integer $M$ such that for $m > M$, $d(A^m x, A^{m+1} x) < \frac{1}{3} d(S_1, S_2)$ and $d(A^m x, S_1, S_2) < \frac{1}{3} d(S_1, S_2)$. Since $S_1 \cup S_2$ is compact there exists a $q \in S_1 \cup S_2$ such that $d(A^m x, q) = d(A^m x, q)$. If $q \in S_1$, then $d(A^m x, S_1) < d(A^m x, q) < \frac{1}{3} d(S_1, S_2)$. Therefore for any $m > M$, either $d(A^m x, S_1) < \frac{1}{2} d(S_1, S_2)$ or, $d(A^m x, S_2) < \frac{1}{3} d(S_1, S_2)$. Both these inequalities cannot hold simultaneously, because in that case $d(S_1, S_2) < d(S_1, A^m x) + d(S_2, A^m x) < \frac{2}{3} d(S_1, S_2)$ which is absurd. Now it is clear that the set of positive integers $m > M$ for which $d(A^m x, S_1) < \frac{1}{3} d(S_1, S_2)$ is also nonempty. Let, for $m_1 > M$, $d(A^m x, S_1) < \frac{1}{2} d(S_1, S_2)$. There exist integers $n > m_1$ such that $d(A^n x, S_2) < \frac{1}{3} d(S_1, S_2)$. Let $k + 1$ be the least such integer. Then $d(A^{k+1} x, S_2) < \frac{1}{3} d(S_1, S_2)$ and $d(A^k x, S_1) < \frac{1}{2} d(S_1, S_2)$. We have

$$d(S_1, S_2) < d(S_1, A^k x) + d(A^k x, A^{k+1} x) + d(A^{k+1} x, S_2)$$

$$< \frac{1}{2} d(S_1, S_2) + \frac{1}{3} d(S_1, S_2) + \frac{1}{3} d(S_1, S_2).$$

This is absurd. Therefore the hypothesis that $\mathcal{L}(x) = S_1 \cup S_2$ with $S_1$ and $S_2$ nonempty, disjoint, closed subsets of $\mathcal{L}(x)$ leads to a contradiction. Hence $\mathcal{L}(x)$ is connected.

By Theorem 1 in Berge [1, p. 96] it follows that $\mathcal{L}(x)$ is either a singleton or is uncountable. We have proved above that $\lim_{m \to \infty} d(A^m x, \mathcal{L}(x)) = 0$, so that
when \( \mathcal{L}(x) \) is a singleton \( \{ \xi \} \), say, \( \lim_{m \to \infty} d(A^m x, \xi) = 0 \). Thus \( \lim_{m \to \infty} A^m x = \xi \in F(A) \).

To prove that \( \mathcal{L}(x) \), if uncountable, lies on the boundary of \( F(A) \), we observe that in this case \( A^k x \not\in F(A) \), \( k = 0, 1, 2, \ldots \). If \( \xi \in \mathcal{L}(x) \subset F(A) \) is an interior point of \( F(A) \), \( A^k x \in F(A) \) for some \( k \), as \( F(A) \) is a neighborhood of \( \xi \) and some subsequence of \( \{ A^m x \} \) converges to \( \xi \). This is a contradiction.

**Remark 1.** We observe here that Theorem 2 of Diaz and Metcalf [6] is a corollary of our theorem if we replace \( q > \) by \( d \). For this we have only to show that when \( \mathcal{L}(x) \) is nonempty, the assumptions of Diaz and Metcalf imply the compactness of \( \overline{O}(x, A) \). This has been shown in [8]. The following example shows that our theorem is indeed a generalization of the theorem of Diaz and Metcalf.

Take \( X = \{ a, b, c, d, e \} \) with the metric \( d(x, y) = 1 \) if \( x \neq y \) and \( d(x, y) = 0 \) if \( x = y \). Take the mapping \( A : X \to X \) such that \( Aa = b \), \( Ab = c \), \( Ac = d \), \( Ad = e = Ae \). Here \( F(A) = \{ e \} \) and it is easy to see that for \( x \neq A x \), \( d(A x, F(A)) < d(x, F(A)) \) is not satisfied for \( x = a, b \) or \( c \). Therefore we cannot invoke the theorem of Diaz and Metcalf to show that \( \mathcal{L}(a) \) is a closed, connected subset of \( F(A) \). We now define a function \( \varphi : X \times X \to R_0 \) the schematic representation of which is given by

\[
\begin{array}{c|ccccc}
  & a & b & c & d & e \\
\hline
a & 0 & 4 & 5 & 9 & 10 \\
b & 40 & 0 & 3 & 6 & 8 \\
c & 50 & 30 & 0 & 2 & 7 \\
d & 90 & 60 & 20 & 0 & 1 \\
e & 100 & 80 & 70 & 10 & 0 \\
\end{array}
\]

where the value of \( \varphi(x, y) \) occurs at the intersection of the row containing \( x \) with the column containing \( y \). We have

\[
1 = \varphi(d, F(A)) = \varphi(A c, F(A)) < \varphi(c, F(A)) = 7 = \varphi(A b, F(A)) < \varphi(b, F(A)) = 8 = \varphi(A a, F(A)) \varphi(a, F(A)) = 10.
\]

Further \( \varphi \) is continuous on \( X \times X \) because it has the discrete topology. Also \( F(A) \) is nonempty and compact and so is \( \overline{O}(a, A) \). Thus we may invoke our theorem to show that \( \mathcal{L}(a) \) is a nonempty, compact and connected subset of \( F(A) \).

**Remark 2.** Suppose, in addition to the hypotheses of Theorem 1, that \( F(A) \) is an at most countable set. In this case \( \lim_{m \to \infty} A^m x \) exists and belongs to \( F(A) \), because \( \mathcal{L}(x) \) is a singleton here and \( \lim_{m \to \infty} d(A^m x, \mathcal{L}(x)) = 0 \).
COROLLARY 1. Let \( A : X \to X \) be such that \( A^k \) is continuous for some \( k \). Suppose
(i) \( F(A^k) \) is nonempty and compact,
(ii) for each \( x \in X, \overline{O}(x, A^k) \) is compact,
(iii) there exists a continuous real-valued function \( \varphi : X \times X \to R_0 \) such that for
\( x \in X - F(A^k), \varphi(A^kx, F(A^k)) < \varphi(x, F(A^k)). \)

Then, for \( x \in X \), the set \( L_k(x) \) of subsequential limit points of the sequence of iterates \( \{A^mx\}_{m=1}^{\infty} \) is a nonempty, compact and connected subset of \( F(A^k) \). Further
the set \( L_1(x) \) of subsequential limit points of the sequence of iterates \( \{A^mx\}_{m=1}^{\infty} \) is the union of the \( k \) nonempty, compact, connected subsets \( L_k(A^jx), j = 0, 1, 2, \ldots, k - 1. \)

The proof is omitted because it is a minor modification of the proof of
Theorem 2\(^k\) of Diaz and Metcalf [6].

COROLLARY 2. Suppose, in addition to the hypotheses of Corollary 1, that \( F(A^k) \)
is an at most countable set. Then for \( x \in X, L_1(x) \) contains at most \( k \) points. This is
because each \( L_k(A^jx) \) is a singleton.

Dotson [7] calls a mapping \( A \) quasi-nonexpansive if \( F(A) \neq \emptyset \) and for each
\( x \in X - F(A), p \in F(A), d(Ax, p) \leq d(x, p) \). We call \( A \) quasi-contractive if the
strict inequality sign holds. The concept of quasi-contractiveness has been dis-
cussed by Diaz and Metcalf [6]. We define the mapping \( A \) to be \( \varphi \)-quasi-nonex-
pansive if \( F(A) \neq \emptyset \) and for \( x \in X - F(A), p \in F(A) \) we have \( \varphi(Ax, p) \leq \varphi(x, p) \) where \( \varphi : X \times X \to R_0 \). We say that \( A \) is \( \varphi \)-quasi-contractive if the strict
inequality sign holds. In this connection we now prove

THEOREM 2. Let \( A : X \to X \) be a continuous self-map of a metric space \((X, d)\).
Suppose that \( A \) is \( \varphi \)-quasi-contractive, where \( \varphi \) is a continuous function from
\( X \times X \to R_0 \). Then \( L(x) \subset F(A) \). If \( \varphi(x, y) = 0 \iff x = y \), then \( L(x) \) consists of
at most one point. If \( \overline{O}(x, A) \) is compact in addition, then \( \lim_{m \to \infty} A^mx \) exists and
belongs to \( F(A) \).

PROOF. If \( L(x) \) is empty there is nothing to prove. Therefore we shall assume
that \( L(x) \neq \emptyset \) and that \( A^kx \notin F(A), k = 0, 1, 2, \ldots \), as in Theorem 1. Then for
any \( p \in F(A) \), the sequence of positive numbers \( \{\varphi(A^n x, p)\} \) is monotonically
decreasing, because \( \varphi(A^{n+1} x, p) < \varphi(A^n x, p) \) by hypothesis. Hence
\( \lim_{n \to \infty} \varphi(A^n x, p) \) exists and is \( r \geq 0 \). Let \( \xi \in L(x) \) and let the subsequence
\( \{A^nx\}_{n=1}^{\infty} \) converge to \( \xi \). If possible, let \( \xi \neq A\xi \). Now
\[ r = \lim_{i \to \infty} \varphi(A^{1+n_i}x, p) = \varphi\left( \lim_{i \to \infty} A^{1+n_i}x, p \right) = \varphi(A\xi, p) \]

\[ < \varphi(\xi, p) = \varphi\left( \lim_{i \to \infty} A^{n_i}x, p \right) = \lim_{i \to \infty} \varphi(A^{n_i}x, p) = \lim_{i \to \infty} \varphi(A^n x, p) = r. \]

This contradiction proves that \( \xi = A\xi \) and so \( \xi \in F(A) \). We have thus proved that \( \mathcal{L}(x) \subset F(A) \). Obviously \( \mathcal{L}(x) \) is closed.

Assume now that \( \varphi(x, y) = 0 \) if and only if \( x = y \). Let \( p, q \in \mathcal{L}(x) \subset F(A) \). Obviously \( \varphi(A^m x, p) \to 0 \) as \( m \to \infty \). If the subsequence \( \{A^m x\}_{m=1}^\infty \) converges to \( q \), then \( \varphi(A^m x, p) \to \varphi(p, q) \). Hence \( \varphi(q, p) = 0 \) so that \( p = q \). Therefore \( \mathcal{L}(x) \) can consist of at most one point. If \( \partial(x, A) \) is compact, then \( \mathcal{L}(x) \) is obviously nonempty and so is a singleton. Let \( \mathcal{L}(x) = \{p\} \). If \( A^n x \to p \) as \( n \to \infty \) then for some \( \varepsilon > 0 \) there exists a subsequence \( \{A^m x\}_{m=1}^\infty \) with \( d(A^m x, p) \geq \varepsilon > 0 \). The compactness of \( \partial(x, A) \) implies the existence of a subsequence of \( \{A^n x\}_{n=1}^\infty \) converging to \( p \) as \( \mathcal{L}(x) = \{p\} \). This contradicts our hypothesis that \( d(A^n x, p) \geq \varepsilon \). Hence \( d(A^n x, p) \to 0 \) as \( n \to \infty \), implying \( A^n x \to p \).

Browder and Petryshyn [2, 3] define a self-map \( A \) of a Banach space to be asymptotically regular if \( A^{n+1}x - A^n x \to 0 \) strongly as \( n \to \infty \). We shall say that a mapping \( A \) is \( \varphi \)-asymptotically regular if \( \varphi(A^n x, A^{n+1} x) \to 0 \) as \( n \to \infty \). We are now in a position to give our

**Theorem 3.** Let \( A \) be a continuous self-map of a metric space \((X, d)\). Suppose

(i) \( F(A) \) is nonempty and compact,

(ii) there exists a continuous function \( \varphi: X \times X \to R_0 \) such that \( \varphi(y, z) = 0 \) if and only if \( y = z \),

(iii) \( A \) is \( \varphi \)-asymptotically regular,

(iv) \( \partial(x, A) \) is compact.

Then \( \mathcal{L}(x) \) is a nonempty, compact and connected subset of \( F(A) \). Either \( \mathcal{L}(x) \) is singleton or uncountable. In the case \( \mathcal{L}(x) \) is a singleton \( \lim_{n \to \infty} A^n x \) exists and belongs to \( F(A) \). In the case \( \mathcal{L}(x) \) is uncountable it is contained in the boundary of \( F(A) \).

**Proof.** If \( A^k x \in F(A) \) for some \( k \), then the proof is trivial. Therefore, assume \( A^k x \notin F(A) \) for all \( k \). The sequence \( \{\varphi(A^m x, F(A))\}_{m=1}^\infty \) is non-increasing and bounded below by zero and so converges to \( r \geq 0 \). Since \( \partial(x, A) \) is compact, \( \mathcal{L}(x) \neq \emptyset \). Let \( \xi \in \mathcal{L}(x) \) with \( A^m x \to \xi \) as \( i \to \infty \). Then \( \varphi(A^m x, A^{i+m} x) \to \varphi(\xi, A\xi) = 0 \), since \( A \) is \( \varphi \)-asymptotically regular. Hence \( \xi = A\xi \) and \( \xi \in F(A) \). Therefore \( \mathcal{L}(x) \subset F(A) \). Obviously \( \mathcal{L}(x) \) is closed. Since \( F(A) \) is compact and \( \mathcal{L}(x) \) is closed, \( \mathcal{L}(x) \) itself is compact. In view of the proof of Theorem 1, to
prove that $\mathcal{L}(x)$ is connected we need prove only $d(A^m x, F(A)) \to 0$ as $m \to \infty$ and this follows from the compactness of $\overline{O}(x, A)$ and the fact that $\mathcal{L}(x) \subset F(A)$. The remaining part of the proof is as in Theorem 1.

We may relax the compactness conditions on $F(A)$ by assuming $A$ to be $\varphi$-quasi-nonexpansive. This we state as

**Theorem 4.** Let $A$ be a continuous self-map of a metric space $(X, d)$. Suppose

(i) $F(A)$ is nonempty,

(ii) $A$ is $\varphi$-asymptotically regular where $\varphi$ is a continuous function: $X \times X \to R_0$ and $\varphi(y, z) = 0$ if and only if $y = z$.

Then $\mathcal{L}(x) \subset F(A)$. If, in addition, $A$ is $\varphi$-quasi-nonexpansive, then $\mathcal{L}(x)$ consists of at most one point. If $\overline{O}(x, A)$ is compact, then $\lim_{m \to \infty} A^m x = p$, where $\mathcal{L}(x) = \{ p \}$

**Proof.** The fact that $A$ is $\varphi$-asymptotically regular and vanishes only on the diagonal shows that $\mathcal{L}(x) \subset F(A)$. If $A$ is $\varphi$-quasi-nonexpansive and $p, q \in \mathcal{L}(x)$ with $p \neq q$, then $\varphi(A^m x, q) \to r \geq 0$. Also there exist subsequences $\{ A^{m_1} x \}$, $\{ A^{n_1} x \}$ such that $A^{m_1} x \to p$, $A^{n_1} x \to q$. Hence $\varphi(A^{m_1} x, A^{n_1} x) \to \varphi(p, q)$. Keeping $i$ fixed and letting $j \to \infty$, we have $\varphi(A^{m_1} x, A^{n_1} x) \to \varphi(A^{m_1} x, q)$. We can extract a subsequence $\{ m'_j \}$ from $\{ m_i \}$ such that $m'_j > n_j$. Since $\lim_{i \to \infty} \varphi(A^{m_1} x, q) = r \geq 0$, we have,

$$r = \lim_{i \to \infty} \varphi(A^{m_1} x, q) = \lim_{m'_j \to \infty} \varphi(A^{m'_j} x, q) \leq \lim_{n_j \to \infty} \varphi(A^{n_1} x, q) = \varphi(q, q) = 0.$$ 

But $\varphi(A^{m_1} x, q) \to \varphi(p, q)$. Therefore $\varphi(p, q) = 0$ whence $p = q$. Thus $\mathcal{L}(x)$ consists of at most one point. If $\overline{O}(x, A)$ is compact, then $\mathcal{L}(x)$ is nonempty and so $\mathcal{L}(x) = \{ p \}$, say. Now, proceeding as in Theorem 2 we can show that $\lim_{n \to \infty} A^n x = p$.

We now take the range of $A$ to be compact and derive

**Theorem 5.** Let $A: X \to X$ be continuous. Suppose

(i) $A(X)$ is compact,

(ii) $A$ is $\varphi$-asymptotically regular where $\varphi$ is a continuous function: $X \times X \to R_0$ and $\varphi(x, y) = 0$ if and only if $x = y$.

Then, for $x \in X$, the set $\mathcal{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathcal{L}(x)$ contains exactly one point or is uncountable. In the case $\mathcal{L}(x)$ is a singleton, $\lim_{m \to \infty} A^m x$ exists and belongs to $F(A)$. In the case $\mathcal{L}(x)$ is uncountable, it is contained in the boundary of $F(A)$.

**Proof.** Since $A$ is continuous, $F(A)$ is closed and so is compact as $F(A) \subset A(X)$, which is compact. Since $\overline{O}(Ax, A) \subset A(X)$, we have $\overline{O}(Ax, A)$ is compact.
Condition (ii) now implies that $\mathcal{L}(Ax) \subset F(A)$. But $\mathcal{L}(x) = \mathcal{L}(Ax)$. Hence $\mathcal{L}(x) \subset F(A)$. The compactness of $O(x, A)$ implies that $d(A^n x, F(A)) \to 0$ as $n \to \infty$. The remaining conclusions can be derived as in Theorem 1.

Following the idea of Singh and Zorzitto [9] we have

**Theorem 6.** Let $A$ be a continuous self-map of a metric space $(X, d)$. Suppose

(i) $F(A)$ is nonempty and compact

(ii) there exists a continuous function $\varphi: X \times X \to R_0$ such that $\varphi(y, z) = 0$ if and only if $y = z$ and for $y \in X - F(A)$, $\varphi(Ay, F(A)) \leq \varphi(y, F(A))$ and $\varphi(A^n y, F(A)) < \varphi(y, F(A))$ for an integer $m = m(y)$,

(iii) $O(x, A)$ is compact.

Then $\mathcal{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathcal{L}(x)$ is a singleton or is uncountable. In the case $\mathcal{L}(x)$ is a singleton, $\lim_{m \to \infty} A^m x$ exists and belongs to $F(A)$. In the case $\mathcal{L}(x)$ is uncountable it is contained in the boundary of $F(A)$.

**Proof.** Since $O(x, A)$ is compact, $\mathcal{L}(x)$ is nonempty. It is enough to prove that $\mathcal{L}(x) \subset F(A)$. The remaining portion of the proof can be derived as in Theorem 1. Assume $A^k x \notin F(A)$ for all $k$. Let $\xi \in (x)$. If possible let $\xi \neq A^k \xi$. Hence there exists a subsequence $\{A^n x\}$ such that $A^n x \to \xi$. Obviously, $\lim_{n \to \infty} \varphi(A^n x, F(A))$ exists and is equal to $r \geq 0$. Now $r = \lim_{\ell \to \infty} \varphi(A^{m(\ell)+n} x, F(A)) \leq \varphi(\lim_{\ell \to \infty} A^{m(\ell)+n} x, F(A)) = \varphi(A^m \xi, F(A)) < \varphi(\xi, F(A))$. Proceeding as in Theorem 1, we can show that $\varphi(\xi, F(A)) = r$. Thus $r = \varphi(A^{m(\ell)} \xi, F(A)) < \varphi(\xi, F(A)) \leq r$, which is absurd. Thus $x = A^k \xi$ and $\mathcal{L}(x) \subset F(A)$.

Corresponding to Theorem 2 we state the following theorem without proof as it can be derived by combining the methods of Theorem 2 and Theorem 6.

**Theorem 7.** Let $A: X \to X$ be a continuous self-map of a metric space $(X, d)$. Suppose that $F(A) \neq \emptyset$ and $A$ is $\varphi$-quasi-inonexpansive where $\varphi: X \times X \to R_0$ is continuous. Assume further that for $y \in X - F(A)$ and $p \in F(A)$ there exists an integer $m = m(y, p)$ such that $\varphi(A^m y, p) < \varphi(y, p)$. Then for $x \in X$, $\mathcal{L}(x) \subset F(A)$. If $\varphi(y, z) = 0 \iff y = z$, then, for any $x \in X$, $\mathcal{L}(x)$ consists of at most one point. If, in addition, $O(x, A)$ is compact then $\lim_{m \to \infty} A^m x$ exists and belongs to $F(A)$.

Now we shall use conditions similar to those of Caristi [4] to derive the same conclusions as those of Diaz and Metcalf [6].
THEOREM 8. Let $A$ be a self-map of a metric space $(X, d)$. Suppose
(i) $A$ is continuous at each point of $F(A)$,
(ii) $F(A)$ is nonempty and compact,
(iii) there exists a function $\psi: X \to R_0$ such that for $y \in X$, $d(Ay, F(A)) < \psi(y) - \psi(Ay)$.

Then, for $x \in X$, $\mathcal{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathcal{L}(x)$ is a singleton or is uncountable. In the case $\mathcal{L}(x)$ is a singleton, $\lim_{m \to \infty} A^m x$ exists and belongs to $F(A)$. In the case $\mathcal{L}(x)$ is uncountable, it is contained in the boundary of $F(A)$.

PROOF. We assume at the outset $A^k x \notin F(A)$ for all $k$, because otherwise the proof is trivial. This implies that $\psi(A^{k+1} x) < \psi(A^k x)$ for all $k$. Thus $\{\psi(A^k x)\}_{k=1}^{\infty}$ is a monotonically decreasing sequence of reals bounded below by zero and so converges to $r \geq 0$. We have

\[
d(A^m x, F(A)) \leq \psi(A^{m-1} x) - \psi(A^m x),
\]

\[
d(A^{m+1} x, F(A)) \leq \psi(A^m x) - \psi(A^{m+1} x),
\]

\[
d(A^n x, F(A)) \leq \psi(A^{n-1} x) - \psi(A^n x),
\]

whence, by adding, we get

\[
d(A^m x, F(A)) + d(A^{m+1} x, F(A)) + \cdots + d(A^n x, F(A)) \\
\leq \psi(A^{m-1} x) - \psi(A^n x).
\]

For $m, n$ sufficiently large, the right hand side can be made less than any preassigned $\epsilon > 0$, since $\{\psi(A^k x)\}_{k=1}^{\infty}$ is a convergent sequence of reals. Hence $d(A^m x, F(A)) \to 0$ as $m \to \infty$. Now, since $F(A)$ is compact we can find a $p_m \in F(A)$ such that $d(A^m x, F(A)) = d(A^m x, p_m)$. The sequence $\{p_m\}_{m=1}^{\infty}$ will have a convergent subsequence $\{p_{m_i}\}_{i=1}^{\infty}$ converging to $p \in F(A)$. Now

\[
d(p, A^m x) \leq d(p, p_{m_i}) + d(p_{m_i}, A^m x) \\
= d(p, p_{m_i}) + d(A^m x, F(A)) \to 0 \text{ as } i \to \infty.
\]

Thus $A^m x \to p$ as $i \to \infty$ and so $p \in \mathcal{L}(x)$ and hence $\mathcal{L}(x)$ is nonempty. If $\xi \in \mathcal{L}(x)$, then there is a subsequence $\{A^{n_i} x\}_{i=1}^{\infty}$ converging to $\xi$. Now $d(\xi, F(A)) = d(\lim A^{n_i} x, F(A)) = \lim d(A^{n_i} x, F(A)) = 0$ and as $F(A)$ is closed (being compact), $\xi \in F(A)$. Thus $\mathcal{L}(x) \subset F(A)$. Obviously $\mathcal{L}(x)$ is a closed subset of the compact set $F(A)$ and hence is itself compact. Now we can proceed as in proving Theorem 1 to establish the remaining conclusions.

COROLLARY 3. If $A^k$ satisfies the conditions of Theorem 8, then the set $\mathcal{L}_k(x)$ of subsequential limit points of the sequence $\{A^{mk} x\}_{m=1}^{\infty}$ is a nonempty, compact, and
connected subset of \( F(A^k) \). The set \( \mathcal{L}_1(x) \) of the subsequential limit points of \( \{A^m x\}_{m=1}^\infty \) is the union of the \( k \) closed and connected sets \( \mathcal{L}_j(A^i x) \), \( j = 0, 1, 2, \ldots, k - 1 \). If \( F(A^k) \) is at most countable, then \( \mathcal{L}_1(x) \) consists of at most \( k \) points.

**Corollary 4.** The conclusions of Theorem 8 remain valid if condition (iii) is replaced by

(iii)' there exists a monotonically decreasing sequence \( \{r_n\} \) of positive reals such that \( d(A^{n+1} x, F(A)) \leq r_n - r_{n+1} \), or

(iii)'' there exists a sequence \( \{s_n\} \) of positive reals converging to zero such that \( d(A^n x, F(A)) \leq s_n \).

Tarafdar [10] has extended some results of Diaz and Metcalf [2] to uniform spaces. We shall show that the results of Tarafdar still hold when our condition replaces his inequality. Our notations will conform to those of Thron [11].

Let \( (X, h) \) be a uniform space, \( h \) being the uniformity. The uniform topology induced by \( h \) will be denoted by \( \mathcal{T}_h \). A family \( \{\rho_\alpha \colon \alpha \in I\} \) of pseudometrics on \( X \) is called an associated family for the uniformity \( h \) on \( X \) if the family \( \{H(\alpha, \varepsilon)|\alpha \in I, \varepsilon > 0\} \) where \( H(\alpha, \varepsilon) = \{(x, y)|\rho_\alpha(x, y) < \varepsilon\} \) is a subbase for \( h \). A family \( \{\rho_\alpha|\alpha \in I\} \) of pseudometrics on \( X \) is called an augmented associated family for \( h \) if \( \{\rho_\alpha|\alpha \in I\} \) is an associated family for \( h \) and has the additional property that given \( \alpha, \beta \in I \), there is \( \gamma \in I \) such that \( \rho_\gamma(x, y) \geq \max(\rho_\alpha(x, y), \rho_\beta(x, y)) \) for all \( x, y \in X \). An associated family and an augmented associated family for \( h \) will be denoted respectively by \( \mathcal{F}(h) \) and \( \mathcal{F}^*(h) \).

We are now in a position to give our

**Theorem 9.** Let \( (X, h) \) be a Hausdorff uniform space and \( \mathcal{F}^*(h) = \{\rho_\alpha|\alpha \in I\} \). Let \( A \colon X \to X \) be \( \mathcal{T}_h \)-continuous. Suppose

(i) \( A(X) \) is \( \mathcal{T}_h \)-compact,

(ii) \( A \) is \( \varphi \)-asymptotically regular where \( \varphi \) is a \( \mathcal{T}_h \times \mathcal{T}_h \) continuous function on \( X \times X \to \mathbb{R}_0^+ \) such that \( \varphi(x, y) = 0 \) if and only if \( x = y \).

Then, for each \( x \in X \), the \( \mathcal{T}_h \)-cluster set \( \mathcal{L}(x) \) is a nonempty \( \mathcal{T}_h \)-closed and \( \mathcal{T}_h \)-connected subset of \( F(A) \). In the case \( \mathcal{L}(x) \) is just one point then \( \mathcal{T}_h \)-lim \( A^n x \) exists and belongs to \( F(A) \). In the case \( \mathcal{L}(x) \) contains more than one point then it is contained in the \( \mathcal{T}_h \)-boundary of \( F(A) \).

**Proof.** The sequence \( \{A^n x\}_{n=1}^\infty \) being a net in \( A(X) \), which is compact, \( \mathcal{L}(x) \) is nonempty. If \( y \in \mathcal{L}(x) \), then there is a subnet \( \{A^n x\}_{j \in J} \) of the net \( \{A^n x\}_{n=1}^\infty \) such that \( A^n \to y \) in the \( \mathcal{T}_h \)-topology. Since \( A \) is \( \mathcal{T}_h \)-continuous, \( A^{1+n} x \to Ay \) in the \( \mathcal{T}_h \)-topology. Hence \( \varphi(A^n x, A^{1+n} x) \to \varphi(y, Ay) \) as \( \varphi \) is \( \mathcal{T}_h \times \mathcal{T}_h \) continuous. Since \( A \) is \( \varphi \)-asymptotically regular, \( \varphi(y, Ay) = 0 \) and hence by condition (ii),
\[ y = Ay. \] Therefore \( \mathcal{L}(x) \subset F(A) \). Obviously \( \mathcal{L}(x) \) is closed. Now, we can proceed as in Tarafdar [10, Theorem 2.1] to prove that \( \mathcal{L}(x) \) is \( \mathcal{T}_h \)-connected. The other parts of the conclusion are to be established likewise.

Corresponding to Theorem 2.2 of Tarafdar [10] we have

**THEOREM 10.** Let \((X, h)\) be a Hausdorff uniform space and let \( \{ \rho_\alpha | \alpha \in I \} = \mathcal{F}^*(h) \). Let \( A: X \to X \) be \( \mathcal{T}_h \)-continuous. Suppose

1. \( F(A) \) is nonempty and compact,
2. there exists a \( \mathcal{T}_h \times \mathcal{T}_h \) continuous function \( \varphi: X \times X \to \mathbb{R}_0^+ \) such that for \( y \neq Ay, \varphi(Ay, F(A)) < \varphi(y, F(A)) \),
3. \( \overline{O}(x, A) \) is compact.

Then \( \mathcal{L}(x) \) is a closed subset of \( F(A) \). If \( \mathcal{L}(x) \) consists of more than one point, then \( \mathcal{L}(x) \) is contained in the \( \mathcal{T}_h \)-boundary of \( F(A) \).

The proof is omitted. A careful perusal of the proof of Theorem 1 shows that no metric properties of the space have been used in proving that \( \mathcal{L}(x) \) is a subset of \( F(A) \). Therefore we have the following theorem for Hausdorff topological spaces.

**THEOREM 11.** Let \((X, \mathcal{T})\) be a Hausdorff topological space and \( A \), a continuous self-map. Suppose

1. \( F(A) \) is nonempty and compact,
2. there exists a continuous function \( \varphi: X \times X \to \mathbb{R}_0^+ \) such that for \( y \neq Ay, \varphi(Ay, F(A)) < \varphi(y, F(A)) \).

Then \( \mathcal{L}(x) \), the set cluster points of \( \{ A^n x \}_{n=1}^{\infty} \) is a closed subset of \( F(A) \). If \( \mathcal{L}(x) \) consists of more than one point then it is contained in the boundary of \( F(A) \). If we further assume that \( \varphi(x, y) = 0 \) if and only if \( x = y \), then \( \mathcal{L}(x) \) is at most a singleton. If \( \mathcal{L}(x) \) is a singleton and \( \overline{O}(x, A) \) is compact, then \( \lim_{n \to \infty} A^n x \) exists and belongs to \( F(A) \).

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