CORRESPONDENCE

The Joint Editors, *12 April* 1950
The Journal of the Institute of Actuaries Students' Society

The Inherent Rate of Natural Increase

Sirs,

In his article 'Demography' (Vol. IX, November 1949), Mr Elfryn Jones sets out, on p. 124, to derive the fundamental equation containing the inherent rate of natural increase r. In doing so, he makes the assumption that the stable population, and the births in it, will be increasing at a geometric rate and from this he develops the fundamental equation for r. This whole process is rather circuitous, for it is necessary to establish first the fundamental equation *before* it can be inferred that the ultimately stable population will increase at some rate r.

The situation may be clarified by the following development, which is adapted from Lotka (*American Journal of Hygiene*, Vol. VIII, p. 875, November 1928) and was published by the writer in the *Transactions of the Actuarial Society of America*, Vol. XLI, p. 512, October–November 1941. Mr Jones's notation is used.

Having established the basic relationship

$$B(t)=\int_0^\infty B(t-x)\,p^F(x)f(x)\,dx,$$

in which $p^F(x)$ and $f(x)$ are prescribed, we are asked to find the course of the birth curve $B(t)$. To solve this integral equation, we may adopt a device familiar in differential equations by using the trial solution

$$B(t)=\sum_{n=1}^{\infty} Q_n e^{r_n t}.$$

Substituting the trial in the basic relationship, we have

$$\sum_{n=1}^{\infty} Q_n e^{r_n t}=\int_0^\infty \sum_{n=1}^{\infty} Q_n e^{r_n(t-x)} p^F(x)f(x)\,dx$$

$$=\sum_{n=1}^{\infty} Q_n e^{r_n t}\int_0^\infty e^{-r_n x} p^F(x)f(x)\,dx.$$

The last equation is satisfied if

$$1=\int_0^\infty e^{-r_n x}\,p^{\mathrm{F}}(x)f(x)\,dx,$$

which is the fundamental equation. With the product $p^{\mathrm{F}}(x)f(x)$ everywhere non-negative between 0 and ∞, the fundamental equation will have one real root and infinitely many complex roots. The real root is the inherent rate of increase. Having found the coefficients Q_n from initial conditions, substitution of r_n and Q_n in the trial solution for $B(t)$ will yield a curve with damped oscillations. In the ultimate state, where t is very large, the oscillations are negligible, and the curve $B(t)$ depends almost wholly upon the real root r. At that stage, we have

$$B(t)=B(t-x)\,e^{rx},$$

which is the premise used by Mr Jones.

Yours faithfully,

MORTIMER SPIEGELMAN

Metropolitan Life Insurance Company
1 Madison Avenue
New York