Seventh Meeting, June 10th, 1898.

Dr MACKAY in the Chair.

Permutations: Alternative Proofs of Elementary Formulas.

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1. The number of the r-permutations of n things is the same as the number of ways of adding r things to a row which originally contains $n-r$ other things.

For suppose the n letters A, B, C . . . N to be placed in a row, then each permutation consisting of r letters may be indicated by placing below those r letters in the row, the digits 1, 2, 3 . . . r, to indicate the order they have in the permutation. We may suppose zeros placed below all the remaining $n-r$ letters. Thus it is clear that the number of the r-permutations of n things is the same as the number of ways in which r numbers can be added to a row originally containing $n-r$ zeros.

2. Hence we can prove the formula

$$_nP_r = n(n-1)(n-2) \ . \ . \ . \ (n-r+1).$$

For starting with $n-r$ zeros, we can put the first number at either end of the row, or in any of the $n-r-1$ places between two successive zeros, *i.e.*, we can add the first number to the row in $n-r+1$ different ways. The next number can then be added in $n-r+2$ ways, giving a total of $(n-r+1).(n-r+2)$ ways of adding two things to the row. Proceeding thus, we see that the total number of ways of adding r things to the row which originally contained $n-r$ is $(n-r+1)(n-r+2) \ . \ . \ . \ n$, which may also be written $n(n-1)(n-2) \ . \ . \ . \ (n-r+1)$.

This then is also the number of the r-permutations of n letters, all different.

3. The number of ways in which two rows of things containing p and q things respectively, can be combined into a single row containing them all, the order of the things in each component set being unaltered, is $= {}_{p+q}C_p$. For it is clearly the same as the number of ways of choosing p out of $p+q$ places for the things of the first set. Of course it is also $= {}_{p+q}C_q$.

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Cor. The result is obviously the same as the number of ways of arranging p like things A, A . . . in a row along with q like things B, B . . . ; for the order of the A's amongst one another and that of the B's amongst one another, being indistinguishable, it is the same as if they had a fixed order.

4. The number of ways in which p A's, q B's, r C's and t other different letters can be arranged in order is $\dfrac{(t+p+q+r+\ldots)!}{p!q!r!\ldots}$.

For first the t different ways can be arranged in $t!$ orders. Taking any one of these, the p A's can be added in ${}_{p+t}C_p$ or $\dfrac{(p+t)!}{t!p!}$ ways to this row. Hence the total number of ways of forming a row consisting of the p A's and the t other things is $t!\dfrac{(p+t)!}{t!p!}$. Adding next the q B's and so on, we finally get at the result

$$t!.\frac{(t+p)!}{t!p!}.\frac{(t+p+q)!}{(t+p)!q!}.\frac{(t+p+q+r)!}{(t+p+q)!r!}\ldots$$

which simplifies at once to

$$\frac{(t+p+q+r+\ldots)!}{p!q!r!\ldots}, \qquad i.e. \quad \frac{n!}{p!q!r!\ldots}$$

where n is the total number of the things.