A FOURIER FORMULA FOR SERIES SUMMABLE $(C, 1)$

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1. Introduction. Recently Bullen **(1)** developed a J_3-integral that gives a Fourier representation for trigonometric series when both the series and its conjugate are summable $(C, 1)$ everywhere. Earlier Cross **(4)** had shown that the $C_2 P$-integral **(3)** was strong enough to integrate such series and, indeed, that any trigonometric series for which the series and its conjugate were summable (C, k) were Fourier series where the integral used in the formula for the coefficients was the $C_{k+1} P$-integral **(3)**. In **(1)** Bullen defines a J_4-integral and says, without proof, that a Fourier formula in terms of this integral may be obtained for series for which the series and its conjugate are summable $(C, 2)$ and the coefficients $o(n)$. In this paper a "symmetric" integral is developed, which successfully integrates (a) trigonometric series for which the series and its conjugate are summable $(C, 1)$ and (b) trigonometric series that are summable $(C, 2)$ whose coefficients are $O(n^\alpha)$, $0 \leqslant \alpha < 1$. The advantage of this development is the comparative simplicity of the proofs.

2. A symmetric integral of order two.

DEFINITION 2.1. *If $F(x)$ is a Lebesgue-integrable function, define*

$$SCD^2 F(x) = \lim_{h\to 0} \left\{ \frac{1}{h^4} \left[\int_x^{x+h} \left(\int_t^{t+h} F(u)du - \int_{t-h}^{t} F(u)du \right) dt \right] - \left[\int_{x-h}^{x} \left(\int_t^{t+h} F(u)du - \int_{t-h}^{t} F(u)du \right) dt \right] \right\},$$

if this limit exists.

DEFINITION 2.2. *Let $f(x)$ be finite-valued everywhere. If there exists a Lebesgue-integrable function, say $F(x)$, such that*

(a) $SCD^2 F(x) = f(x)$ *in* $[a - 2\pi, a + 2\pi]$,

(b) $F(a + 2\pi) = F(a - 2\pi)$,

(c) $F(a) = 0$,

(d) $F(x)$ *is C-continuous* **(2)** *at the points* $a, a + 2\pi, a - 2\pi$,

(e) $\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} F(t)\, dt$ *exists everywhere,*

then $f(x)$ is said to be SCP^2-integrable over $[a, a + 2\pi]$, the value of the integral being $F(a + 2\pi)$. The following notation will be adopted:

$$F(a + 2\pi) = SCP^2 \int_a^{a+2\pi} f(t)\, dt.$$

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$F(x)$ will be called an SCP^2-primitive of $f(x)$ on the interval $[a, a + 2\pi]$.

THEOREM 2.1. *If*

$$G(t) = \int_a^t F(t)\,dt, \qquad H(x) = \int_a^x G(t)\,dt,$$

then $SCD^2\, F(x)$ exists if and only if $D^4\, H(x)$ exists, and $SCD^2\, F(x) = D^4\, H(x)$ where the derivatives exist. The definition of the SCP^2-integral is unique.

Proof. The proof of the first statement in the theorem is elementary and is omitted.

If there are two functions $F(x)$ and $L(x)$ satisfying the conditions of Definition 2.2, then

$$SCD^2\,[F(x) - L(x)] = 0 \text{ in } [a - 2\pi, a + 2\pi]$$

and so $D^4\,\theta(x) = 0$ in $[a - 2\pi, a + 2\pi]$, where

$$\theta(x) = \int_a^x \int_a^u (F(t) - L(t))\,dt\,du.$$

Since $\theta''(x)$ exists everywhere (by Definition 2.2 (e)), it follows (6) that $\theta(x)$ is a cubic. Hence $\theta''(x)$ is linear and so

$$\theta''(x) = F(x) - L(x) = px + q, \qquad \text{a.e.}$$

From the C-continuity of $F(x)$ and $L(x)$ at $a + 2\pi$ and $a - 2\pi$, it follows that

$$\begin{aligned} F(a + 2\pi) - L(a + 2\pi) &= \lim_{h\to 0} \frac{1}{h} \int_{a+2\pi}^{a+2\pi+h} (px + q)\,dx \\ &= p(a + 2\pi) + q = F(a - 2\pi) - L(a - 2\pi) \\ &= \lim_{h\to 0} \frac{1}{h} \int_{a-2\pi}^{a-2\pi+h} (px + q)\,dx = p(a - 2\pi) + q, \end{aligned}$$

which implies that $p = 0$. A similar argument, utilizing the C-continuity of $F(x) - L(x)$ at $x = a$, shows that $q = 0$, which gives

$$F(x) - L(x) = 0, \qquad \text{a.e.}$$

Furthermore, again using the property of C-continuity of $F(x)$ and $L(x)$ at $a + 2\pi$, it is clear that

$$F(a + 2\pi) = \lim_{h\to 0} \frac{1}{h} \int_{a+2\pi}^{a+2\pi+h} F(t)\,dt = \lim_{h\to 0} \frac{1}{h} \int_{a+2\pi}^{a+2\pi+h} L(t)\,dt = L(a + 2\pi).$$

The definition of the SCP^2-integral thus guarantees a unique value for the integral over $[a, a + 2\pi]$.

3. A Fourier formula for series summable (*C*, 1). In the following theorem the hypothesis is that the trigonometric series

$$\tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + b_n \sin nx$$

and its conjugate

$$\sum_1^\infty a_n \sin nx - b_n \cos nx$$

are both summable $(C, 1)$. For convenience this condition is expressed in terms of the corresponding complex series

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

but should not be confused with the condition that the symmetric partial sums only of this series are summable $(C, 1)$.

THEOREM 3.1 *Suppose that the series*

$$\sum_{-\infty}^{\infty} c_n e^{inx} \tag{3.1}$$

is summable $(C, 1)$ *everywhere to sum* $f(x)$. *Then*

$$c_n = \frac{1}{2\pi^2} SCP^2 \int_a^{a+2\pi} f(t) e^{-int}\, dt, \tag{3.2}$$

where a is a suitably chosen number.

Proof. By hypothesis,

$$c_0 + \sum_{-\infty}^{\infty}{}' c_n e^{inx} \equiv f(x), \qquad (C, 1), \tag{3.3}$$

$$c_0 x + \sum_{-\infty}^{\infty}{}' \frac{c_n e^{inx}}{in} \equiv g(x), \qquad (C, 0), \tag{3.4}$$

$$\frac{c_0 x^2}{2} + \sum_{-\infty}^{\infty}{}' \frac{c_n e^{inx}}{(in)^2} \equiv F(x), \qquad (C, 0), \tag{3.5}$$

$$\frac{c_0 x^3}{6} + \sum_{-\infty}^{\infty}{}' \frac{c_n e^{inx}}{(in)^3} \equiv G(x), \qquad (C, 0), \tag{3.6}$$

$$\frac{c_0 x^4}{24} + \sum_{-\infty}^{\infty}{}' \frac{c_n e^{inx}}{(in)^4} \equiv H(x), \qquad (C, 0), \tag{3.7}$$

everywhere. By the Riesz-Fischer theorem, the series in (3.5) is the Fourier series of a function in L^2, since the coefficients are $o(1/n)$. The sum function $F(x)$ is thus Lebesgue-integrable and hence C-continuous almost everywhere. Letting $x = a$ be a point of C-continuity of $F(x)$, and defining

$$h(x) = \frac{c_0(x-a)^2}{2} + \sum_{-\infty}^{\infty}{}' \frac{c_n e^{inx}}{(in)^2},$$

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it will be shown that $h(x) - h(a)$ is an SCP^2-primitive of $f(x)$ on the interval $[a, a + 2\pi]$ according to Definition 2.2.

Since $SCD^2\, F(x) = D^4\, H(x)$ by Theorem 2.1, the fact that

$$SCD^2[h(x) - h(a)] = f(x)$$

follows from **(1)** since $SCD^2[h(x) - h(a)] = SCD^2\, F(x)$. Again, since under the hypothesis of the theorem, $G'(x)$ exists everywhere **(1)**, it is clear that

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} (h(t) - h(a))\, dt$$

exists everywhere. (The other conditions in Definition 2.2 are obviously satisfied.) It then follows that

$$h(a + 2\pi) - h(a) = SCP^2 \int_a^{a+2\pi} f(t)\, dt,$$

whence is obtained

$$c_0 = \frac{1}{2\pi^2}\, SCP^2 \int_a^{a+2\pi} f(t)\, dt.$$

The expression for c_n, $n \neq 0$, may be obtained in the usual way, expressing $f(x)e^{-inx}$ as the $(C, 1)$ sum of a trigonometric series with constant term c_n **(5)**. Then the same kind of calculation as above applied to the new series yields formula (3.2) for $n \neq 0$.

The following result is also valid.

THEOREM 3.2. *Suppose that the series*

$$\tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + b_n \sin nx \tag{3.8}$$

is summable $(C, 2)$ *everywhere to sum* $f(x)$ *and that* $a_n, b_n = 0(n^\alpha)$, $0 \leqslant \alpha < 1$. *Then*

$$a_n = \frac{1}{\pi^2}\, SCP^2 \int_0^{2\pi} f(t) \cos nt\, dt, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{\pi^2}\, SCP^2 \int_0^{2\pi} f(t) \sin nt\, dt, \qquad n > 0.$$

Proof. In the notation of Theorem 3.1 (applied to the symmetric sums of the series involved) $F(x)$ is continuous and $G'(x)$ exists everywhere since the series in (3.5) is uniformly convergent. It needs only to be noted further that the theorem on Riemann summability **(7**, p. 69**)** is applicable to series 3.8 since $a_n, b_n = o(n^2)$.

4. The J_4-integral and the SCP^2-integral. While the definition of the SCP^2-integral as given in §2 of this paper is quite adequate to obtain a representation for the coefficients of certain trigonometric series in Fourier form, it is necessary to extend the definition slightly so that a comparison with the J_4-integral may be made.

DEFINITION 4.1. *If the conditions of Definition 2.2 are satisfied when 2π is replaced by $2n\pi$, $n = 1, 2, \ldots, k$, k a fixed positive integer, $f(x)$ will be said to be SCP^2-integrable over the interval $[a, a + 2k\pi]$. The notation then is*

$$F(a + 2n\pi) = SCP^2 \int_a^{a+2n\pi} f(t)\,dt, \qquad n = 1, 2, \ldots, k.$$

Uniqueness may be demonstrated as before.

THEOREM 4.1. *Let $f(x)$ be periodic with period 2π. If $f(x)$ is SCP^2-integrable over $[0, 4\pi]$, then $f(x)$ is J_4-integrable over $[-4\pi, 4\pi]$ and*

$$\frac{3}{4\pi^2} \int_{-4\pi,-2\pi,2\pi,4\pi}^{0} f(t)\,d_4t = SCP^2 \int_0^{2\pi} f(t)\,dt.$$

Proof. Let $F(t)$ be an SCP^2-primitive of $f(x)$ on $[0, 4\pi]$. If

$$G(x) = \int_0^x F(t)\,dt, \qquad H(x) = \int_0^x G(t)\,dt,$$

then $D^4H(x) = SCD^2F(x)$ everywhere, by Theorem 2.1, and $H''(x)$ exists everywhere in $[-4\pi, 4\pi]$. It follows that $f(x)$ is J_4-integrable on $[-4\pi, 4\pi]$ and

$$(4.1) \qquad \int_{-4\pi,-2\pi,2\pi,4\pi}^{0} f(t)\,d_4t = H(0) + \frac{H(4\pi)}{6} - \frac{2H(2\pi)}{3} - \frac{2H(-2\pi)}{3} + \frac{H(-4\pi)}{6}.$$

By hypothesis,

$$SCD^2F(x) = f(x) = f(x + 2\pi) = SCD^2F(x + 2\pi),$$

provided (as in the following) that x and $x + 2\pi$ are both in $[-4\pi, 4\pi]$. This implies (compare the proof of the uniqueness of the SCP^2-integral) that

$$(4.2) \qquad F(x + 2\pi) - F(x) = ax + b, \qquad \text{a.e.}$$

for some constants a and b. It is then possible to write

$$\begin{aligned} G(x + 2\pi) &= \int_0^{2\pi} F(t)\,dt + \int_{2\pi}^{2\pi+x} F(t)\,dt \\ &= \int_0^{2\pi} F(t)\,dt + \int_0^x (F(t) + at + b)\,dt \\ &= G(x) + \tfrac{1}{2}ax^2 + bx + c \end{aligned}$$

and

$$H(x+2\pi) = \int_0^{2\pi} G(t)\,dt + \int_{2\pi}^{2\pi+x} G(t)\,dt$$
$$= \int_0^{2\pi} G(t)\,dt + \int_0^{x} (G(t) + \tfrac{1}{2}at^2 + bt + c)\,dt,$$

or

$$(4.3) \qquad H(x+2\pi) - H(x) = \tfrac{1}{6}ax^3 + \tfrac{1}{2}bx^2 + cx + d.$$

The right-hand side of (4.1) may be written as

$$\tfrac{1}{6}(H(4\pi) - H(2\pi)) - \tfrac{1}{2}(H(2\pi) - H(0)) + \tfrac{1}{2}(H(0) - H(-2\pi)) - \tfrac{1}{6}(H(-2\pi) - H(-4\pi)),$$

which, in virtue of equation (4.3), reduces to $4\pi^3a/3$. But equation (4.2) together with the definition of the SCP^2-primitive and integral yields $F(2\pi) = b$, $-F(-2\pi) = b - 2\pi a$. But $F(2\pi) = F(-2\pi)$ and hence $a = b/\pi = F(2\pi)/\pi$. This shows that

$$\frac{3}{4\pi^2}\int_{-4\pi,-2\pi,2\pi,4\pi}^{0} f(t)\,d_4t = F(2\pi) = SCP^2\int_0^{2\pi} f(t)\,dt.$$

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