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An Optimally Regularized Estimator of Multilevel Latent Variable Models, with Improved MSE Performance

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Abstract

We propose an optimally regularized Bayesian estimator of multilevel latent variable models that aims to outperform traditional maximum likelihood (ML) estimation in mean squared error (MSE) performance. We focus on the between-group slope in a two-level model with a latent covariate. Our estimator combines prior information with data-driven insights for optimal parameter estimation. We present a "proof of concept" by computer simulations, involving varying numbers of groups, group sizes, and intraclass correlations (ICCs), which we conducted to compare the newly proposed estimator with ML. Additionally, we provide a step-by-step tutorial on applying the regularized Bayesian estimator to real-world data using our `MultiLevelOptimalBayes` package.

Encouragingly, our results show that our estimator offers improved MSE performance, especially in small samples with low ICCs. These findings suggest that the estimator can be an effective means for enhancing estimation accuracy.

Keywords: regularized estimation, multilevel latent variable model, mean squared error, small sample, intraclass correlation

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1. Introduction

Multilevel latent variable models have been widely adopted in psychology, education, and related sciences to analyze hierarchical data while accounting for unobserved effects (Lüdtke et al., 2008; Skrondal & Rabe-Hesketh, 2009; Bollen et al., 2022; Zitzmann, Wagner, et al., 2022). Unlike traditional multilevel regression models (Raudenbush & Bryk, 2002; Snijders & Bosker, 2012), which rely on observed variables at each level, multilevel latent variable models introduce latent constructs that improve measurement accuracy and reduce bias in parameter estimates (Muthén & Asparouhov, 2012; Zitzmann et al., 2016). These models allow for more precise estimations of relationships at different levels of analysis by correcting for measurement error and providing a more flexible framework for capturing complex dependencies in nested data.

Over the past two decades, multilevel latent variable models have been widely applied in educational research to model student achievement and classroom effects (Lüdtke et al., 2008; Marsh, 1987), psychological research for latent personality and cognitive processes (Bollen et al., 2022; Muthén & Asparouhov, 2012), and health sciences for hierarchical patient-reported outcomes (Hamaker & Klugkist, 2011).

Compared to mixed-effects models (Raudenbush & Bryk, 2002; Snijders & Bosker, 2012), which typically assume that all predictors are observed and measured without error, multilevel latent variable models provide greater flexibility in handling measurement error and latent constructs. This makes them particularly valuable in psychological and educational research, where many key variables (e.g., cognitive ability, motivation, instructional quality) cannot be directly observed. Moreover, multilevel latent variable models allow researchers to separate within-group and between-group variance more effectively than traditional mixed-effects models, leading to more reliable inferences.

Multilevel models can be classified based on whether variables are assessed at the individual or group level (Croon & van Veldhoven, 2007; Snijders & Bosker, 2012). One

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relevant example in education is the study of student learning outcomes as a function of class-level characteristics such as class size. The "classic" multilevel models (also called random intercept models) used for this purpose are often estimated using software such as HLM (Raudenbush et al., 2011) or lme4 (Bates et al., 2015).

However, various works (e.g., Asparouhov & Muthén, 2007; Lüdtke et al., 2008) have argued that this type of aggregation can lead to severely biased estimates of the effect of the context characteristic. One possible solution is to use a specialized multilevel model in which the context variable is formed through latent rather than manifest aggregation (for a discussion of latent aggregation, see Lüdtke et al., 2008, 2011). Unfortunately, such a model with a latent predictor cannot be specified in HLM or lme4 and is therefore often estimated using Mplus (Muthén & Muthén, 2012). However, these models place high demands on the data, and convergence problems or inaccurate estimates of effects at the class level (accuracy issues) can occur.

Similar methods also play a role in other modeling contexts, such as regression analysis (Hoerl & Kennard, 1970; Tibshirani, 1996; see also McNeish, 2015) and structural equation models (Yuan & Chan, 2008; see also Yuan & Chan, 2016). In the latter, a small value is typically added to the estimated variance, and it has been suggested that a similar effect can be achieved by selecting an appropriate prior distribution (e.g., Chung et al., 2015; McNeish, 2016; Zitzmann et al., 2016).

Bayesian approaches have gained increasing popularity in multilevel modeling due to their ability to enhance estimation accuracy by incorporating prior information (Hamaker & Klugkist, 2011; Lüdtke et al., 2013; Muthén & Asparouhov, 2012; Zitzmann et al., 2015, 2016). The possibility of adding prior information is a fundamental aspect of Bayesian estimation. It combines information from the data at hand, captured by the likelihood function, with additional information from prior distribution, resulting in inferences based on the posterior distribution (Gelman, 2006). However, specifying priors can pose challenges, particularly in small samples with a low intraclass correlation (ICC), where the choice of prior is crucial (Hox et al., 2012). Small sample sizes are very common in psychology and related sciences due to limitations in funding and resource constraints

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(Browne & Draper, 2006). In such cases, between-group estimates may approach zero and become unstable, significantly increasing sensitivity to prior specification. This makes prior misspecification one of the biggest challenges in applying Bayesian approaches to latent variable models (Natarajan & Kass, 2000; Zitzmann et al., 2015). However, this effect of the prior can also be exploited. Recent research by Smid et al. (2020) has shed light on the importance of constructing "thoughtful priors" based on previous knowledge to enhance estimation accuracy (see also Zitzmann, Lüdtke, et al., 2021). In the Bayesian approach proposed in this paper, the prior parameters are determined through a theoretically derived automated procedure that minimizes the estimated Mean Squared Error (MSE). This removes the need for the user to manually specify a prior, thereby eliminating the risk of user-induced misspecification.

While Smid et al. (2020) focused on addressing small-sample bias, it has been argued that evaluating the quality of a method should consider not only bias but also the variability of the estimator, particularly in small samples with low ICCs (Greenland, 2000; Zitzmann, Lüdtke, et al., 2021). In cases of low ICCs, within-group variability dominates, and small sample sizes lead to unstable group-level estimates, resulting in higher variance when estimating between-group slopes. This highlights a crucial point — approaches solely dedicated to minimizing bias may, in fact, perform less optimally than those focused on reducing variability alone. Thus, it is important to consider both bias and variability in optimizing analytical strategies. In this regard, alternative suggestions for specifying priors have aimed at reducing the MSE, which combines both bias and variability (e.g., Zitzmann et al., 2015, 2016). Note that in cases of small samples and low ICCs, MSE is largely driven by the variability of the estimator. Therefore, minimizing variability remains an important goal when optimizing MSE.

In the same spirit, in this article, we derive a distribution for the Bayesian estimator of between-group slopes, building on the model originally established by Lüdtke et al. (2008). Specifically, we use this distribution to develop an optimally regularized Bayesian estimator that automatically selects priors to minimize MSE, thereby avoiding misspecification caused by user-specified priors. We then report the results from computational

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simulations conducted across a broad spectrum of conditions to evaluate the estimator. They demonstrate the advantages of this approach compared to ML estimation, particularly in scenarios of small samples and a low ICC.

2. Theoretical Derivation

Before delving into detailed aspects, we will briefly summarize Lüdtke et al.'s (2008) model, which we use to exemplify our approach. This model was proposed as one way to provide unbiased estimates of between-group slopes in contextual studies. It proposes predicting the dependent variable Y at the group level by using a latent variable. This latent variable represents a group's latent mean, offering a more reliable alternative than the traditional manifest mean approach. Known as the "multilevel latent covariate model", this model allows for the integration of latent group means into the more complex frameworks of multilevel structural equation models, which are prevalent in psychological research and related research (see also Zitzmann, Lohmann, et al., 2022).

Zitzmann, Lüdtke, et al. (2021) have proposed and discussed a Bayesian estimator for the between-group slope in this model (see also Zitzmann & Helm, 2021). Their approach introduced a method for incorporating prior information in estimating between-group slopes. However, this method required manual specification of prior distributions, which could be challenging, particularly in small samples where misspecified priors may lead to biased or unstable estimates. In contrast, our approach extends this work by upgrading their Bayesian estimator to a regularized Bayesian estimator that automatically selects optimal priors, thereby preventing user misspecification and improving estimation stability.

Since our method regularizes the estimator introduced by Zitzmann, Lüdtke, et al. (2021), we maintain their notation for consistency. More precisely, in the model, it is assumed that the individual-level predictor X is decomposed into two independent, normally distributed components: X_b, representing the latent group mean, and X_w, representing individual deviations from X_b. Thus, for an individual $i = 1, \ldots n$ within a group $j = 1, \ldots J$, the decomposition can be stated:

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$$X_{ij} = X_{b,j} + X_{w,ij} \tag{1}$$

$$X_{b,j} \sim N(\mu_X, \tau_X^2) \tag{2}$$

$$X_{w,ij} \sim N(0, \sigma_X^2) \tag{3}$$

Note that further, we assume that each of J groups includes n persons, therefore the overall sample size is nJ.

Hereafter, we will refer to σ_X^2 and τ_X^2 as the within-group and between-group variances of X, respectively. Similarly, σ_Y^2 and τ_Y^2 are the within-group and between-group variances of Y, respectively.

The individual-level and group-level regressions read:

$$\text{Level 1: } Y_{ij} = \beta_{0j} + \beta_w X_{w,ij} + \varepsilon_{ij} \tag{4}$$

$$\text{Level 2: } \beta_{0j} = \alpha + \beta_b X_{b,j} + \delta_j \tag{5}$$

In Equation 4, β_w represents the within-group slope that characterizes the relationship between the predictor and the dependent variable at the individual level, while β_{0j} describes the random intercept. Normally distributed residuals are denoted as $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$.

Moreover, we denote between-group slope in Equation 5 as β_b and the overall intercept as α. $\delta_j \sim N(0, \tau_\delta^2)$ represents normally distributed residuals. See Figure 1 for a visual representation of the model. Note that the between-group component Y_b in Figure 1 corresponds to the random intercept β_{0j} in Equation 4, whereas the within-group component Y_w in Figure 1 corresponds to $(\beta_w X_{w,ij} + \varepsilon_{ij})$ in Equation 4.

We focus on the between-group slope β_b, which is the most important parameter in numerous multilevel model applications, such as when analyzing contextual effects. For balanced data (where each group has an equal number of individuals), the maximum likelihood (ML) estimator of β_b is given by:

$$\hat{\beta}_b = \frac{\hat{\tau}_{YX}}{\hat{\tau}_X^2} \tag{6}$$

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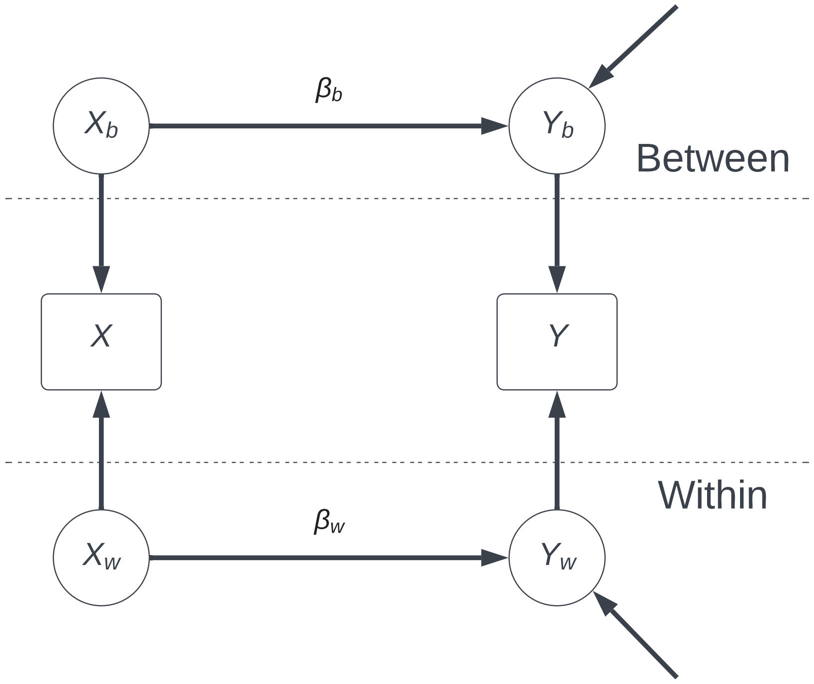
Figure 1

A multilevel structural equation model using the within-between framework that decomposes the variables X and Y into within-group and between-group components



Note. The within-group components are denoted by subscript w, and the between-group components are denoted by subscript b. The between-group components (X_b and Y_b) are connected through a regression, where Y_b serves as the dependent variable and X_b as the predictor. Similarly, the within-group components (X_w and Y_w) are related to each other in an analogous manner. The notation includes β_b for the between-group slope and β_w for the within-group slope.

In this equation, $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ are sample estimators of the group-level variance of X and the group-level covariance between X and Y, respectively.

While the asymptotic properties of the ML estimator (6) are advantageous, it tends to exhibit bias in finite sample sizes and displays significant variability, leading to a substantial Mean Squared Error (MSE) in such scenarios (as demonstrated by, e.g., McNeish, 2016). This poses a challenge to the practical utility of the ML estimator for rather small

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samples with low ICCs, as results from individual studies could be notably imprecise. Consequently, researchers have recommended alternative estimators that demonstrate lower variability, leading to increased accuracy and a reduced MSE, although potentially at the cost of some more bias compared to the ML estimator. Notable among these are the estimators proposed by Chung et al. (2013), Zitzmann et al. (2015), Zitzmann, Lüdtke, et al. (2021); see also Zitzmann & Helm (2021). Next, we will develop a regularized version of Zitzmann, Lüdtke, et al.'s Bayesian estimator for the between-group slope, drawing on so-called indirect strategy approach of constructing the estimator outlined by Zitzmann, Lüdtke, et al. (2021). The details of this development are provided in Appendix A.

Zitzmann, Lüdtke, et al.'s (2021) Bayesian estimator starts with the prior gamma distribution and its two parameters, ν_0 and τ_0^2 (see Appendix A). A specific choice of prior parameters is not required, as our forthcoming Bayesian estimator is designed to find the optimal values to minimize MSE. Combining priors with the ML estimator, Zitzmann, Lüdtke, et al. (2021) derived the Bayesian estimator as:

$$\tilde{\beta}_b = \frac{\hat{\tau}_{YX}}{(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2} \tag{7}$$

where ω is the weighting parameter defined as a function of the gamma-distributed priors. The denominator in Equation 7 accounts for both the prior variance τ_0^2 and the observed between-group variance τ_X^2, with weights adjusted by ω to control the influence of prior information as J increases.

Practically, $\omega \in [0, 1]$ can be interpreted as the relative weight given to the prior versus the data-base estimate: $\omega = 1$ corresponds to the standard ML estimator (Equation 6), $\omega = 0$ corresponds to full shrinkage toward the prior mean, and intermediate values balance the two sources of information.

The derivation of the Bayesian estimator (Equation 7) is described in detail in Appendix A. Note that Equation 7 is essentially a Stein-type estimator (Stein, 1956).

We specify the weighting parameter (prior) ω in a manner similar to that of Zitz-

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mann, Lüdtke, et al. (2021):

$$\omega = \frac{\frac{J-1}{2}}{\frac{\nu_0}{2} + \frac{J}{2} - 1} \tag{8}$$

So ω is defined as a function of the gamma-distributed prior ν_0 and the number of groups J. The weighting factor ω is derived such that as $J \to \infty$, ω approaches 1, ensuring that the Bayesian estimator converges to the ML estimator. Note that the weighting parameter ω in Equation 8 differs from the one introduced by Zitzmann, Helm, and Hecht (2021) because we further optimize it (see Appendix A).[1]

The Bayesian estimator $\tilde{\beta}_b$ is not yet regularized. To this end, the two parameters τ_0^2 and ω need to be identified. As mentioned, ω is defined as a function of sample size and converges to 1 when $J \to \infty$. Therefore, the Bayesian estimator $\tilde{\beta}_b$ is asymptotically unbiased and coincides with the ML estimator $\hat{\beta}_b$ in Equation 6 when samples are sufficiently large. In finite samples, however, the Bayesian estimator is biased.

To obtain the optimally regularized $\tilde{\beta}_b$, it is essential to find the values for τ_0^2 and ω based on an optimality criterion. The MSE serves as the natural choice for this criterion. It is defined as:

$$MSE(\tilde{\beta}_b) = Var(\tilde{\beta}_b) + (E(\tilde{\beta}_b) - \beta_b)^2 \tag{9}$$

As can be seen from the equation, this measure is simply the sum of the variance and the squared bias of the estimator. As the ML estimator in Equation 6 is unbiased in theory, its MSE shortens just to the variance of this estimator. The Bayesian estimator as defined in Equation 7 does not share the same unbiasedness property. Rather, it reduces the MSE by reducing its variance at the cost of some bias. We will show how to construct the estimator in such a way that a substantially reduced MSE is achieved compared to the ML estimator $\hat{\beta}_b$ in small samples with low ICCs. In infinite samples, the MSE of $\hat{\beta}_b$ reaches its global minimum of 0 (as both variance and bias converge to

[1] In this case, optimized stands for ω that minimizes the total error of an approximated denominator of the Bayesian estimator in Equation 7.

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0), and due to the weighting parameter ω, the Bayesian estimator $\tilde{\beta}_b$ achieves the same outcome.

To find the optimal values of the parameters τ_0^2 and ω, it is necessary to express the between-group (co)variance estimators from Equation 7, $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}^2$, in terms of the normal distributions of the between- and within-group components of the predictor and the dependent variable, namely X_b, X_w, Y_b and Y_w (see Appendix B for more details). We derived the expression for $\hat{\tau}_X^2$ under the restriction that it should have an easily definable distribution. For the derivation, see Appendix B. This resulted in:

$$\hat{\tau}_X^2 = H_X' S_X V_X' A V_X S_X H_X \tag{10}$$

where $H_X \sim N(0, \mathbf{I}_{nJ+J+1})$. The coefficient matrix A is defined in Equation 102 of Appendix F. Additionally, matrices V_X and S_X are the matrices of eigenvectors and eigenvalues, respectively. They are defined in Equation 57 of Appendix B. The internal part of Equation 10, $S_X V_X' A V_X S_X$, is a diagonal coefficient matrix. This means that in Equation 10, we express τ_X^2 as a weighted sum of squares of independent normally distributed random variables, that is, a weighted sum of χ_1^2-distributed random variables, which are transformed from X_b, X_w, Y_b, and Y_w.

To express $\hat{\tau}_{YX}$, we use a similar transformation as for $\hat{\tau}_X^2$. This transformation is described in detail in Appendix C. The result is:

$$\hat{\tau}_{YX} = H_2' S_H V_H' Q V_H S_H H_2 \tag{11}$$

where $H_2 \sim N(0, \mathbf{I}_{2(nJ+J+1)})$ is a multivariate standard normally distributed random vector. Coefficient matrix Q is computed in Equation 75 of Appendix C. Matrices V_H and S_H are the matrices of eigenvectors and eigenvalues, respectively. They are defined in Equation 72 of Appendix C. Furthermore, the internal part of Equation 11, $S_H V_H' Q V_H S_H$, is a diagonal coefficient matrix. With Equation 11, the estimator of the group-level covariance $\hat{\tau}_{YX}$ is represented as a weighted sum of squares of independent normally distributed random variables, that is, a weighted sum of χ_1^2-distributed random variables.

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As a consequence, we express each of the estimators of group-level (co)variances $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ as a sum of squares of independent and identically distributed normal random variables in Equations 10 and 11, respectively. Every term of these sums is χ_1^2-distributed, thus following the Gamma$\left(\frac{1}{2}, 2\right)$ distribution. Notice that a gamma distribution can be scaled: if a variable ψ follows the Gamma(k, θ) distribution, then $c*\psi$ is Gamma$(k, c*\theta)$-distributed. Therefore, we can represent the estimators of group covariances, $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$, as gamma-distributed random variables:

$$\begin{aligned}\hat{\tau}_X^2 &\sim \text{Gamma}(k_{sum1}, \theta_{sum1})\\ k_{sum1} &= \frac{\left(\sum_i \theta_{X,i}\right)^2}{2\sum_i \theta_{X,i}^2}, \theta_{sum1} = \frac{\sum_i \theta_{X,i}^2}{\sum_i \theta_{X,i}}\end{aligned} \tag{12}$$

$$\begin{aligned}\hat{\tau}_{YX} &\sim \text{Gamma}(k_{sum2}, \theta_{sum2})\\ k_{sum2} &= \frac{\left(\sum_i \theta_{YX,i}\right)^2}{2\sum_i \theta_{YX,i}^2}, \theta_{sum2} = \frac{\sum_i \theta_{YX,i}^2}{\sum_i \theta_{YX,i}}\end{aligned} \tag{13}$$

The scales $\theta_{X,i}$ and $\theta_{YX,i}$ are the elements of the diagonal matrices $S_X V_X' A V_X S_X$ (for $\hat{\tau}_X^2$) and $S_H V_H' Q V_H S_H$ (for $\hat{\tau}_{YX}$) in Equations 10 and 11.

In the next step, we make use of the distributions of the sample covariances $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ to calculate the distributions of the ML estimator $\tilde{\beta}_b$ and the Bayesian estimator $\hat{\beta}_b$. The estimators $\tilde{\beta}_b$ and $\hat{\beta}_b$ are defined using an F distribution, because ratios of gamma-distributed random variables follow F distributions. The full procedures of deriving the distributions of $\hat{\beta}_b$ and $\tilde{\beta}_b$ are presented in Appendix D. The results of these derivations are the following distributions:

$$\frac{k_{sum1}\theta_{sum1}}{k_{sum2}\theta_{sum2}}\hat{\beta}_b \sim F(2k_{sum2}, 2k_{sum1}) \tag{14}$$

$$\frac{k_B(\omega, \tau_0^2)\theta_B(\omega, \tau_0^2)}{k_{sum2}\theta_{sum2}}\tilde{\beta}_b \sim F(2k_{sum2}, 2k_B(\omega, \tau_0^2)) \tag{15}$$

where the coefficients k_{sum1}, θ_{sum1}, k_{sum2}, θ_{sum2}, k_B, θ_B are defined and fully described in Equations 81, 82, 87, and 88 of Appendix D. Note that k_B and θ_B are functions of

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the prior parameters ω and τ_0^2. Using these distributions, we compute the variances and expected values of both estimators and combine them into the final formulas for their MSEs:

$$MSE(\hat{\beta}_b) = \frac{k_{sum2}\theta_{sum2}^2(k_{sum1} + k_{sum2} - 1)}{\theta_{sum1}^2(k_{sum1} - 1)^2(k_{sum1} - 2)} + \left(\frac{k_{sum2}\theta_{sum2}}{(k_{sum1} - 1)\theta_{sum1}} - \beta_b\right)^2 \quad (16)$$

$$MSE(\tilde{\beta}_b) = \frac{k_{sum2}\theta_{sum2}^2(k_B(\omega,\tau_0^2) + k_{sum2} - 1)}{\theta_B^2(\omega,\tau_0^2)(k_B(\omega,\tau_0^2) - 1)^2(k_B(\omega,\tau_0^2) - 2)} + \left(\frac{k_{sum2}\theta_{sum2}}{(k_B(\omega,\tau_0^2) - 1)\theta_B(\omega,\tau_0^2)} - \beta_b\right)^2 \quad (17)$$

As a byproduct, we obtain their standard errors from the estimators' distributions as:

$$SE\left(\hat{\beta}_b\right) = \frac{\theta_{sum2}}{\theta_{sum1}\left(k_{sum1} - 1\right)}\sqrt{\frac{k_{sum2}\left(k_{sum1} + k_{sum2} - 1\right)}{k_{sum1} - 2}} \quad (18)$$

$$SE\left(\widetilde{\beta}_b\right) = \frac{\theta_{sum2}}{\theta_B(\omega,\tau_0^2)\left(k_B(\omega,\tau_0^2) - 1\right)}\sqrt{\frac{k_{sum2}\left(k_B(\omega,\tau_0^2) + k_{sum2} - 1\right)}{k_B(\omega,\tau_0^2) - 2}} \quad (19)$$

Using these standard errors, one can describe the uncertainty associated with the estimation or use them for statistical testing. However, when samples are rather small, we recommend to use resampling procedures for obtaining standard errors, such as the delete-d jackknife (Shao & Wu, 1989; for applications in multilevel modeling, see Zitzmann, 2018; Zitzmann, Lohmann, et al., 2022; Zitzmann et al., 2023, 2024).

Having obtained the MSE of $\tilde{\beta}_b$ (Equation 17), we can minimize it with respect to the parameters ω and τ_0^2 in order to obtain our regularized Bayesian estimator. To find the optimal choices for the prior parameters, we employ a numerical approach, which is algorithmic in nature, making it well-suited for implementation in software platforms like R or MatLab. The algorithm is a grid search over the parameters, with $0 \leq \omega \leq 1$ and $0 > \tau_0^2 > d * \hat{\tau}_X^2$. Since it is impossible to find the global minimum in the general case (Lakshmanan, 2019), the algorithm we implement performs only a local optimum search. We propose to choose parameter d to be at least five times the standard deviation of the

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estimated group-level variance of X, that is, $5 * \sqrt{Var(\hat{\tau}_X^2)}$. The value of $Var(\hat{\tau}_X^2)$ may be obtained from the derived distribution of $\hat{\tau}_X^2$ in Equation 81 of Appendix D, or even more exactly, by using the procedures of Mathai (1993) or Fateev et al. (2016). This 5-sigma region guarantees that the minimum estimated MSE falls inside this region with high probability. The probability of the minimum estimated MSE being within this interval is at least 0.9857 for $J = 3$, 0.9996 for $J = 5$, and > 0.99998 for $J \geq 7$. In this case, our grid search will find the inner solution for the optimal values of ω and τ_0^2 that minimize the estimated MSE. Note that the grid search algorithm minimizes the estimated MSE but not the unknown true MSE.

It is important to note that the MSE in Equations 16 and 17 incorporates the unknown between-group coefficient β_b. We propose using its ML estimate, $\hat{\beta}_b$, as a substitute, thereby giving our technique an empirical Bayes flavor. Such uses of "plug-in estimates" are not uncommon in statistics and often very useful (Liang & Tsou, 1992; see also Zitzmann et al., 2024).

We have demonstrated an approach for minimizing the MSE of the between-group parameter, leading to what we refer to as the optimally regularized Bayesian estimator $\tilde{\beta}_b$ for this parameter. Notice that our estimator uses the ML estimator $\hat{\beta}_b$ during MSE optimization and even includes ML as a special case when $\omega = 1$. This means, in small samples, we can do better than the ML estimator in terms of MSE. However, when working with large sample sizes, the costs due to using approximate distributions and the plug-in procedure to compute the regularized Bayesian estimator may be larger than the benefits. Such a scenario is likely to occur with larger group sample sizes combined with high levels of the intraclass correlation of the predictor. In the next section, we demonstrate some of these properties using simulated data.

3. Simulation Studies

We begin with the description of the data-generating mechanism, including its parameters such as group size n, number of groups J, intraclass correlation coefficient ICC_X, and the coefficients β_b and β_w. We utilized the generated data to compute estimates us-

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ing both the proposed optimally regularized Bayesian estimator and, for benchmarking purposes, also the ML estimator. The full algorithm used to actually yield $\tilde{\beta}_b$ is detailed in Appendix E. Finally, we present the results graphically. Detailed results can be found in Appendix G, which allows for a more comprehensive evaluation of the estimation accuracy under varying input parameters.

3.1. Data Generation

Next, we detail the data generation process and outline the specifics of our simulation setup. We base our simulations on the data-generating process used by Zitzmann, Helm, and Hecht (2021), Zitzmann, Lüdtke, et al. (2021). Specifically, we conducted simulations for each unique combination of the following parameters:

- ICC_X: Intraclass Correlation (0.05, 0.1, 0.3, 0.5)
- J: Number of groups (5, 10, 20, 30, 40)
- n: Number of individuals per group (5, 15, 30)
- β_b: Between-group parameter (0.2, 0.5, 0.6)
- β_w: Within-group parameter (0.2, 0.5, 0.7)

In total, this resulted in $4 \times 5 \times 3 \times 3 \times 3 = 540$ scenarios, each of which was replicated 5,000 times. The relatively small number of groups was chosen to reflect reasonable two-level scenarios in the social sciences (i.e., typically < 30 students per class, < 30 schools per district), and to align with examples from Gelman & Hill (2006).

The values of β_b and β_w follow ranges used in prior simulation studies on the multilevel latent covariate framework and related models. For example, Lüdtke et al. (2008) used values $\{0.2, 0.7\}$, Grilli & Rampichini (2011) considered values including $\{0.25, 0.5, 0.75, 1, 1.5\}$, and Zitzmann & Helm (2021) used the value of 0.7. The combination $\beta_b = \beta_w = 0.7$ is infeasible under our fixed $\text{ICC}_Y = 0.2$ design, so β_b was reduced to 0.6 in that case. Similarly, near-zero β_b values were not included because for $\text{ICC}_Y = 0.2$, they would violate ICC constraints:

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$$\frac{ICC_Y}{\beta_b^2} > ICC_X > 1 - \frac{1 - ICC_Y}{\beta_w^2} \tag{20}$$

The intraclass correlation of the dependent variable, denoted as ICC_Y, was preset to 0.2 within the code to study scenarios with ICC values that lie at the center of the typical ICC range observed in empirical studies (Gulliford et al., 1999). Additionally, we incorporated another validity check in order to identify and exclude incorrectly specified inputs, such as non-integer values for J or n.

3.2. Evaluation Criteria

The goal of our simulations was to assess how well the regularized Bayesian estimator can estimate the true parameter value β_b across various scenarios. To this end we assessed its performance in terms of the MSE and bias. Note that in addition to the presented estimator, a variant thereof was studied. Both variants were compared against the ML estimator.

We consider the following variants of the regularized Bayesian estimator: our proposed Bayesian estimator with the MSE optimization based on plugged-in ML-estimate $\hat{\beta}_b$; Bayesian estimator with MSE optimization based on the true value of β_b.

It is important to note that only the variant-1 Bayesian estimator (with MSE optimization based on the ML estimate $\hat{\beta}_b$) and the ML estimator are practically applicable to real data. In contrast, the second Bayesian estimator (with MSE optimization based on the true β_b) serves only as a theoretical benchmark.

Further, as evaluation measures, we use the square root of the MSE, denoted as RMSE, and the relative bias. First, MSE is computed as the mean of the squared differences between the estimated parameter and the true between-group parameter, β_b. Second, the square root is taken to obtain RMSE from MSE. RMSE then allows for comparisons similar to those made with MSE[2] while presenting the error in the original units of measurement. Our preference for RMSE over MSE stems from its scalability and

[2]The method with the smallest MSE also has the smallest RMSE, and the reverse is also true.

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straightforward interpretability. These attributes enhance the visualization of our analysis, facilitating clearer insights into the estimators' performance. The RMSE describes the overall accuracy of parameter estimation, indicating the proximity of estimated values to the true parameter values. Relative bias, in contrast, assesses the average deviation of the estimated parameters from the true value. It is computed as the ratio of the mean difference between the estimated parameter and the true between-group parameter to the true between-group parameter, β_b. The mean difference is calculated over repeated replications of each scenario in our simulation study. A small relative bias indicates that the estimator produces results that, on average, are closer to the true parameter value, while a larger relative bias suggests systematic over- or underestimation.

3.3. Simulation Results

Here, we report the results of our simulation study, focusing on the characteristics of the simulated data, their alignment with theoretical expectations, and the comparisons between our proposed estimator, the variant thereof, and the ML estimator. To facilitate a better understanding, we present visual analyses in Figures 2, 3, and 4, which illustrate the differential behaviors of the estimators as a function of the group-level sample size and the ICC. For a better differentiation between methods, we chose to show the logged RMSE in Figures 2 and 3. Note that log is a monotone increasing function for RMSE>0.

For more details about the RMSE and relative bias across 540 unique scenarios, see Tables 2 – 9 (see Appendix G).

Figure 2 provides a visual representation of the log of the RMSE patterns for the three estimators of the slope. The first line (blue dashed line) in Figure 2 is from the second alternative variant of the Bayesian estimator; that is, the Bayesian estimator based on the true value of β_b and thus the direct implementation of Equation 17. As mentioned, this estimator cannot be used on the real data, as the β_b is unknown, but it works as a benchmark for comparison with our proposed Bayesian estimator. This latter estimator (red solid line) is the Bayesian estimator with the plug-in ML estimate $\hat{\beta}_b$ in place of β_b. The third estimator (black dash dot line) is the ML estimator. Recall that

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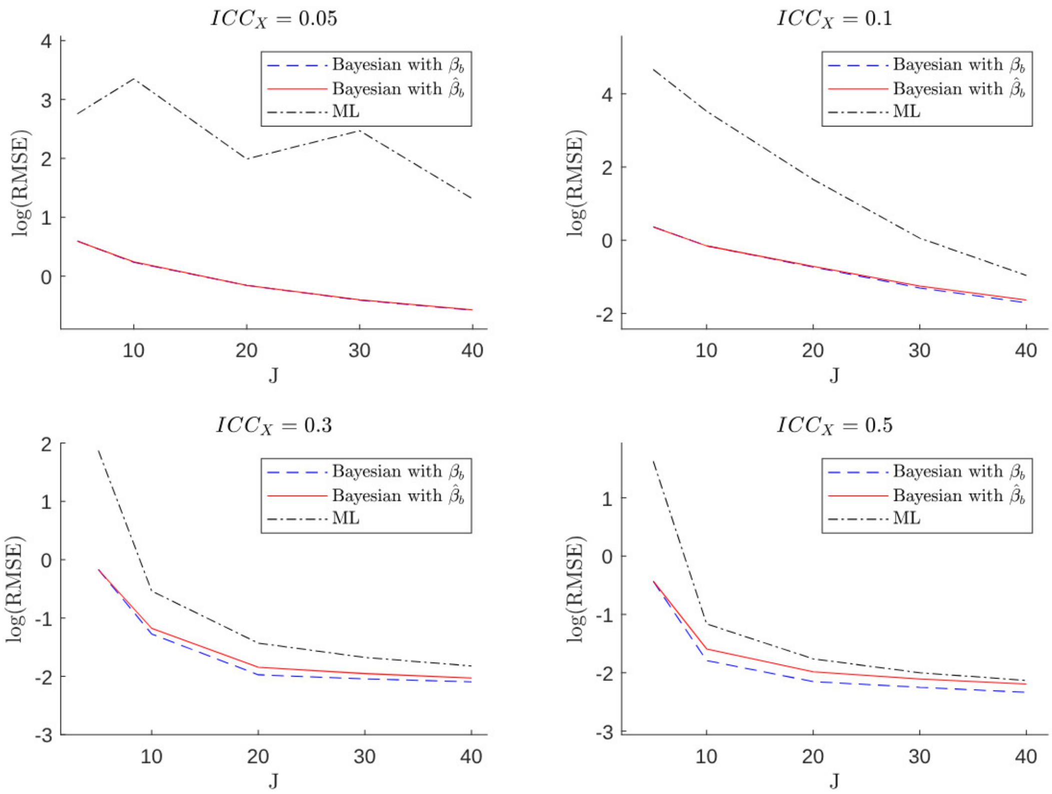
Figure 2

Log of root mean squared error (RMSE) in estimating the between-group slope β_b for the ML and the two Bayesian estimators as a function of the sample size at the group level (J) and the intraclass correlation of the predictor ICC_X



Note. The scale of the y-axis differs between the four subplots. Results are shown for $n = 15$ people per group, and constant within-group and between-group slopes of $\beta_w = 0.5$ and $\beta_b = 0.2$, respectively.

among the three estimators, only the second and third are applicable to the real data.

Our theoretical expectations align with the observed trends, as both Bayesian estimators exhibit lower RMSE compared to the ML estimator. This RMSE reduction is more pronounced for smaller group sizes (J), with the effect amplified by lower intraclass correlations (ICC_X). Additionally, RMSE consistently decreases with increasing J for all methods and ICC levels. However, an exception is observed for the ML estimator in

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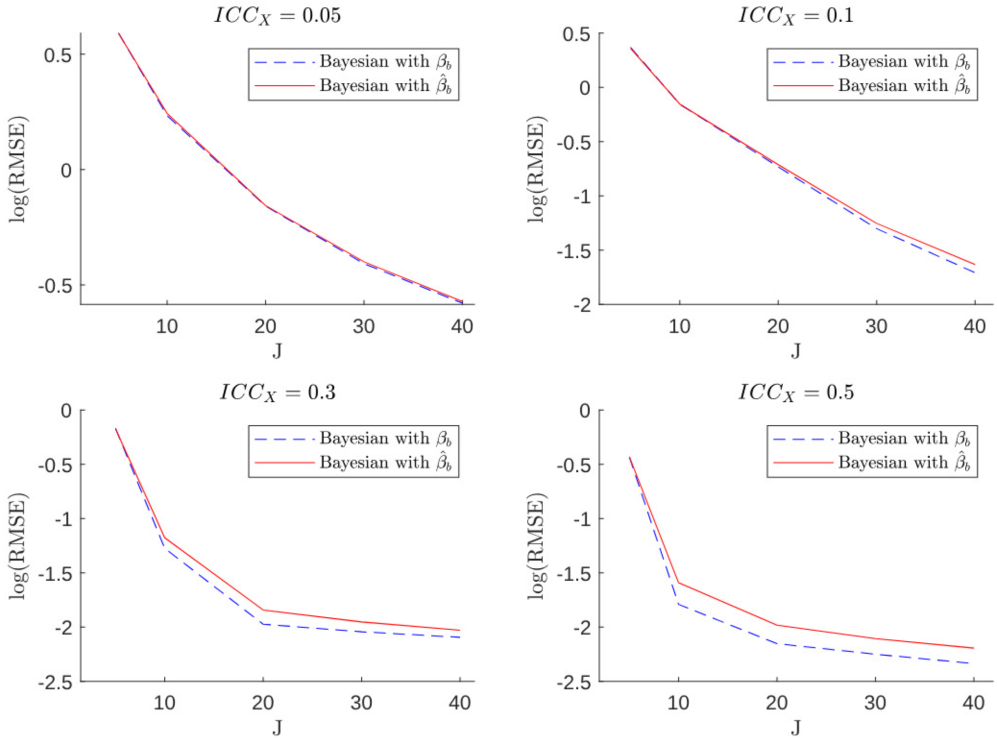
Figure 3

Log of root mean squared error (RMSE) in estimating the between-group slope β_b for the two Bayesian estimators as a function of the sample size at the group level (J) and the intraclass correlation of the predictor ICC_X



Note. The scale of the y-axis differs between the four subplots. Results are shown for $n = 15$ people per group, and constant within-group and between-group slopes of $\beta_w = 0.5$ and $\beta_b = 0.2$, respectively.

the upper left plot of Figure 2, where RMSE does not follow this expected trend. At low ICC_X and small J, between-group variance $\hat{\tau}_X^2$ is often estimated near zero, causing the ML estimator (Equation 6) to inflate and produce occasional extreme values. This yields a finite-sample distribution that mixes regular estimates with such extremes. Because RMSE is highly sensitive to these rare events, the population RMSE can display non-monotonic patterns across adjacent J values even with very large numbers of replications. In contrast, the regularized Bayesian estimators replace $\hat{\tau}_X^2$ with $(1 - \omega)\tau_0^2 + \omega\hat{\tau}_X^2$

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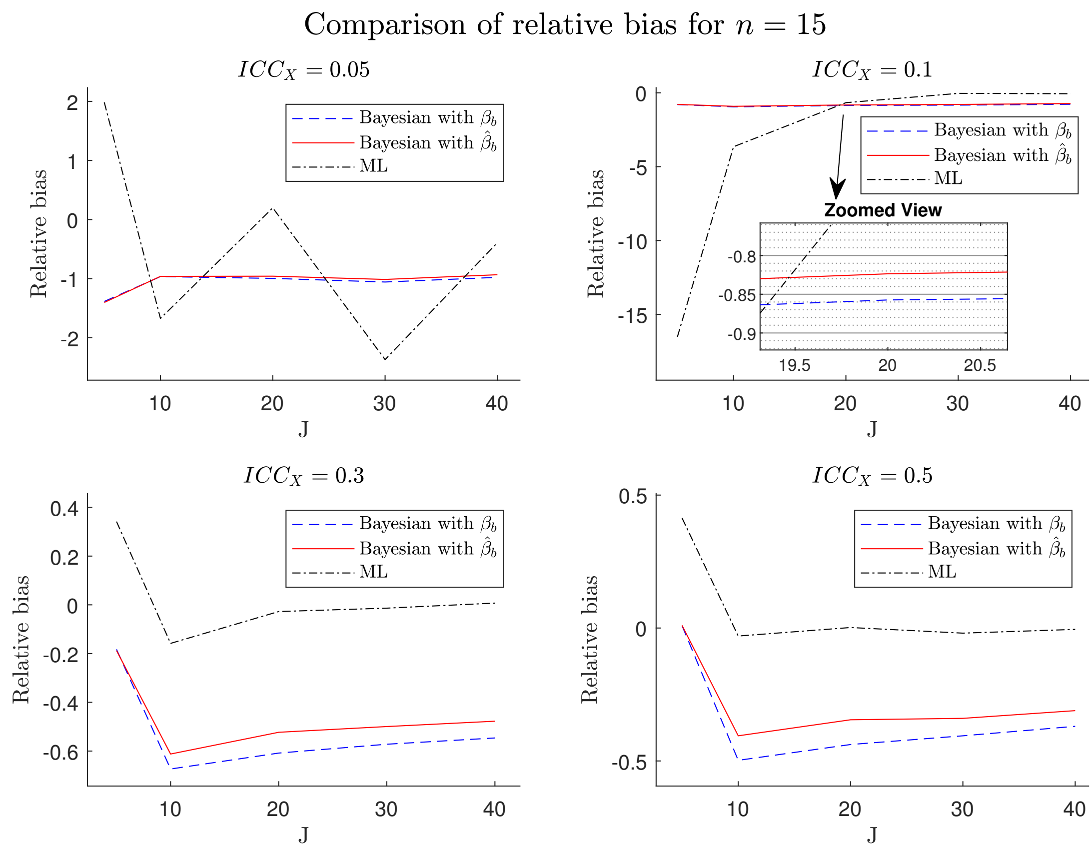
Figure 4

Relative bias in estimating the between-group slope β_b for the ML and the two Bayesian estimators as a function of the sample size at the group level (J) and the intraclass correlation of the predictor ICC_X



Note. The scale of the y-axis differs between the four subplots. Results are shown for $n = 15$ people per group, and constant within- and between-group slopes of $\beta_w = 0.5$ and $\beta_b = 0.2$, respectively.

in the denominator, bounding it away from zero and producing smooth, strictly decreasing RMSE curves. Despite this, the overall comparison remains valid, as ML consistently underperforms the regularized Bayesian estimators across all analyzed scenarios in Figure 2.

Figure 3 further adds to the understanding of the performance differences. This

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figure demonstrates that the differences in RMSE between Bayesian estimators based on inserting the true versus estimated values of β_b are only negligible, speaking for the usefulness of the Bayesian estimator with the plugged in ML estimate of β_b.

Figure 4 shows the behavior of the estimators with respect to the relative bias. The first thing to mention is that both variants of the Bayesian estimator (blue dashed and red solid lines) do not converge to a bias of zero with an increasing, but finite number of groups J, while the ML estimator does (black dash dot line). This bias is not due to misspecified priors but is the intended result of MSE-optimal shrinkage in the Bayesian estimator (Equation 7), where bias is deliberately traded for reduced variability. However, as $J \to \infty$, and $\omega \to 1$, the regularized Bayesian estimator converges to ML, and the bias disappears. Secondly, with an increasing intraclass correlation ICC_X, the relative bias of all three estimators decreases (plots 1-4 of Figure 4). Thirdly, despite being asymptotically unbiased, the ML estimator exhibits small-sample bias, especially for small ICC values (see upper left plot in Figure 4). This bias is inherent to ML estimation and results from denominator instabilities when $\hat{\tau}_X^2$ (Equation 6) is estimated near zero under low ICC, which can lead to sporadic extreme values and a heavy-tailed error distribution. This effect occurs only with the ML estimator, whereas the regularized Bayesian approaches remain stable across all scenarios because the denominator uses the weighted sum $(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2$ (Equation 7).

Table 1 presents RMSE and relative bias values computed across all 540 scenarios and averaged within each combination of group size n and number of groups J. It consolidates information from Tables 2 - 9 in Appendix G. Specifically, Table 1 compares three estimators: Maximum Likelihood (ML), regularized Bayesian with β_b, and regularized Bayesian with $\hat{\beta}_b$. Highlighted cells identify the estimator with the smallest RMSE (and therefore the smallest MSE) and the smallest relative bias. Results clearly illustrate that, across all examined cases, the regularized Bayesian estimators consistently provide lower RMSE values compared to the ML approach. However, as both group size and the number of groups increase, the relative bias of the ML estimator approaches zero, as it is a consistent estimator. At the same time, relative bias of the regularized Bayesian

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Table 1

Average RMSE and Relative Bias values of the ML ($RMSE_{ML}$ and $Bias_{ML}$, respectively), the Bayesian estimator with β_b ($RMSE_{Bay}$ and $Bias_{Bay}$, respectively), and the Bayesian estimator with $\hat{\beta}_b$ ($RMSE_{BML}$ and $Bias_{BML}$, respectively) for different values of n and J

n	**J**	**$RMSE_{ML}$**	**$RMSE_{Bay}$**	**$RMSE_{BML}$**	**$Bias_{ML}$**	**$Bias_{Bay}$**	**$Bias_{BML}$**
5	5	138.948	2.165	**2.139**	-541.286	-85.861	**-85.565**
5	10	65.035	**1.230**	1.231	**20.007**	-79.511	-77.130
5	20	101.584	**0.771**	0.781	253.325	-67.544	**-64.531**
5	30	20.412	**0.602**	0.611	**33.315**	-59.854	-56.847
5	40	25.685	**0.519**	0.526	-60.882	-57.754	**-54.792**
15	5	456.334	1.131	**1.129**	-2815.721	-31.872	**-31.855**
15	10	107.527	**0.653**	0.662	-564.371	-51.219	**-48.227**
15	20	19.847	**0.443**	0.451	-79.606	-54.971	**-51.664**
15	30	7.720	**0.362**	0.368	**-5.551**	-55.591	-52.659
15	40	3.561	**0.315**	0.320	**-4.161**	-55.796	-53.163
30	5	84.649	**0.949**	0.950	-88.566	-20.531	**-20.521**
30	10	19.940	**0.546**	0.556	**16.845**	-52.950	-49.524
30	20	4.110	**0.341**	0.347	**-12.779**	-57.784	-54.571
30	30	0.473	**0.279**	0.283	**-1.565**	-57.588	-54.760
30	40	0.386	**0.257**	0.260	**-1.737**	-56.888	-54.412

estimators remains around 60%. Consequently, for larger n, the ML estimator often has the smallest highlighted relative bias. Nevertheless, even when the ML estimator exhibits less bias than both regularized Bayesian estimators, the regularized Bayesian estimators achieve a substantial reduction in MSE and RMSE values, especially when n and J are small. Thus, Table 1 emphasizes that, according to our simulation studies, regularized Bayesian estimation - where only the regularized Bayesian estimator with $\hat{\beta}_b$ is applicable in the real world - may deliver more biased estimations, compared to ML, but is highly preferable in terms of MSE, especially in scenarios with small n and J.

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In conclusion, our optimally regularized Bayesian estimator with the ML estimate plugged-in demonstrates its power to refine the accuracy of estimators for the between-group slope β_b in small samples. While acknowledging inherent bias (see Table 3 in Appendix G for details), this estimator generated through our approach demonstrates enhanced accuracy when juxtaposed with the ML estimator, particularly in situations characterized by a finite sample size. Next, we provide a summary of our introduced approach, reflect on the theoretical advancements, highlight new findings, address limitations, and offer insights into the broader implications of our work.

4. Step-by-Step Tutorial Using `MLOB` R Package

To illustrate the practical application of the newly developed estimator, we created the MultiLevelOptimalBayes (`MLOB`) package, which includes the estimation function `mlob()`. In this section, we provide step-by-step instructions on using the regularized Bayesian estimator with the `MLOB` package in R. The estimator is applied to the PASSNYC dataset—a real-world dataset on educational equity in New York City that includes data from 1,272 schools across 32 districts.

4.1. Loading `MLOB` Package

First, install and load the `MLOB` package, which is available on CRAN:

```
install.packages("MultiLevelOptimalBayes")
```

Alternatively, the development version can be installed from GitHub:

```
install.packages("devtools")
devtools::install_github("MLOB-dev/MLOB")
library("MultiLevelOptimalBayes")
```

4.2. Loading and Preparing the Dataset

As mentioned earlier, we demonstrate how to use the `MLOB` package based on the PASSNYC dataset. The PASSNYC dataset is available on Kaggle.[3] In the next step,

[3] `https://www.kaggle.com/datasets/passnyc/data-science-for-good/data`

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load, clean, and convert the relevant variables of the PASSNYC dataset to numeric values:

```
# Load data (set up the correct folder in R using setwd())
data <- read.table("2016 School Explorer.csv", sep = ',', header = TRUE)

# Create a subset excluding N/A values in Average.Math.Proficiency
data_subset <- data[data$Average.Math.Proficiency != 'N/A', ]

# Convert the Average Math Proficiency variable to numeric
data_subset$math <- as.numeric(data_subset$Average.Math.Proficiency)

# transform variable Economic.Need.Index to numeric variable ENI
data_subset$ENI = as.numeric(data_subset$Economic.Need.Index)
```

4.3. Estimating the Between-Group Effect

We seek to obtain the contextual effect of economic need on average math proficiency using the regularized Bayesian estimator. For user convenience, the `mlob()` function follows a similar notation and works as simply as the linear regression function `lm()` in R. We specify `District` as the grouping variable. To ensure reproducibility, we set a random seed before processing the dataset. Since the dataset is unbalanced (i.e., the number of individuals per group varies), our procedure balances the data by randomly removing entities from larger groups to achieve equal group sizes. Setting a seed ensures that the same entities are removed each time the procedure is run, making the results fully replicable.

```
# Set seed for reproducibility
set.seed(123)

# Apply the mlob function
result <- mlob(math ~ ENI, data = data_subset, group = 'District', balancing.limit = 0.35)
```

Warnings may indicate that the data are unbalanced and that a balancing procedure has been applied. The function also alerts the user if estimates may be unreliable due to

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a highly unbalanced structure. By default, if more than 20% of the data would need to be deleted to achieve balance (threshold adjustable via the *balancing.limit* parameter), the function stops and issues a warning. While this procedure preserves the estimator's assumptions, removing many observations or groups may affect the generalizability of the results.

4.4. Summary of Results

The output of the customized **summary()** function follows the format of the **summary(lm())** function and provides the estimated between-group effect (β_b) obtained with the regularized Bayesian estimator. For comparison, the **summary()** function also includes ML estimation results:

```
summary(result)

Call:
mlob(math ~ ENI, data = data_subset, group = "District", balancing.limit = 0.35)

Summary of Coefficients:
         Estimate  Std. Error  Lower CI (95%)  Upper CI (95%)   Z value   Pr(>|z|)  Significance
beta_b   -1.0379      0.0183       -1.0737         -1.0020     -56.6769   0.00e+00      ***

For comparison, summary of coefficients from unoptimized analysis (ML):
         Estimate  Std. Error  Lower CI (95%)  Upper CI (95%)   Z value   Pr(>|z|)  Significance
beta_b   -1.7415      0.7580       -3.2271         -0.2560      -2.2977   0.0216         *

Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

4.5. Interpretation

The results indicate that the regularized Bayesian estimator provides an estimate with a significantly lower standard error compared to the ML estimator. Notably, the between-group coefficient estimated by the regularized Bayesian estimator ($\tilde{\beta}_b = -1.0379$) is smaller in absolute terms than the one estimated by ML ($\hat{\beta}_b = -1.7415$). The reduction

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in absolute magnitude suggests that ML may overestimate the effect due to its higher variance, whereas the regularized Bayesian estimator produces more reliable estimates, particularly in small samples. The between-group effect in this context represents how economic need, averaged at the district level, influences math proficiency across the districts of New York City. The negative coefficient suggests that districts with higher economic need tend to have lower average math proficiency. Given that the PASSNYC dataset is relatively small, containing 1,272 schools across 32 districts, the primary small-sample issue arises from the limited number of districts rather than the total number of schools. Since hierarchical models rely on the number of groups to estimate between-group effects, a small number of districts leads to increased variance in the estimated between-group coefficient. In this setting, the lower variance of the Bayesian estimator is particularly beneficial, as it enhances the reliability of the estimates. This highlights the advantages of the regularized Bayesian estimator in two-level latent variable models, especially with small datasets such as PASSNYC.

To draw a parallel with the previous section, we refer to Table 1, which summarizes the average RMSE and relative bias across different n and J and illustrates when regularized Bayesian or ML estimation is the preferable choice. A green color code is used to indicate the superior estimator for each scenario. Notably, in all analyzed cases, the newly developed estimator outperformed ML in terms of RMSE, further demonstrating its reliability in multilevel latent variable modeling. Therefore, even when the sample is sufficiently large, we recommend using our `MLOB` package, which offers both ML and regularized Bayesian estimations, allowing users to select the most appropriate method for their data. It is also important to consider degenerate cases where either the between-group or within-group effect is zero. In such cases, the `mlob()` function recommends using simpler models, such as ordinary least squares (OLS) or ML.

5. Discussion and Conclusion

In this article, we thoroughly described and analyzed a regularized Bayesian estimator for multilevel latent variable models, which we optimized with respect to MSE

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performance, using the multilevel latent covariate model as an example. In addition, we derived an analytical expression for the standard error.

However, given our specific focus on small sample size, rather than using this standard error, it might be more reasonable to employ a resampling technique for accurately determining the standard error. As mentioned, one such effective method is a delete-d jackknife procedure. The main achievement lies in deriving an optimally regularized Bayesian estimator by seamlessly integrating the minimization of MSE with respect to the parameters of the prior distribution. Through graphical representations of the results, we highlighted the pronounced improvements that our approach garners over ML estimation, particularly in small samples.

The following contributions to the theoretical landscape are noteworthy. Primarily, we derived a distribution of the Bayesian estimator, enabling us to achieve further optimization of the MSE with respect to the parameters of the prior distribution for this estimator. Moreover, we proposed an algorithm to construct our optimally regularized Bayesian estimator. These theoretical achievements are mirrored by the results from our simulation study as detailed in the previous section. In a nutshell, from these results, significant performance improvements emerged for the optimally regularized Bayesian estimator compared to the ML estimator, particularly in situations characterized by small sample sizes and low ICCs. These advantages can be attributed to the way the estimator is constructed, which allows for some bias while actively minimizing the MSE.

Although our work focuses on Bayesian estimation, the utilization of prior information to enhance estimation is not exclusive to Bayesian methods. Similar means are taken by frequentist approaches. For example, the Bayesian estimator's weighting parameter ω in Equation 8 achieves an effect analogous to the penalty in regularized structural equation modeling, as seen in Jacobucci et al. (2016). Similarly, the weighting parameter in the denominator of Equation 7 aligns with the concept of regularized consistent partial least squares estimation (e.g., Jung & Park, 2018).

While our research offers significant contributions, we also acknowledge limitations. The advantages of our method over ML estimation become less pronounced with larger

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sample sizes, indicating that our approach may be most beneficial in contexts with smaller samples. Another limitation of our approach lies in the locality of the search for the optimal MSE. Our optimization strategy within a $5 * \sigma$ region ensures that the minimum MSE falls within this region with almost 100% probability, although this is not guaranteed. Additionally, since the true MSE remains unknown, we rely on the estimated MSE, which provides a reliable approximation within the defined bound. However, the extrema of the real and estimated MSE do not always coincide. As a result, misspecification of the regularized Bayesian estimation is possible but extremely unlikely. Moreover, by reducing the $5 * \sigma$ search region, we can control bias and select an optimal estimator within the reduced region. While this decreases the probability of finding the globally optimal MSE, it ensures that the estimator has a relative bias within a predefined threshold. In the degenerate case where the search region is zero, we obtain an exact ML estimator. This is a potential area for future research.

One more limitation is the assumption of equal group sizes, which simplifies the statistical problem. However, in practice, group sizes often vary (e.g., the number of students in classes). While our current approach does not directly account for unequal group sizes, one possible solution would be to average the group sizes and apply our estimator. It is important to note that our regularized Bayesian estimator formulas extend to non-integer values of n, allowing for this flexibility. This is also a potential area for future research. Nevertheless, our `MLOB` R package includes a built-in data-balancing mechanism that provides a practical solution for handling unequal group sizes. Notably, if more than 20% of the data would need to be deleted to achieve balance, the function stops and alerts the user.

Beyond these limitations, the regularized Bayesian estimator can be extended to three- and higher-level models. While our estimator has not yet been fully developed for such multilevel structures, these models could be implemented through an iterative application of the two-level estimator. One approach is to iteratively apply the regularized Bayesian estimator by reducing the model to two levels at a time, computing estimates, and then proceeding to the next pair of levels.

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An extension for future simulation work is to explore a broader range of between-group parameter values, including near-zero β_b settings, to more fully assess performance under weak between-group effects. Future designs could also relax the constraints on ICC_Y to investigate the estimator's behavior in such scenarios.

Another possible extension is incorporating time as a predictor, enabling a longitudinal modeling framework for analyzing time-related trends. For example, the application of our regularized Bayesian estimator to the longitudinal dataset `ChickWeight` is included as a standard example in the `MLOB` R package. Such extensions provide promising directions for future research and further refinement of the regularized Bayesian estimator.

To conclude, our optimized Bayesian estimator, which sophistically balances bias reduction and variance minimization, offers improved precision in parameter estimation, particularly in small samples. Thus, our findings hold promising implications for multilevel latent variable modeling, and the demonstrated accuracy improvements due to optimized regularization underscore the practical value of our estimator. We aspired to empower researchers in psychology and related fields to utilize the benefits of our proposed estimator and use the newly developed `mlob` package in R, as demonstrated in the Section Step-by-Step Tutorial when dealing with small samples in fitting multilevel latent variable models.

By highlighting the efficacy of Bayesian strategies, we hope to inspire a paradigm shift in estimation techniques for small-sample scenarios. This shift could lead to more robust and informed modeling practices in the research community.

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Appendix A

To derive a Bayesian estimator following Zitzmann, Helm, and Hecht (2021) indirect strategy, we start by adopting a gamma prior distribution for the inverse of the group-level variance of the predictor variable τ_X^2:

$$\frac{1}{\tau_X^2} \sim \text{Gamma}(a, b) \tag{21}$$

where a and b are the parameters of the Gamma distribution. For better interpretability, we employ a reparameterization of $a = \frac{\nu_0}{2}$ and $b = \frac{\nu_0 \tau_0^2}{2}$ leading to:

$$\frac{1}{\tau_X^2} \sim \text{Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0 \tau_0^2}{2}\right) \tag{22}$$

Similarly, the likelihood for the inverse of the group-level variance is:

$$\frac{1}{\tau_X^2} \sim \text{Gamma}\left(\frac{J}{2}, \frac{J \hat{\tau}_X^2}{2}\right) \tag{23}$$

with $\hat{\tau}_X^2$ being an estimate of the group-level variance. To get an inverse-gamma posterior, we combine Equations 22 and 23 and yield:

$$\frac{1}{\tau_X^2} \sim \text{Gamma}\left(\frac{\nu_0 + J}{2}, \frac{\nu_0 \tau_0^2 + J \hat{\tau}_X^2}{2}\right) \tag{24}$$

As demonstrated by Zitzmann, Lüdtke, et al. (2021) in Appendix C, an approximation for the mean of this distribution can be derived as follows:

$$\overline{\tau}_X^2 \approx (1 - \omega)\tau_0^2 + \omega \hat{\tau}_X^2 \tag{25}$$

With the Equation 25, the Bayesian expected a posteriori (EAP) estimate is defined. We specify the weighting parameter ω from the Equation 25 as:

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$$\omega = \frac{\frac{J-1}{2}}{\frac{\nu_0}{2} + \frac{J}{2} - 1} \tag{26}$$

This formula minimizes the total error of the approximation of $\overline{\tau}_X^2$ from Equation 25, making it optimal.

Note that ω is defined in Equation 26 as a function of sample size, or more precisely, as a function of the number of groups J.

Asymptotically, when $J \to \infty$, ω converges to 1. Thus, $\overline{\tau}_X^2$ becomes equal to $\hat{\tau}_X^2$ in this case.

To derive the new estimator, we take Equation 6 and replace $\hat{\tau}_X^2$, with its Bayesian EAP as defined in Equation 25. This gives:

$$\tilde{\beta}_b = \frac{\hat{\tau}_{YX}}{(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2} \tag{27}$$

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Appendix B

To compute an estimate of the group-level covariances, we apply the formulas from Zitzmann, Lüdtke, et al. (2021), starting from the decompositions:

$$X_{ij} = X_{b,j} + X_{w,ij} \tag{28}$$

$$Y_{ij} = Y_{b,j} + Y_{w,ij} \tag{29}$$

We assume that $X_{b,j}$ and $X_{w,ij}$ are uncorrelated and both independently identically normally distributed. The same assumptions are considered for Y.

Next, we define (manifest) group means for both X and Y as:

$$\overline{X}_{\bullet j} = \frac{1}{n}\sum_{i=1}^{n}(X_{b,j} + X_{w,ij}) = X_{b,j} + \frac{1}{n}\sum_{i=1}^{n} X_{w,ij} \tag{30}$$

$$\overline{Y}_{\bullet j} = \frac{1}{n}\sum_{i=1}^{n}(Y_{b,j} + Y_{w,ij}) = Y_{b,j} + \frac{1}{n}\sum_{i=1}^{n} Y_{w,ij} \tag{31}$$

Then, the overall means are:

$$\overline{X}_{\bullet\bullet} = \frac{1}{nJ}\sum_{j=1}^{J}\sum_{i=1}^{n}(X_{b,j} + X_{w,ij}) = \frac{1}{J}\sum_{j=1}^{J} X_{b,j} + \frac{1}{nJ}\sum_{j=1}^{J}\sum_{i=1}^{n} X_{w,ij} \tag{32}$$

$$\overline{Y}_{\bullet\bullet} = \frac{1}{nJ}\sum_{j=1}^{J}\sum_{i=1}^{n}(Y_{b,j} + Y_{w,ij}) = \frac{1}{J}\sum_{j=1}^{J} Y_{b,j} + \frac{1}{nJ}\sum_{j=1}^{J}\sum_{i=1}^{n} Y_{w,ij} \tag{33}$$

The sums of squared deviations of the group means from the overall mean (SSA) and of the individual values from the group means (SSD) for X are:

$$SSA = n\sum_{j=1}^{J}(\overline{X}_{\bullet j} - \overline{X}_{\bullet\bullet})^2 = n\sum_{j=1}^{J}\overline{X}_{\bullet j}^2 - nJ\overline{X}_{\bullet\bullet}^2 \tag{34}$$

$$SSD = \sum_{j=1}^{J}\sum_{i=1}^{n}(X_{ij} - \overline{X}_{\bullet j})^2 = \sum_{j=1}^{J}\sum_{i=1}^{n} X_{ij}^2 - n\sum_{j=1}^{J}\overline{X}_{\bullet j}^2 \tag{35}$$

The same equations hold for Y. And the cross products of Y and X are:

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$$SPA = n\sum_{j=1}^{J}(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet})(\overline{X}_{\bullet j} - \overline{X}_{\bullet\bullet}) = n\sum_{j=1}^{J}\overline{Y}_{\bullet j}\overline{X}_{\bullet j} - nJ\overline{Y}_{\bullet\bullet}\overline{X}_{\bullet\bullet} \quad (36)$$

$$SPD = \sum_{j=1}^{J}\sum_{i=1}^{n}(Y_{ij} - \overline{Y}_{\bullet j})(X_{ij} - \overline{X}_{\bullet j}) = \sum_{j=1}^{J}\sum_{i=1}^{n}Y_{ij}X_{ij} - n\sum_{j=1}^{J}\overline{Y}_{\bullet j}\overline{X}_{\bullet j} \quad (37)$$

Zitzmann, Lüdtke, et al. (2021) derived the relations between the sum of squared deviations of X and the within- and between-group variances as:

$$SSA = n(J-1)\hat{\tau}_X^2 - (J-1)\hat{\sigma}_X^2 \quad (38)$$

$$SSD = (n-1)J\hat{\sigma}_X^2 \quad (39)$$

Combining Equations 38 and 39 with Equations 34 and 35, we yield an estimate of the group-level variance of X:

$$\hat{\tau}_X^2 = -\frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n}X_{ij}^2 + \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\overline{X}_{\bullet j}^2 - \frac{J}{J-1}\overline{X}_{\bullet\bullet}^2 \quad (40)$$

Note that this estimator may not be optimal, because estimates may not be positive. To address this issue, Chung et al. (2013) introduced a maximum penalized likelihood (MPL) approach for the estimating this parameter. This method mitigates the problem of boundary estimates, specifically preventing the occurrence of negative estimated group-level variances. In our approach we used the estimator from Equation 40, due to the transformation in the further steps and no anomalies found during the extensive simulations.

Zitzmann, Lüdtke, et al. (2021) also derived how the sum of squared deviations of cross products of X and Y can be expressed in terms of their within- and between-group covariances:

$$SPA = n(J-1)\hat{\tau}_{YX} + (J-1)\hat{\sigma}_{YX} \quad (41)$$

$$SPD = (n-1)J\hat{\sigma}_{YX} \quad (42)$$

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This means that the estimator for the group-level covariance $\hat{\tau}_{YX}$ can be obtained from Equations 36, 37, 41 and 42 as:

$$\hat{\tau}_{YX} = -\frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} Y_{ij}X_{ij} + \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\overline{Y}_{\bullet j}\overline{X}_{\bullet j} - \frac{J}{J-1}\overline{Y}_{\bullet\bullet}\overline{X}_{\bullet\bullet} \quad (43)$$

So far, we have derived both the numerator and the denominator of the ML estimator and, partly, of Bayesian estimator in Equation 7. But how can we use these derivations? Our aim is to minimize the MSE of the Bayesian estimator, and to do this, we need to know the mean and the variance of the estimator. One way to find them is to compute the estimator's distribution.

We begin with the derivation of the distributions of group-level variance of X and the group-level covariance between X and Y. To this end, two new variables are defined. The Z_X merges all the elements of predictor sample together with its means into one vector of length $(nJ+J+1)$, and Z_Y combines all the elements of the dependent variable and its means:

$$Z_X = \left(X_{11}, \dots X_{n1}, X_{12}, \dots X_{nJ}, \overline{X}_{\bullet 1}, \dots \overline{X}_{\bullet J}, \overline{X}_{\bullet\bullet}\right)' \quad (44)$$

$$Z_Y = \left(Y_{11}, \dots Y_{n1}, Y_{12}, \dots Y_{nJ}, \overline{Y}_{\bullet 1}, \dots \overline{Y}_{\bullet J}, \overline{Y}_{\bullet\bullet}\right)' \quad (45)$$

Using these newly defined variables, we can rewrite the estimators for the group-level variance and the covariance $\hat{\tau}_{YX}$ in matrix form:

$$\hat{\tau}^2_X = Z'_X A Z_X \quad (46)$$

$$\hat{\tau}_{YX} = Z'_X A Z_Y \quad (47)$$

With the same coefficient matrix A for both defined in Equation 102 of Appendix F. Note that matrix A is diagonal.

Thus, $\hat{\tau}_{YX}$ and $\hat{\tau}^2_X$ are quadratic forms of the sample elements and their means. If the equations consist only of second order terms of normally distributed random variables, we

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can interpret $\hat{\tau}_{YX}$ and $\hat{\tau}_X^2$ as the weighed sums of χ^2, and thus gamma-distributed random variables. However, the distribution of such a quadratic form is highly complicated in the general case. Therefore, we apply a transformation to yield weighted sum of squares (without interaction terms) of iid normal random variables.

Firstly, we compute the distribution of Z_X and Z_Y, using the previously made assumptions about X and Y:

$$Z_X \sim N(\mathbb{1}_{nJ+J+1} * \mu_X, \Sigma_X) \tag{48}$$

$$Z_Y \sim N(\mathbb{1}_{nJ+J+1} * \mu_Y, \Sigma_Y) \tag{49}$$

Where $\mathbb{1}_{nJ+J+1}$ is a vector of ones of size $(nJ+J+1)$. Also, note the following important facts:

- each element of Z_X and Z_Y has the same mean
- the sum of coefficients defined by matrix A in Equation 102 of Appendix F is zero

As a result, when we demean Equations 46 and 47, these means sum up to zero. To demonstrate it, define Z_X^* and Z_Y^* and all their elements as the demeaned counterparts of Z_X and Z_Y, respectively:

$$\begin{aligned} Z_X &= Z_X^* + \mathbb{1}_{nJ+J+1} * \mu_X \\ Z_Y &= Z_Y^* + \mathbb{1}_{nJ+J+1} * \mu_Y \end{aligned} \tag{50}$$

Show that $Z_X^{*\prime} * A * \mathbb{1}_{nJ+J+1}$ and $Z_Y^{*\prime} * A * \mathbb{1}_{nJ+J+1}$ are both zeros:

$$\begin{gathered} Z_X^{*\prime} * A * \mathbb{1}_{nJ+J+1} = -\frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} X_{ij}^* + \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\overline{X}_{\bullet j}^* - \\ \frac{J}{J-1}\overline{X}_{\bullet\bullet}^* = \sum_{j=1}^{J}\sum_{i=1}^{n} X_{ij}^*\left(-\frac{1}{n(n-1)J} + \frac{nJ-1}{n(n-1)(J-1)J} - \frac{J}{nJ(J-1)}\right) = \\ \sum_{j=1}^{J}\sum_{i=1}^{n} X_{ij}^* \frac{-J+1+nJ-1-nJ+J}{nJ(n-1)(J-1)} = 0 \end{gathered} \tag{51}$$

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$$
\begin{aligned}
Z_Y^{*\prime} * A * \mathbb{1}_{nJ+J+1} = -\frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} Y_{ij}^* + \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\overline{Y}_{\bullet j}^* - \\
\frac{J}{J-1}\overline{Y}_{\bullet\bullet}^* = \sum_{j=1}^{J}\sum_{i=1}^{n} Y_{ij}^*\left(-\frac{1}{n(n-1)J} + \frac{nJ-1}{n(n-1)(J-1)J} - \frac{J}{nJ(J-1)}\right) = 0
\end{aligned} \tag{52}
$$

Plug the expressions from Equation 50 into the Equations 46 and 47, and remind that the sum of coefficients of matrix A is zero:

$$
\begin{gathered}
\hat{\tau}_X^2 = Z_X' A Z_X = (Z_X^* + \mathbb{1}_{nJ+J+1} * \mu_X)' A (Z_X^* + \mathbb{1}_{nJ+J+1} * \mu_X) = Z_X^{*\prime} A Z_X^* + \\
\underbrace{Z_X^{*\prime} A * \mathbb{1}_{nJ+J+1}}_{=0} * \mu_X + \mu_X * \underbrace{\mathbb{1}_{nJ+J+1}' A Z_X^*}_{=0} + \mu_X * \underbrace{\mathbb{1}_{nJ+J+1}' A \mathbb{1}_{nJ+J+1}}_{=0} * \mu_X \rightarrow \\
\hat{\tau}_X^2 = Z_X^{*\prime} A Z_X^*
\end{gathered} \tag{53}
$$

$$
\begin{gathered}
\hat{\tau}_{YX} = Z_X' A Z_Y = (Z_X^* + \mathbb{1}_{nJ+J+1} * \mu_X)' A (Z_Y^* + \mathbb{1}_{nJ+J+1} * \mu_Y) = Z_X^{*\prime} A Z_Y^* + \\
\underbrace{Z_X^{*\prime} A * \mathbb{1}_{nJ+J+1}}_{=0} * \mu_Y + \mu_X * \underbrace{\mathbb{1}_{nJ+J+1}' A Z_Y^*}_{=0} + \mu_X * \underbrace{\mathbb{1}_{nJ+J+1}' A \mathbb{1}_{nJ+J+1}}_{=0} * \mu_Y \rightarrow \\
\hat{\tau}_{YX} = Z_X^{*\prime} A Z_Y^*
\end{gathered} \tag{54}
$$

Hence it is irrelevant for $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ whether Z_X and Z_Y have non-zero means or not, they always cancel out. So, we do not loose generality by assuming $\mu_X = 0$ and $\mu_Y = 0$.

Σ_X and Σ_Y are defined in the Equations 104 and 105 of Appendix F. These matrices are symmetric and positive semi-definite as covariance matrices. Therefore, their square roots will have only real entries (Horn & Johnson, 2013). Using the matrices, we can transform $\hat{\tau}_X^2$ to:

$$
\hat{\tau}_X^2 = Z_X' A Z_X = Z_X' \Sigma_X^{-1/2} \Sigma_X^{1/2} A \Sigma_X^{1/2} \Sigma_X^{-1/2} Z_X = W_X' \Sigma_X^{1/2} A \Sigma_X^{1/2} W_X \tag{55}
$$

Where $W_X = \Sigma_X^{-1/2} Z_X \sim N(0, \mathbf{I}_{nJ+J+1})$ follows the standard (multivariate) normal distribution, which has the identity matrix $\mathbf{I}$ as the covariance matrix. Following the rationale

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that led to Equation (55), we define a square root of the covariance matrix Σ_X by using its spectral decomposition as:

$$\Sigma_X = V_X D_X V_X' \tag{56}$$

Where V_X is a matrix of eigenvectors and it is orthogonal ($V_X' = V_X^{-1}$), because Σ_X is a real symmetric matrix by its nature (Horn & Johnson, 2013). Matrix D_X is a diagonal matrix of eigenvalues. These eigenvalues are non-negative, because Σ_X is positive-semidefinite (Horn & Johnson, 2013). Thus, we may denote the square root of D_X as S_X, which is just a diagonal matrix with real square roots of each element of D_X. This helps us to define the matrix $\Sigma_X^{1/2}$:

$$\Sigma_X^{1/2} = V_X S_X V_X' \tag{57}$$

Indeed, we have:

$$\Sigma_X^{1/2}\Sigma_X^{1/2} = V_X S_X \overbrace{V_X' V_X}^{=\mathbf{I}} S_X V_X' = V_X \overbrace{S_X S_X}^{=D_X} V_X' = V_X D_X V_X' = \Sigma_X \tag{58}$$

The eigenvalues of Σ_X are the following:

- $\lambda_i = 0$, $(J+1)$ eigenvalues
- $\lambda_i = \sigma_X^2$, $((n-1)J)$ eigenvalues
- $\lambda_i = (n+1)\left(\tau_X^2 + \frac{1}{n}\sigma_X^2\right)$, $(J-1)$ eigenvalues
- $\lambda_{nJ+J+1} = \frac{nJ+J+1}{J}\left(\tau_X^2 + \frac{1}{n}\sigma_X^2\right)$, 1 eigenvalue

D_X, a diagonal matrix, is composed of the eigenvalues in this order. Matrix $V_X = V$ is presented in Equation 103 of Appendix F. Due to its bulkiness, we provide V_X for the case $n = 3$ and $J = 4$, but it could be expanded upon demand.

We can now plug the decomposition of Σ_X into Equation 55 so that it becomes:

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$$\hat{\tau}_X^2 = W_X' \Sigma_X^{1/2} A \Sigma_X^{1/2} W_X = W_X' V_X S_X V_X' A V_X S_X V_X' W_X \tag{59}$$

$$\hat{\tau}_X^2 = H_X' S_X V_X' A V_X S_X H_X \tag{60}$$

where $H_X = V_X' W_X \sim N(0, V_X' \mathbf{I}_{nJ+J+1} V_X) = N(0, \mathbf{I}_{nJ+J+1})$. Thus, the orthogonality of matrix V_X kept the standard normal distribution of the new variable H_X. Since the internal right-hand side of Equation 60, $S_X V_X' A V_X S_X$, is diagonal, we indeed managed to represent τ_X^2 as a weighted sum of squares of independent normally distributed random variables, that is, a weighted sum of χ_1^2-distributed random variables.

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Appendix C

Similarly to the transformation of the group-level variance of X, which was introduced in Appendix B, we continue with the description of the transformation of the group-level covariance of X and Y as this is partially similar. We start from Equation 47 in Appendix B and use the previously defined covariance matrices Σ_X and Σ_Y (Equations 48 and 49 in Appendix B):

$$\hat{\tau}_{YX} = Z'_X A Z_Y = Z'_X \Sigma_X^{-1/2} \Sigma_X^{1/2} A \Sigma_Y^{1/2} \Sigma_Y^{-1/2} Z_Y = W'_X \Sigma_X^{1/2} A \Sigma_Y^{1/2} W_Y \tag{61}$$

where $W_Y = \Sigma_Y^{-1/2} Z_Y \sim N(0, \mathbf{I}_{nJ+J+1})$ is a new random vector that follows the multivariate standard normal distribution. For further transformation, we also introduce the spectral decomposition of covariance matrix Σ_Y and its square root as:

$$\Sigma_Y = V_Y D_Y V'_Y \tag{62}$$

$$\Sigma_Y^{1/2} = V_Y S_Y V'_Y \tag{63}$$

where V_Y is a matrix of eigenvectors of Σ_Y. It turns out to be equal to V_X, therefore sharing its property of orthogonality. We will further refer to them as $V = V_X = V_Y$ (see Equation 103 in Appendix F).

Matrix D_Y consists of (non-negative) eigenvalues of Σ_Y on the diagonal (because of the positive-semidefiniteness of Σ_Y). Its square root matrix, S_Y, is also diagonal, with non-negative square roots of eigenvalues on the main diagonal. We can compute the eigenvalues of Σ_Y in closed-form and thus define matrix D_Y by:

- $\lambda_i = 0$, $(J+1)$ eigenvalues
- $\lambda_i = \sigma_Y^2$, $((n-1)J)$ eigenvalues
- $\lambda_i = (n+1)\left(\tau_Y^2 + \frac{1}{n}\sigma_Y^2\right)$, $(J-1)$ eigenvalues
- $\lambda_{nJ+J+1} = \frac{nJ+J+1}{J}\left(\tau_Y^2 + \frac{1}{n}\sigma_Y^2\right)$, 1 eigenvalue

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For the next step we plug in the decompositions Equation (57) of Appendix B and Equation (63) into the Equation (61) and obtain:

$$\hat{\tau}_{YX} = W_X' \Sigma_X^{1/2} A \Sigma_Y^{1/2} W_Y = W_X' V S_X V' A V S_Y V' W_Y \tag{64}$$

$$\hat{\tau}_{YX} = H_X' S_X V' A V S_Y H_Y \tag{65}$$

where $H_Y = V'W_Y \sim N(0, V'\mathbf{I}_{nJ+J+1}V) = N(0, \mathbf{I}_{nJ+J+1})$. Thus, the distribution of the new variable H_Y is standard normal because of the orthogonality of the matrix V. Additionally, the inner right-hand side of Equation 65, $S_X V' A V S_Y$, is diagonal due to its construction. Comparing Equations 60 and 65, one might be inclined to see the distinct similarities and the claim to also represent τ_{YX} as a weighted sum of squares of independent normally distributed random variables. However, this is not true. H_X and H_Y are different random vectors, and thus, we continue the transformation by defining a new aggregated variable:

$$H = \begin{pmatrix} H_X \\ H_Y \end{pmatrix} \tag{66}$$

with the distribution of H being $N(0, \Sigma_H)$. Its covariance matrix Σ_H is defined as follows:

$$\Sigma_H = \begin{pmatrix} Var(H_X) & Cov(H_X, H_Y) \\ Cov(H_X, H_Y) & Var(H_Y) \end{pmatrix} \tag{67}$$

We already showed that $Var(H_X) = \mathbf{I}_{nJ+J+1}$ and $Var(H_Y) = \mathbf{I}_{nJ+J+1}$ as well. Before calculation of $Cov(H_X, H_Y)$, we additionally define Σ_{YX} in Equation 106 of Appendix F in a manner similar to Equations (104) and (105). Then the spectral decomposition of Σ_{YX} become:

$$\Sigma_{YX} = V D_{YX} V' \tag{68}$$

$$\Sigma_{YX}^{1/2} = V S_{YX} V' \tag{69}$$

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where matrix V is the same as in decompositions of Σ_X in Equation 56 from Appendix B and Σ_Y in Equation 62. Matrix D_{YX} is diagonal with non-negative eigenvalues of positive-semidefinite matrix Σ_{YX} (Horn & Johnson, 2013). Thus, the square root matrix, S_{YX}, is diagonal with non-negative square roots of eigenvalues on the main diagonal. The eigenvalues of Σ_{YX} that define matrix D_{YX} are in the closed-form:

- $\lambda_i = 0$, $(J+1)$ eigenvalues
- $\lambda_i = \sigma_{YX}$, $((n-1)J)$ eigenvalues
- $\lambda_i = (n+1)\left(\tau_{YX} + \frac{1}{n}\sigma_{YX}\right)$, $(J-1)$ eigenvalues
- $\lambda_{nJ+J+1} = \frac{nJ+J+1}{J}\left(\tau_{YX} + \frac{1}{n}\sigma_{YX}\right)$, 1 eigenvalue

Next, we use the generalized inverses of matrices S_X and S_Y, as described by Penrose (1955), since they include zero eigenvalues and are not invertible. These matrices are denoted as S_X^+ and S_Y^+ and include the inverse of diagonal elements that are invertible and zeros otherwise.

Using all this, the covariance $Cov(H_X, H_Y)$ is computed as:

$$Cov(H_X, H_Y) = Cov(V'W_X, V'W_Y) = V'Cov(W_X, W_Y)V =$$
$$V'Cov(\Sigma_X^{-1/2}Z_X, \Sigma_Y^{-1/2}Z_Y)V = V'\Sigma_X^{-1/2}\underbrace{Cov(Z_X, Z_Y)}_{\Sigma_{YX}}\Sigma_Y^{-1/2}V =$$
$$V'\Sigma_X^{-1/2}\Sigma_{YX}\Sigma_Y^{-1/2}V = V'VS_X^+V'VD_{YX}V'VS_Y^+V'V \rightarrow$$

$$Cov(H_X, H_Y) = S_X^+ D_{YX} S_Y^+ \tag{70}$$

This result is used to fully define the covariance matrix of H:

$$\Sigma_H = \begin{pmatrix} \mathbf{I} & S_X^+ D_{YX} S_Y^+ \\ S_X^+ D_{YX} S_Y^+ & \mathbf{I} \end{pmatrix} \tag{71}$$

and its spectral decomposition:

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$$\Sigma_H = V_H D_H V_H' \tag{72}$$

where the closed-form solutions for both the matrix of eigenvalues D_H and the orthogonal matrix of eigenvectors V_H. D_H is:

$$D_H = \begin{pmatrix} \mathbf{I} + S_X^+ D_{YX} S_Y^+ & 0 \\ 0 & \mathbf{I} - S_X^+ D_{YX} S_Y^+ \end{pmatrix} \tag{73}$$

Matrix V_H is defined in Equation 107 of Appendix F. Both matrices follow the same properties as their predecessor: D_H is diagonal with non-negative eigenvalues, and V_H is orthogonal.

After exposing the new composite vector H and its covariance matrix Σ_H, we can rewrite Equation 65 as:

$$\hat{\tau}_{YX} = H'QH \tag{74}$$

with coefficient matrix Q defined as:

$$Q = \begin{pmatrix} 0 & \frac{1}{2} S_X V' A V S_Y \\ \frac{1}{2} S_X V' A V S_Y & 0 \end{pmatrix} \tag{75}$$

Note that Q is designed to keep the symmetry of Equation 74. Including the square root of the covariance matrix leads to:

$$\hat{\tau}_{YX} = H'QH = H'\Sigma_H^{-1/2}\Sigma_H^{1/2} Q \Sigma_H^{1/2}\Sigma_H^{-1/2} H = H_1'\Sigma_H^{1/2} Q \Sigma_H^{1/2} H_1 \tag{76}$$

where $H_1 = \Sigma_H^{-1/2} H \sim N(0, \mathbf{I}_{2(nJ+J+1)})$ is a vector of independent normally distributed variables. Using the decomposition of Σ_H from Equation 72, denoting a square root of D_H as S_H, and plugging both of terms into in Equation 76 yields:

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$$\hat{\tau}_{YX} = H_1'\Sigma_H^{1/2}Q\Sigma_H^{1/2}H_1 = H_1'V_HS_HV_H'QV_HS_HV_H'H_1 \quad (77)$$

$$\hat{\tau}_{YX} = H_2'S_HV_H'QV_HS_HH_2 \quad (78)$$

with $H_2 = V_H'H_1 \sim N(0, \mathbf{I}_{2(nJ+J+1)})$ - a multivariate standard normally distributed random vector, as V_H is orthogonal. Furthermore, since matrix $S_HV_H'QV_HS_H$ is diagonal, the estimator of the group-level covariance $\hat{\tau}_{YX}$ is now represented as a weighted sum of squares of independent normally distributed random variables, that is, a weighted sum of χ_1^2-distributed random variables. Thus, at this point we achieved our aim of transforming $\hat{\tau}_{YX}$.

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Appendix D

Here, we derive the distributions of the ML and the Bayesian estimator. To this end, we start by calculating the distributions of sample group-level covariances $\hat{\tau_X}^2$ and $\hat{\tau}_{YX}$ in Equations 10 and 11, respectively. According to Welch (1947) and Satterthwaite (1946), we can approximate these sums as generic Gamma distribution with parameters:

$$k_{sum} = \frac{\left(\sum_i \theta_i k_i\right)^2}{\sum_i \theta_i^2 k_i} \tag{79}$$

$$\theta_{sum} = \frac{\sum_i \theta_i k_i}{k_{sum}} \tag{80}$$

Notice that each element in the sums $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ is scaled. The scales are defined by diagonal matrices $S_X V_X' A V_X S_X$ (for $\hat{\tau}_X^2$) and $S_H V_H' Q V_H S_H$ (for $\hat{\tau}_{YX}$). Let us denote their diagonal elements as $\theta_{X,i}$ and $\theta_{YX,i}$ respectively. Then, we can express the distributions of $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ as:

$$\hat{\tau}_X^2 \sim \text{Gamma}(k_{sum1}, \theta_{sum1}) \tag{81}$$

$$k_{sum1} = \frac{\left(\sum_i \theta_{X,i}\right)^2}{2\sum_i \theta_{X,i}^2}, \theta_{sum1} = \frac{\sum_i \theta_{X,i}^2}{\sum_i \theta_{X,i}}$$

$$\hat{\tau}_{YX} \sim \text{Gamma}(k_{sum2}, \theta_{sum2}) \tag{82}$$

$$k_{sum2} = \frac{\left(\sum_i \theta_{YX,i}\right)^2}{2\sum_i \theta_{YX,i}^2}, \theta_{sum2} = \frac{\sum_i \theta_{YX,i}^2}{\sum_i \theta_{YX,i}}$$

Using these distributions, we can find the distribution of the ML estimator. It is well known that the ratio of two independent gamma-distributed random variables follows F distribution. The independence of $\hat{\tau}_X^2$ and $\hat{\tau}_{YX}$ is not directly clear, but it follows from the approximation of the sum of Gamma-distributions. Therefore, the ML estimator's distribution is:

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$$\frac{k_{sum1}\theta_{sum1}}{k_{sum2}\theta_{sum2}}\hat{\beta}_b \sim F(2k_{sum2}, 2k_{sum1}) \quad (83)$$

Next, we derive the distribution of Bayesian estimator. Since it includes the two parameters τ_0^2 and ω, we need to adjust the process of derivation and find the distribution of denominator first.

The denominator is $(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2$ and consists of a stochastic part $\omega\hat{\tau}_X^2$ and deterministic part $(1-\omega)\tau_0^2$. To sum them up, we replace the deterministic part with the sequence of random variables t_m, that converges (in probability) to this deterministic part:

$$t_m \sim \text{Gamma}\left(m\tau_0^2, \frac{1}{m}\right) \quad (84)$$

Further we substitute τ_0^2 with t_m and yield a sum of gamma-distributed random variables. Using once more the approach from Welch (1947) and Satterthwaite (1946), we compute a sum as a new sequence of random variables that follows a Gamma distribution with parameters $k_{B,m}$ and $\theta_{B,m}$:

$$k_{B,m} = \frac{(\omega\theta_{sum1}k_{sum1} + (1-\omega)\tau_0^2)^2}{\omega^2\theta_{sum1}^2 k_{sum1} + \frac{(1-\omega)^2}{m^2} m\tau_0^2} \quad (85)$$

$$\theta_{B,m} = \frac{\omega\theta_{sum1}k_{sum1} + (1-\omega)\tau_0^2}{k_{B,m}} \quad (86)$$

The limit is the Gamma(k_B, θ_B) distribution with parameters:

$$k_B = \lim_{m\to\infty} k_{B,m} = \frac{(\omega\theta_{sum1}k_{sum1} + (1-\omega)\tau_0^2)^2}{\omega^2\theta_{sum1}^2 k_{sum1}} \quad (87)$$

$$\theta_B = \lim_{m\to\infty} \theta_{B,m} = \frac{\omega^2\theta_{sum1}^2 k_{sum1}}{\omega\theta_{sum1}k_{sum1} + (1-\omega)\tau_0^2} \quad (88)$$

Using the derived distribution of the denominator, similarly to the ML estimator, we yield the total distribution of the Bayesian estimator:

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$$\frac{k_B \theta_B}{k_{sum2} \theta_{sum2}} \tilde{\beta}_b \sim F(2k_{sum2}, 2k_B) \tag{89}$$

After computing the distributions of the ML estimator (Equation 83) and the Bayesian estimator (Equation 89), we use them to calculate biases and variances of the estimators and thus their MSEs as:

$$MSE(\hat{\beta}_b) = \frac{k_{sum2}\theta_{sum2}^2(k_{sum1} + k_{sum2} - 1)}{\theta_{sum1}^2(k_{sum1} - 1)^2(k_{sum1} - 2)} + \left(\frac{k_{sum2}\theta_{sum2}}{(k_{sum1} - 1)\theta_{sum1}} - \beta_b\right)^2 \tag{90}$$

$$MSE(\tilde{\beta}_b) = \frac{k_{sum2}\theta_{sum2}^2(k_B + k_{sum2} - 1)}{\theta_B^2(k_B - 1)^2(k_B - 2)} + \left(\frac{k_{sum2}\theta_{sum2}}{(k_B - 1)\theta_B} - \beta_b\right)^2 \tag{91}$$

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Appendix E: Estimation Algorithm

Finally, we introduce a novel and practical algorithm based on the theoretical investigations made in the main part of the paper. This algorithm aims to provide an efficient and effective solution for computing the regularized Bayesian estimator:

1. Input data: n, J, X_{ij} and Y_{ij}
2. Define matrix A from Equation 102 of Appendix F
3. Calculate the (manifest) group means: $\overline{X}_{\bullet j}$ of X from Equation 30 in Appendix B and $\overline{Y}_{\bullet j}$ of Y from Equation 31 in Appendix B
4. Calculate the overall means: $\overline{X}_{\bullet\bullet}$ of X from Equation 32 in Appendix B and $\overline{Y}_{\bullet\bullet}$ of Y from Equation 33 in Appendix B
5. Compute $\hat{\tau}_X^2$ from Equation 40 in Appendix B and $\hat{\tau}_{YX}^2$ from Equation 43 in Appendix B as well as:

$$\hat{\tau}_Y^2 = -\frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} Y_{ij}^2 + \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\overline{Y}_{\bullet j}^2 - \frac{J}{J-1}\overline{Y}_{\bullet\bullet}^2 \tag{92}$$

$$\hat{\sigma}_X^2 = \frac{1}{(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} X_{ij}^2 - \frac{n}{(n-1)J}\sum_{j=1}^{J}\overline{X}_{\bullet j}^2 \tag{93}$$

$$\hat{\sigma}_{YX} = \frac{1}{(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} X_{ij}Y_{ij} - \frac{n}{(n-1)J}\sum_{j=1}^{J}\overline{X}_{\bullet j}\overline{Y}_{\bullet j} \tag{94}$$

$$\hat{\sigma}_Y^2 = \frac{1}{(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n} Y_{ij}^2 - \frac{n}{(n-1)J}\sum_{j=1}^{J}\overline{Y}_{\bullet j}^2 \tag{95}$$

6. Find the ML estimator $\hat{\beta}_b$ from Equation 6
7. Compute diagonal matrices of eigenvalues D_X (page 44), D_Y (page 46), D_{YX} (page 48) and matrix of eigenvectors V from Equation 103 of Appendix F
8. Calculate the square root matrices $S_X = \sqrt{D_X}$ and $S_Y = \sqrt{D_Y}$
9. Compute the diagonal matrix of coefficients $L_1 = S_X V' A V S_X$
10. Calculate matrix Q from Equation 75 in Appendix C
11. Compute the diagonal matrix of eigenvalues D_H from Equation 73 of Appendix C and eigenvectors matrix V_H from Equation 107 of Appendix F

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12. Calculate the square root matrix $S_H = \sqrt{D_H}$
13. Compute the diagonal matrix of coefficients $L_2 = S_H V_H' Q V_H S_H$
14. Compute the coefficients k_{sum1}, θ_{sum1}, k_{sum2} and θ_{sum2} (note that $\mathbb{1}$ is a vector of ones):

$$k_{sum1} = \frac{\left(\mathbb{1}'_{nJ+J+1} L_1\right)^2}{2L_1' L_1} \tag{96}$$

$$\theta_{sum1} = \frac{L_1' L_1}{\mathbb{1}'_{nJ+J+1} L_1} \tag{97}$$

$$k_{sum2} = \frac{\left(\mathbb{1}'_{2(nJ+J+1)} L_2\right)^2}{2L_2' L_2} \tag{98}$$

$$\theta_{sum2} = \frac{L_2' L_2}{\mathbb{1}'_{2(nJ+J+1)} L_2} \tag{99}$$

15. Define vectors W and T_{02}, with the values of ω and τ_0^2 that specify grid search region. For example, W goes from 0 to 1 by steps of 0.01, and T_{02} goes from 0.1 to 10 by steps of 0.1
16. Compute the MSE for each value of W and T_{02}, whereby β_b should be substituted with $\hat{\beta}_b$. The final formula is delineated as:

$$\begin{aligned} MSE_{ML}(i,j) = &\left\{k_{sum2}\theta^2_{sum2}(k_{sum2}+1)\left((1-W(i))T_{02}(j) + W(i)\mathbb{1}'_{nJ+J+1}L_1\right)\right\} \\ &/\left\{\left(((1-W(i))T_{02}(j) + W(i)\mathbb{1}'_{nJ+J+1}L_1)^2 - 2W(i)^2(L1'L1)\right) * \right. \\ &\left.\left(((1-W(i))T_{02}(j) + W(i)\mathbb{1}'_{nJ+J+1}L_1)^2 - 4W(i)^2(L1'L1)\right)\right\} - \\ &\frac{2\hat{\beta}_b k_{sum2}\theta_{sum2}\left((1-W(i))T_{02}(j) + W(i)\mathbb{1}'_{nJ+J+1}L_1\right)}{\left(((1-W(i))T_{02}(j) + W(i)\mathbb{1}'_{nJ+J+1}L_1)^2 - W(i)^2 \cdot (L1'L1)\right)} + \hat{\beta}_b \end{aligned} \tag{100}$$

17. Find the minimum MSE and indexes i^* and j^* that provide this minimum
18. Define the optimal parameters $\omega^* = W(i^*)$ and $\tau_0^{2*} = T_{02}(j^*)$
19. Compute the optimally regularized Bayesian estimator as:

$$\tilde{\beta}_b = \frac{\hat{\tau}_{YX}}{(1-\omega^*)\tau_0^{2*} + \omega^*\hat{\tau}_X^2} \tag{101}$$

■

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Appendix F: Matrices

$$A = \begin{pmatrix} -\frac{1}{n(n-1)J} & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & -\frac{1}{n(n-1)J} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \frac{nJ-1}{(n-1)(J-1)J} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & \frac{nJ-1}{(n-1)(J-1)J} & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & -\frac{J}{J-1} \end{pmatrix} \tag{102}$$

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$$
V = \left(\begin{matrix}
-\frac{1}{\sqrt{n\,(n+1)}} & 0 & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & \dots \\
-\frac{1}{\sqrt{n\,(n+1)}} & 0 & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & \dots \\
0 & -\frac{1}{\sqrt{n\,(n+1)}} & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & -\frac{\sqrt{2}}{2} & \dots \\
0 & -\frac{1}{\sqrt{n\,(n+1)}} & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & \frac{\sqrt{2}}{2} & \dots \\
0 & -\frac{1}{\sqrt{n\,(n+1)}} & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & -\frac{1}{\sqrt{n\,(n+1)}} & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & -\frac{1}{\sqrt{n\,(n+1)}} & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & -\frac{1}{\sqrt{n\,(n+1)}} & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & -\frac{1}{\sqrt{n\,(n+1)}} & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & -\frac{1}{\sqrt{n\,(n+1)}} & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & -\frac{1}{\sqrt{n\,(n+1)}} & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
\sqrt{\frac{n}{n+1}} & 0 & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & \sqrt{\frac{n}{n+1}} & 0 & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & \sqrt{\frac{n}{n+1}} & 0 & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & \sqrt{\frac{n}{n+1}} & -\frac{1}{\sqrt{(J+J\,n)\,(J+J\,n+1)}} & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & \sqrt{\frac{J+J\,n}{J+J\,n+1}} & 0 & 0 & \dots &
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} & \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
-\frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
-\frac{\sqrt{6}}{6} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
\frac{\sqrt{2}\,\sqrt{3}}{3} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 & \frac{\sqrt{6}}{3\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 & \frac{\sqrt{6}}{3\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & \frac{\sqrt{2}\,\sqrt{3}}{3} & 0 & 0 & 0 & \frac{\sqrt{6}}{3\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & 0 & \frac{\sqrt{3}}{2\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & 0 & 0 & \frac{\sqrt{3}}{2\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & \frac{\sqrt{2}\,\sqrt{3}}{3} & 0 & 0 & \frac{\sqrt{3}}{2\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2\,\sqrt{n+1}} & -\frac{\sqrt{6}}{6\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{3\,\sqrt{n+1}} & -\frac{\sqrt{3}}{6\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2\,\sqrt{n+1}} & \frac{1}{\sqrt{J+J\,n+1}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{J+J\,n+1}}
\end{matrix}\right) \qquad (103)
$$

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$$\Sigma_X =$$

$$\begin{pmatrix}
\tau_X^2+\sigma_X^2 & \tau_X^2 & 0 & \dots & 0 & 0 & \tau_X^2+\frac{1}{n}\sigma_X^2 & 0 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
\tau_X^2 & \tau_X^2+\sigma_X^2 & 0 & \dots & 0 & 0 & \tau_X^2+\frac{1}{n}\sigma_X^2 & 0 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
0 & 0 & \tau_X^2+\sigma_X^2 & \dots & \tau_X^2 & 0 & 0 & 0 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
0 & 0 & \tau_X^2 & \dots & \tau_X^2+\sigma_X^2 & 0 & 0 & 0 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & \tau_X^2+\sigma_X^2 & 0 & \tau_X^2+\frac{1}{n}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
\tau_X^2+\frac{1}{n}\sigma_X^2 & \tau_X^2+\frac{1}{n}\sigma_X^2 & 0 & \dots & 0 & 0 & \tau_X^2+\frac{1}{n}\sigma_X^2 & 0 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & \tau_X^2+\frac{1}{n}\sigma_X^2 & 0 & \tau_X^2+\frac{1}{n}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \\
\frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \cdots & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 & \frac{1}{J}\tau_X^2+\frac{1}{nJ}\sigma_X^2 &
\end{pmatrix} \tag{104}$$

$$\Sigma_Y =$$

$$\begin{pmatrix}
\tau_Y^2+\sigma_Y^2 & \tau_Y^2 & 0 & \dots & 0 & 0 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & 0 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
\tau_Y^2 & \tau_Y^2+\sigma_Y^2 & 0 & \dots & 0 & 0 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & 0 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
0 & 0 & \tau_Y^2+\sigma_Y^2 & \dots & \tau_Y^2 & 0 & 0 & 0 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
0 & 0 & \tau_Y^2 & \dots & \tau_Y^2+\sigma_Y^2 & 0 & 0 & 0 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & \tau_Y^2+\sigma_Y^2 & 0 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
\tau_Y^2+\frac{1}{n}\sigma_Y^2 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & 0 & \dots & 0 & 0 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & 0 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & 0 & \tau_Y^2+\frac{1}{n}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \\
\frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \cdots & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 & \frac{1}{J}\tau_Y^2+\frac{1}{nJ}\sigma_Y^2 &
\end{pmatrix} \tag{105}$$

$$\Sigma_{YX} =$$

$$\begin{pmatrix}
\tau_{YX}+\sigma_{YX} & \tau_{YX} & 0 & \dots & 0 & 0 & \tau_{YX}+\frac{1}{n}\sigma_{YX} & 0 & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
\tau_{YX} & \tau_{YX}+\sigma_{YX} & 0 & \dots & 0 & 0 & \tau_{YX}+\frac{1}{n}\sigma_{YX} & 0 & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
0 & 0 & \tau_{YX}+\sigma_{YX} & \dots & \tau_{YX} & 0 & 0 & 0 & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
0 & 0 & \tau_{YX} & \dots & \tau_{YX}+\sigma_{YX} & 0 & 0 & 0 & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & \tau_{YX}+\sigma_{YX} & 0 & \tau_{YX}+\frac{1}{n}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
\tau_{YX}+\frac{1}{n}\sigma_{YX} & \tau_{YX}+\frac{1}{n}\sigma_{YX} & 0 & \dots & 0 & 0 & \tau_{YX}+\frac{1}{n}\sigma_{YX} & 0 & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & \tau_{YX}+\frac{1}{n}\sigma_{YX} & 0 & \tau_{YX}+\frac{1}{n}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \\
\frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \cdots & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} & \frac{1}{J}\tau_{YX}+\frac{1}{nJ}\sigma_{YX} &
\end{pmatrix} \tag{106}$$

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$$V_H = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix} \tag{107}$$

Appendix G: Tables

Table 2

RMSE values of the ML ($RMSE_{ML}$) and the Bayesian estimators ($RMSE_{Bay}$ represents the Bayesian with β_b and $RMSE_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.05$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$	n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$
5	5	0.2	0.2	57.587	2.725	2.667	15	20	0.5	0.7	6.992	0.913	0.913
5	5	0.2	0.5	529.69	3.065	3.036	15	20	0.6	0.2	24.893	0.868	0.872
5	5	0.2	0.7	93.937	3.475	3.462	15	20	0.6	0.5	31.363	0.863	0.864
5	5	0.5	0.2	29.285	2.644	2.613	15	20	0.6	0.7	271.245	0.883	0.883
5	5	0.5	0.5	32.474	3.07	3.043	15	30	0.2	0.2	24.704	0.67	0.676
5	5	0.5	0.7	108.22	3.458	3.426	15	30	0.2	0.5	14.967	0.702	0.707
5	5	0.6	0.2	37.614	2.79	2.755	15	30	0.2	0.7	66.872	0.769	0.775
5	5	0.6	0.5	38.324	3.036	3.002	15	30	0.5	0.2	85.611	0.679	0.681
5	5	0.6	0.7	246.624	3.4	3.393	15	30	0.5	0.5	17.079	0.703	0.704
5	10	0.2	0.2	103.585	1.868	1.862	15	30	0.5	0.7	18.208	0.749	0.748
5	10	0.2	0.5	18.639	2.121	2.133	15	30	0.6	0.2	15.823	0.744	0.743
5	10	0.2	0.7	68.983	2.348	2.346	15	30	0.6	0.5	11.853	0.72	0.719
5	10	0.5	0.2	21.904	1.881	1.872	15	30	0.6	0.7	9.818	0.741	0.739
5	10	0.5	0.5	172.495	2.051	2.039	15	40	0.2	0.2	6.608	0.516	0.525
5	10	0.5	0.7	85.472	2.383	2.392	15	40	0.2	0.5	3.685	0.546	0.551
5	10	0.6	0.2	65.174	1.911	1.894	15	40	0.2	0.7	18.378	0.585	0.593
5	10	0.6	0.5	19.356	2.129	2.124	15	40	0.5	0.2	15.753	0.61	0.611
5	10	0.6	0.7	553.141	2.315	2.313	15	40	0.5	0.5	5.633	0.607	0.605
5	20	0.2	0.2	32.3	1.37	1.364	15	40	0.5	0.7	25.081	0.606	0.603
5	20	0.2	0.5	186.452	1.486	1.491	15	40	0.6	0.2	9.669	0.652	0.648
5	20	0.2	0.7	31.417	1.633	1.652	15	40	0.6	0.5	6.565	0.61	0.607
5	20	0.5	0.2	528.303	1.302	1.313	15	40	0.6	0.7	13.398	0.632	0.629
5	20	0.5	0.5	15.767	1.38	1.376	30	5	0.2	0.2	346.81	1.549	1.554
5	20	0.5	0.7	81.714	1.614	1.612	30	5	0.2	0.5	697.82	1.646	1.649
5	20	0.6	0.2	84.956	1.347	1.347	30	5	0.2	0.7	841.537	1.734	1.732
5	20	0.6	0.5	22.968	1.379	1.378	30	5	0.5	0.2	44.781	1.552	1.554
5	20	0.6	0.7	70.052	1.57	1.58	30	5	0.5	0.5	41.708	1.557	1.56
5	30	0.2	0.2	25.439	1.087	1.098	30	5	0.5	0.7	116.407	1.712	1.718
5	30	0.2	0.5	39.795	1.142	1.14	30	5	0.6	0.2	89.971	1.519	1.523
5	30	0.2	0.7	157.449	1.337	1.343	30	5	0.6	0.5	51.606	1.591	1.593
5	30	0.5	0.2	17.714	1.107	1.113	30	5	0.6	0.7	111.256	1.604	1.611
5	30	0.5	0.5	65.436	1.175	1.169	30	10	0.2	0.2	125.74	1.067	1.072
5	30	0.5	0.7	20.967	1.249	1.248	30	10	0.2	0.5	44.729	1.087	1.091
5	30	0.6	0.2	112.352	1.104	1.109	30	10	0.2	0.7	28.094	1.133	1.136
5	30	0.6	0.5	24.527	1.181	1.183	30	10	0.5	0.2	11.672	1.057	1.061

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5	30	0.6	0.7	31.858	1.241	1.25	30	10	0.5	0.5	25.1	1.076	1.078
5	40	0.2	0.2	42.185	0.979	0.983	30	10	0.5	0.7	164.174	1.13	1.131
5	40	0.2	0.5	53.56	1.054	1.053	30	10	0.6	0.2	40.099	1.044	1.045
5	40	0.2	0.7	15.499	1.082	1.086	30	10	0.6	0.5	68.118	1.047	1.05
5	40	0.5	0.2	115.685	0.987	0.987	30	10	0.6	0.7	122.808	1.088	1.092
5	40	0.5	0.5	32.061	0.998	0.994	30	20	0.2	0.2	4.08	0.587	0.597
5	40	0.5	0.7	28.321	1.161	1.166	30	20	0.2	0.5	6.696	0.561	0.571
5	40	0.6	0.2	34.223	0.973	0.97	30	20	0.2	0.7	8.71	0.587	0.597
5	40	0.6	0.5	107.287	1.018	1.016	30	20	0.5	0.2	6.382	0.647	0.646
5	40	0.6	0.7	41.419	1.044	1.047	30	20	0.5	0.5	66.702	0.642	0.642
15	5	0.2	0.2	205.555	1.676	1.667	30	20	0.5	0.7	6.12	0.647	0.644
15	5	0.2	0.5	290.677	1.777	1.762	30	20	0.6	0.2	9.591	0.693	0.689
15	5	0.2	0.7	89.916	1.946	1.942	30	20	0.6	0.5	11.39	0.669	0.665
15	5	0.5	0.2	96.434	1.599	1.597	30	20	0.6	0.7	2.793	0.705	0.702
15	5	0.5	0.5	61.309	1.747	1.742	30	30	0.2	0.2	0.981	0.342	0.353
15	5	0.5	0.7	83.573	1.936	1.926	30	30	0.2	0.5	0.794	0.322	0.333
15	5	0.6	0.2	34.357	1.622	1.61	30	30	0.2	0.7	2.281	0.331	0.341
15	5	0.6	0.5	111.232	1.742	1.739	30	30	0.5	0.2	0.973	0.503	0.499
15	5	0.6	0.7	328.599	1.904	1.903	30	30	0.5	0.5	2.255	0.476	0.472
15	10	0.2	0.2	216.574	1.186	1.184	30	30	0.5	0.7	1.147	0.494	0.491
15	10	0.2	0.5	2961.914	1.25	1.249	30	30	0.6	0.2	0.815	0.558	0.55
15	10	0.2	0.7	93.279	1.271	1.278	30	30	0.6	0.5	0.745	0.541	0.533
15	10	0.5	0.2	30.459	1.195	1.195	30	30	0.6	0.7	1.736	0.564	0.558
15	10	0.5	0.5	120.55	1.208	1.209	30	40	0.2	0.2	0.621	0.231	0.24
15	10	0.5	0.7	19.802	1.288	1.291	30	40	0.2	0.5	3.259	0.241	0.252
15	10	0.6	0.2	135.805	1.178	1.181	30	40	0.2	0.7	0.651	0.261	0.272
15	10	0.6	0.5	40.038	1.221	1.226	30	40	0.5	0.2	0.708	0.443	0.435
15	10	0.6	0.7	35.817	1.265	1.268	30	40	0.5	0.5	0.572	0.441	0.435
15	20	0.2	0.2	24.678	0.878	0.883	30	40	0.5	0.7	1.291	0.432	0.427
15	20	0.2	0.5	166.166	0.875	0.878	30	40	0.6	0.2	1.296	0.522	0.514
15	20	0.2	0.7	19.525	0.936	0.939	30	40	0.6	0.5	0.731	0.509	0.502
15	20	0.5	0.2	43.648	0.899	0.903	30	40	0.6	0.7	0.475	0.508	0.501
15	20	0.5	0.5	62.277	0.879	0.878							

Table 3

RMSE values of the ML ($RMSE_{ML}$) and the Bayesian estimators ($RMSE_{Bay}$ represents the Bayesian with β_b and $RMSE_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.1$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$	n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$
5	5	0.2	0.2	33.935	2.436	2.383	15	20	0.5	0.7	2.45	0.511	0.51
5	5	0.2	0.5	612.83	2.858	2.853	15	20	0.6	0.2	24.405	0.578	0.578

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5	5	0.2	0.7	258.045	3.069	3.057	15	20	0.6	0.5	1.927	0.551	0.548
5	5	0.5	0.2	46.967	2.389	2.341	15	20	0.6	0.7	3.717	0.547	0.544
5	5	0.5	0.5	61.524	2.63	2.607	15	30	0.2	0.2	1.268	0.257	0.271
5	5	0.5	0.7	41.284	2.988	2.976	15	30	0.2	0.5	0.733	0.265	0.28
5	5	0.6	0.2	38.72	2.449	2.383	15	30	0.2	0.7	0.807	0.308	0.321
5	5	0.6	0.5	346.286	2.657	2.625	15	30	0.5	0.2	0.723	0.42	0.416
5	5	0.6	0.7	58.937	3.06	3.049	15	30	0.5	0.5	3.031	0.417	0.413
5	10	0.2	0.2	176.892	1.591	1.571	15	30	0.5	0.7	1.083	0.421	0.418
5	10	0.2	0.5	20.44	1.737	1.736	15	30	0.6	0.2	0.657	0.478	0.472
5	10	0.2	0.7	49.498	1.994	1.99	15	30	0.6	0.5	1.69	0.475	0.47
5	10	0.5	0.2	55.096	1.52	1.509	15	30	0.6	0.7	0.588	0.47	0.464
5	10	0.5	0.5	230.062	1.618	1.613	15	40	0.2	0.2	0.577	0.19	0.202
5	10	0.5	0.7	62.571	1.865	1.86	15	40	0.2	0.5	1.869	0.207	0.22
5	10	0.6	0.2	17.002	1.57	1.565	15	40	0.2	0.7	15.892	0.229	0.24
5	10	0.6	0.5	20.908	1.661	1.663	15	40	0.5	0.2	1.213	0.381	0.376
5	10	0.6	0.7	180.241	1.756	1.742	15	40	0.5	0.5	0.391	0.383	0.378
5	20	0.2	0.2	728.749	1.06	1.063	15	40	0.5	0.7	0.373	0.382	0.378
5	20	0.2	0.5	105.743	1.088	1.085	15	40	0.6	0.2	0.396	0.439	0.433
5	20	0.2	0.7	108.22	1.278	1.273	15	40	0.6	0.5	0.339	0.44	0.435
5	20	0.5	0.2	26.338	1.017	1.01	15	40	0.6	0.7	0.34	0.441	0.437
5	20	0.5	0.5	11.918	1.018	1.022	30	5	0.2	0.2	25.285	1.216	1.216
5	20	0.5	0.7	58.23	1.206	1.208	30	5	0.2	0.5	38.692	1.278	1.286
5	20	0.6	0.2	1378.614	1.001	1.005	30	5	0.2	0.7	135.292	1.247	1.248
5	20	0.6	0.5	39.003	1.061	1.057	30	5	0.5	0.2	20.224	1.135	1.136
5	20	0.6	0.7	123.476	1.104	1.11	30	5	0.5	0.5	40.565	1.21	1.212
5	30	0.2	0.2	52.669	0.793	0.801	30	5	0.5	0.7	46.002	1.159	1.163
5	30	0.2	0.5	20.106	0.836	0.832	30	5	0.6	0.2	124.16	1.127	1.129
5	30	0.2	0.7	14.304	0.926	0.928	30	5	0.6	0.5	12.079	1.129	1.13
5	30	0.5	0.2	11.626	0.769	0.763	30	5	0.6	0.7	46.667	1.198	1.201
5	30	0.5	0.5	18.425	0.789	0.786	30	10	0.2	0.2	4.575	0.61	0.621
5	30	0.5	0.7	33.711	0.792	0.8	30	10	0.2	0.5	4.977	0.628	0.64
5	30	0.6	0.2	14.777	0.789	0.793	30	10	0.2	0.7	29.187	0.651	0.664
5	30	0.6	0.5	14.068	0.82	0.819	30	10	0.5	0.2	2.935	0.629	0.63
5	30	0.6	0.7	50.97	0.858	0.854	30	10	0.5	0.5	12.047	0.665	0.665
5	40	0.2	0.2	13.05	0.616	0.625	30	10	0.5	0.7	6.835	0.68	0.681
5	40	0.2	0.5	6.66	0.655	0.661	30	10	0.6	0.2	3.711	0.684	0.684
5	40	0.2	0.7	322.906	0.757	0.759	30	10	0.6	0.5	10.482	0.679	0.676
5	40	0.5	0.2	12.974	0.642	0.642	30	10	0.6	0.7	6.904	0.667	0.667
5	40	0.5	0.5	12.791	0.662	0.655	30	20	0.2	0.2	0.505	0.227	0.243
5	40	0.5	0.7	8.006	0.711	0.712	30	20	0.2	0.5	0.479	0.223	0.242
5	40	0.6	0.2	35.647	0.693	0.699	30	20	0.2	0.7	0.592	0.235	0.252
5	40	0.6	0.5	13.025	0.661	0.66	30	20	0.5	0.2	0.441	0.395	0.391
5	40	0.6	0.7	25.894	0.703	0.701	30	20	0.5	0.5	0.6	0.4	0.395

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15	5	0.2	0.2	32.744	1.411	1.402	30	20	0.5	0.7	0.437	0.394	0.39
15	5	0.2	0.5	823.55	1.494	1.497	30	20	0.6	0.2	18.717	0.458	0.452
15	5	0.2	0.7	13462.32	1.654	1.651	30	20	0.6	0.5	0.577	0.466	0.46
15	5	0.5	0.2	100.543	1.402	1.394	30	20	0.6	0.7	0.451	0.462	0.456
15	5	0.5	0.5	12.623	1.392	1.388	30	30	0.2	0.2	0.344	0.162	0.174
15	5	0.5	0.7	238.948	1.459	1.458	30	30	0.2	0.5	0.345	0.163	0.176
15	5	0.6	0.2	169.018	1.356	1.359	30	30	0.2	0.7	0.347	0.168	0.181
15	5	0.6	0.5	97.213	1.343	1.343	30	30	0.5	0.2	0.341	0.369	0.363
15	5	0.6	0.7	25.553	1.525	1.519	30	30	0.5	0.5	0.511	0.375	0.37
15	10	0.2	0.2	30.52	0.852	0.855	30	30	0.5	0.7	0.326	0.372	0.368
15	10	0.2	0.5	37.813	0.877	0.884	30	30	0.6	0.2	0.332	0.43	0.424
15	10	0.2	0.7	17.617	0.9	0.901	30	30	0.6	0.5	0.319	0.433	0.428
15	10	0.5	0.2	8.591	0.842	0.846	30	30	0.6	0.7	0.308	0.433	0.429
15	10	0.5	0.5	28.307	0.863	0.866	30	40	0.2	0.2	0.292	0.16	0.167
15	10	0.5	0.7	16.876	0.838	0.84	30	40	0.2	0.5	0.292	0.159	0.165
15	10	0.6	0.2	12.698	0.84	0.842	30	40	0.2	0.7	0.293	0.16	0.168
15	10	0.6	0.5	18.314	0.833	0.833	30	40	0.5	0.2	0.279	0.359	0.354
15	10	0.6	0.7	17.259	0.834	0.835	30	40	0.5	0.5	0.272	0.36	0.356
15	20	0.2	0.2	4.809	0.437	0.449	30	40	0.5	0.7	0.269	0.362	0.358
15	20	0.2	0.5	14.818	0.448	0.459	30	40	0.6	0.2	0.266	0.421	0.418
15	20	0.2	0.7	5.329	0.486	0.498	30	40	0.6	0.5	0.268	0.421	0.417
15	20	0.5	0.2	1.404	0.525	0.524	30	40	0.6	0.7	0.261	0.423	0.42
15	20	0.5	0.5	1.637	0.518	0.517							

Table 4

RMSE values of the ML ($RMSE_{ML}$) and the Bayesian estimators ($RMSE_{Bay}$ represents the Bayesian with β_b and $RMSE_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.3$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$	n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$
5	5	0.2	0.2	42.506	1.716	1.658	15	20	0.5	0.7	0.202	0.255	0.261
5	5	0.2	0.5	18.529	1.853	1.828	15	20	0.6	0.2	0.196	0.282	0.287
5	5	0.2	0.7	19.436	1.959	1.943	15	20	0.6	0.5	0.188	0.283	0.288
5	5	0.5	0.2	17.082	1.664	1.634	15	20	0.6	0.7	0.177	0.284	0.291
5	5	0.5	0.5	150.933	1.746	1.725	15	30	0.2	0.2	0.19	0.128	0.141
5	5	0.5	0.7	30.333	1.858	1.821	15	30	0.2	0.5	0.186	0.13	0.142
5	5	0.6	0.2	15.691	1.594	1.555	15	30	0.2	0.7	0.189	0.13	0.142
5	5	0.6	0.5	171.096	1.616	1.592	15	30	0.5	0.2	0.166	0.23	0.236
5	5	0.6	0.7	122.525	1.71	1.69	15	30	0.5	0.5	0.157	0.231	0.238
5	10	0.2	0.2	20.815	0.758	0.761	15	30	0.5	0.7	0.155	0.231	0.237
5	10	0.2	0.5	36.747	0.844	0.835	15	30	0.6	0.2	0.153	0.261	0.266
5	10	0.2	0.7	38.392	0.883	0.878	15	30	0.6	0.5	0.142	0.262	0.267

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5	10	0.5	0.2	8.447	0.699	0.697	15	30	0.6	0.7	0.135	0.263	0.268
5	10	0.5	0.5	13.505	0.713	0.705	15	40	0.2	0.2	0.161	0.122	0.131
5	10	0.5	0.7	12.165	0.799	0.796	15	40	0.2	0.5	0.16	0.125	0.134
5	10	0.6	0.2	15.714	0.763	0.75	15	40	0.2	0.7	0.16	0.125	0.134
5	10	0.6	0.5	6.207	0.675	0.674	15	40	0.5	0.2	0.14	0.22	0.227
5	10	0.6	0.7	22.794	0.74	0.728	15	40	0.5	0.5	0.135	0.219	0.225
5	20	0.2	0.2	1.301	0.325	0.344	15	40	0.5	0.7	0.129	0.219	0.225
5	20	0.2	0.5	0.905	0.315	0.343	15	40	0.6	0.2	0.129	0.252	0.258
5	20	0.2	0.7	4.667	0.371	0.386	15	40	0.6	0.5	0.12	0.251	0.256
5	20	0.5	0.2	6.983	0.368	0.374	15	40	0.6	0.7	0.117	0.253	0.258
5	20	0.5	0.5	0.504	0.366	0.37	30	5	0.2	0.2	2.041	0.705	0.706
5	20	0.5	0.7	0.866	0.367	0.376	30	5	0.2	0.5	2.276	0.707	0.708
5	20	0.6	0.2	2.347	0.39	0.396	30	5	0.2	0.7	57.25	0.727	0.727
5	20	0.6	0.5	0.58	0.365	0.37	30	5	0.5	0.2	2.991	0.579	0.579
5	20	0.6	0.7	2.782	0.368	0.372	30	5	0.5	0.5	4.882	0.583	0.584
5	30	0.2	0.2	1.821	0.176	0.201	30	5	0.5	0.7	5.315	0.658	0.66
5	30	0.2	0.5	0.34	0.184	0.21	30	5	0.6	0.2	2.366	0.54	0.54
5	30	0.2	0.7	0.337	0.192	0.216	30	5	0.6	0.5	1.13	0.542	0.543
5	30	0.5	0.2	0.346	0.282	0.288	30	5	0.6	0.7	126.33	0.525	0.525
5	30	0.5	0.5	0.618	0.277	0.283	30	10	0.2	0.2	0.422	0.165	0.199
5	30	0.5	0.7	0.284	0.281	0.287	30	10	0.2	0.5	0.366	0.184	0.218
5	30	0.6	0.2	0.861	0.309	0.315	30	10	0.2	0.7	0.347	0.16	0.198
5	30	0.6	0.5	2.374	0.307	0.314	30	10	0.5	0.2	0.309	0.294	0.299
5	30	0.6	0.7	0.316	0.302	0.308	30	10	0.5	0.5	0.322	0.29	0.295
5	40	0.2	0.2	0.248	0.145	0.164	30	10	0.5	0.7	0.541	0.3	0.305
5	40	0.2	0.5	0.239	0.143	0.165	30	10	0.6	0.2	0.273	0.324	0.328
5	40	0.2	0.7	0.283	0.144	0.166	30	10	0.6	0.5	1.181	0.327	0.331
5	40	0.5	0.2	0.54	0.249	0.255	30	10	0.6	0.7	0.253	0.331	0.336
5	40	0.5	0.5	0.232	0.257	0.264	30	20	0.2	0.2	0.218	0.133	0.149
5	40	0.5	0.7	0.196	0.254	0.261	30	20	0.2	0.5	0.214	0.134	0.149
5	40	0.6	0.2	0.222	0.279	0.286	30	20	0.2	0.7	0.211	0.134	0.15
5	40	0.6	0.5	0.195	0.28	0.287	30	20	0.5	0.2	0.186	0.243	0.249
5	40	0.6	0.7	0.175	0.282	0.289	30	20	0.5	0.5	0.178	0.243	0.249
15	5	0.2	0.2	12.929	0.902	0.898	30	20	0.5	0.7	0.174	0.246	0.253
15	5	0.2	0.5	24.547	0.906	0.913	30	20	0.6	0.2	0.165	0.277	0.282
15	5	0.2	0.7	23.651	0.925	0.926	30	20	0.6	0.5	0.156	0.276	0.281
15	5	0.5	0.2	5.145	0.802	0.8	30	20	0.6	0.7	0.156	0.278	0.283
15	5	0.5	0.5	33.086	0.776	0.776	30	30	0.2	0.2	0.172	0.126	0.135
15	5	0.5	0.7	11.948	0.795	0.794	30	30	0.2	0.5	0.171	0.126	0.135
15	5	0.6	0.2	15.276	0.732	0.731	30	30	0.2	0.7	0.168	0.126	0.135
15	5	0.6	0.5	3.845	0.742	0.74	30	30	0.5	0.2	0.146	0.228	0.234
15	5	0.6	0.7	12.678	0.736	0.737	30	30	0.5	0.5	0.143	0.226	0.232
15	10	0.2	0.2	1.994	0.291	0.315	30	30	0.5	0.7	0.14	0.227	0.233

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15	10	0.2	0.5	0.616	0.284	0.313	30	30	0.6	0.2	0.129	0.259	0.263
15	10	0.2	0.7	0.95	0.287	0.314	30	30	0.6	0.5	0.123	0.26	0.265
15	10	0.5	0.2	1.122	0.345	0.351	30	30	0.6	0.7	0.12	0.26	0.264
15	10	0.5	0.5	0.693	0.342	0.348	30	40	0.2	0.2	0.146	0.12	0.126
15	10	0.5	0.7	1.563	0.34	0.345	30	40	0.2	0.5	0.144	0.122	0.128
15	10	0.6	0.2	13.285	0.367	0.373	30	40	0.2	0.7	0.143	0.12	0.126
15	10	0.6	0.5	2.519	0.36	0.366	30	40	0.5	0.2	0.123	0.215	0.221
15	10	0.6	0.7	1.467	0.361	0.365	30	40	0.5	0.5	0.118	0.215	0.221
15	20	0.2	0.2	0.245	0.135	0.156	30	40	0.5	0.7	0.118	0.217	0.223
15	20	0.2	0.5	0.344	0.139	0.161	30	40	0.6	0.2	0.111	0.251	0.254
15	20	0.2	0.7	0.245	0.141	0.163	30	40	0.6	0.5	0.105	0.25	0.253
15	20	0.5	0.2	0.213	0.251	0.257	30	40	0.6	0.7	0.101	0.251	0.255
15	20	0.5	0.5	0.206	0.254	0.259							

Table 5

RMSE values of the ML ($RMSE_{ML}$) and the Bayesian estimators ($RMSE_{Bay}$ represents the Bayesian with β_b and $RMSE_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.5$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$	n	J	β_b	β_w	$RMSE_{ML}$	$RMSE_{Bay}$	$RMSE_{BML}$
5	5	0.2	0.2	25.163	1.146	1.126	15	20	0.5	0.7	0.122	0.161	0.175
5	5	0.2	0.5	10.82	1.242	1.221	15	20	0.6	0.2	0.1	0.17	0.177
5	5	0.2	0.7	1591.347	1.281	1.257	15	20	0.6	0.5	0.093	0.173	0.181
5	5	0.5	0.2	35.852	1.098	1.086	15	20	0.6	0.7	0.085	0.175	0.182
5	5	0.5	0.5	9.419	1.04	1.023	15	30	0.2	0.2	0.136	0.103	0.12
5	5	0.5	0.7	10.523	1.092	1.077	15	30	0.2	0.5	0.137	0.103	0.121
5	5	0.6	0.2	16.765	1.05	1.028	15	30	0.2	0.7	0.136	0.104	0.121
5	5	0.6	0.5	27.568	1.048	1.033	15	30	0.5	0.2	0.103	0.137	0.149
5	5	0.6	0.7	14.273	1.031	1.023	15	30	0.5	0.5	0.097	0.139	0.151
5	10	0.2	0.2	3.749	0.334	0.371	15	30	0.5	0.7	0.094	0.139	0.151
5	10	0.2	0.5	1.869	0.356	0.383	15	30	0.6	0.2	0.079	0.154	0.161
5	10	0.2	0.7	41.055	0.396	0.428	15	30	0.6	0.5	0.072	0.154	0.159
5	10	0.5	0.2	1.281	0.345	0.359	15	30	0.6	0.7	0.067	0.154	0.16
5	10	0.5	0.5	2.041	0.323	0.337	15	40	0.2	0.2	0.115	0.093	0.108
5	10	0.5	0.7	12.806	0.344	0.358	15	40	0.2	0.5	0.114	0.094	0.108
5	10	0.6	0.2	179.541	0.346	0.363	15	40	0.2	0.7	0.113	0.094	0.109
5	10	0.6	0.5	0.501	0.323	0.334	15	40	0.5	0.2	0.088	0.128	0.138
5	10	0.6	0.7	2.179	0.311	0.322	15	40	0.5	0.5	0.084	0.128	0.138
5	20	0.2	0.2	0.24	0.134	0.172	15	40	0.5	0.7	0.082	0.129	0.14
5	20	0.2	0.5	0.242	0.136	0.177	15	40	0.6	0.2	0.068	0.143	0.148
5	20	0.2	0.7	0.273	0.148	0.185	15	40	0.6	0.5	0.062	0.144	0.15
5	20	0.5	0.2	0.208	0.194	0.213	15	40	0.6	0.7	0.058	0.145	0.149

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5	20	0.5	0.5	0.186	0.19	0.208	30	5	0.2	0.2	3.068	0.519	0.517
5	20	0.5	0.7	0.179	0.197	0.215	30	5	0.2	0.5	1.998	0.521	0.521
5	20	0.6	0.2	0.212	0.209	0.224	30	5	0.2	0.7	1.584	0.519	0.519
5	20	0.6	0.5	0.167	0.206	0.22	30	5	0.5	0.2	1.417	0.365	0.365
5	20	0.6	0.7	0.156	0.21	0.223	30	5	0.5	0.5	1.486	0.366	0.366
5	30	0.2	0.2	0.181	0.117	0.144	30	5	0.5	0.7	0.624	0.357	0.358
5	30	0.2	0.5	0.177	0.118	0.144	30	5	0.6	0.2	1.038	0.282	0.283
5	30	0.2	0.7	0.178	0.119	0.145	30	5	0.6	0.5	0.281	0.247	0.247
5	30	0.5	0.2	0.157	0.163	0.181	30	5	0.6	0.7	0.434	0.247	0.247
5	30	0.5	0.5	0.145	0.164	0.182	30	10	0.2	0.2	0.261	0.135	0.174
5	30	0.5	0.7	0.136	0.166	0.182	30	10	0.2	0.5	0.25	0.13	0.169
5	30	0.6	0.2	0.142	0.172	0.185	30	10	0.2	0.7	0.258	0.137	0.175
5	30	0.6	0.5	0.125	0.173	0.185	30	10	0.5	0.2	0.181	0.2	0.213
5	30	0.6	0.7	0.117	0.178	0.19	30	10	0.5	0.5	0.174	0.203	0.216
5	40	0.2	0.2	0.153	0.109	0.13	30	10	0.5	0.7	0.171	0.204	0.216
5	40	0.2	0.5	0.155	0.111	0.133	30	10	0.6	0.2	0.131	0.214	0.219
5	40	0.2	0.7	0.152	0.111	0.133	30	10	0.6	0.5	0.116	0.215	0.221
5	40	0.5	0.2	0.13	0.144	0.16	30	10	0.6	0.7	0.113	0.217	0.224
5	40	0.5	0.5	0.123	0.148	0.166	30	20	0.2	0.2	0.158	0.112	0.132
5	40	0.5	0.7	0.115	0.149	0.164	30	20	0.2	0.5	0.158	0.111	0.131
5	40	0.6	0.2	0.119	0.157	0.169	30	20	0.2	0.7	0.159	0.113	0.132
5	40	0.6	0.5	0.106	0.162	0.173	30	20	0.5	0.2	0.11	0.152	0.164
5	40	0.6	0.7	0.097	0.162	0.172	30	20	0.5	0.5	0.109	0.152	0.164
15	5	0.2	0.2	4.898	0.637	0.637	30	20	0.5	0.7	0.106	0.156	0.168
15	5	0.2	0.5	1.439	0.601	0.602	30	20	0.6	0.2	0.077	0.168	0.173
15	5	0.2	0.7	1.517	0.618	0.613	30	20	0.6	0.5	0.073	0.17	0.175
15	5	0.5	0.2	0.851	0.466	0.467	30	20	0.6	0.7	0.071	0.171	0.176
15	5	0.5	0.5	2.01	0.456	0.46	30	30	0.2	0.2	0.127	0.099	0.115
15	5	0.5	0.7	1.7	0.475	0.477	30	30	0.2	0.5	0.126	0.1	0.114
15	5	0.6	0.2	1.784	0.41	0.411	30	30	0.2	0.7	0.125	0.1	0.115
15	5	0.6	0.5	2.641	0.396	0.395	30	30	0.5	0.2	0.088	0.136	0.146
15	5	0.6	0.7	3.921	0.372	0.374	30	30	0.5	0.5	0.088	0.139	0.149
15	10	0.2	0.2	0.291	0.155	0.195	30	30	0.5	0.7	0.086	0.137	0.146
15	10	0.2	0.5	0.284	0.15	0.193	30	30	0.6	0.2	0.062	0.154	0.157
15	10	0.2	0.7	0.281	0.145	0.189	30	30	0.6	0.5	0.057	0.153	0.157
15	10	0.5	0.2	0.209	0.21	0.225	30	30	0.6	0.7	0.055	0.155	0.158
15	10	0.5	0.5	0.21	0.216	0.229	30	40	0.2	0.2	0.108	0.089	0.103
15	10	0.5	0.7	0.197	0.216	0.23	30	40	0.2	0.5	0.106	0.09	0.103
15	10	0.6	0.2	0.26	0.227	0.236	30	40	0.2	0.7	0.107	0.091	0.105
15	10	0.6	0.5	0.782	0.226	0.235	30	40	0.5	0.2	0.077	0.128	0.137
15	10	0.6	0.7	2.011	0.228	0.237	30	40	0.5	0.5	0.076	0.128	0.136
15	20	0.2	0.2	0.176	0.115	0.139	30	40	0.5	0.7	0.074	0.127	0.135
15	20	0.2	0.5	0.172	0.117	0.141	30	40	0.6	0.2	0.052	0.145	0.148

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15	20	0.2	0.7	0.172	0.115	0.138	30	40	0.6	0.5	0.05	0.147	0.15
15	20	0.5	0.2	0.131	0.155	0.169	30	40	0.6	0.7	0.048	0.146	0.15
15	20	0.5	0.5	0.124	0.16	0.174							

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Table 6

Relative bias in % of the ML ($Bias_{ML}$) and the Bayesian estimators ($Bias_{Bay}$ represents the Bayesian with β_b and $Bias_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.05$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$	n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$
5	5	0.2	0.2	939.706	-175.218	-177.665	15	20	0.5	0.7	-23.76	-67.163	-65.211
5	5	0.2	0.5	-2576.35	-344.055	-340.826	15	20	0.6	0.2	-102.766	-32.639	-31.891
5	5	0.2	0.7	1012.501	-456.048	-461.978	15	20	0.6	0.5	-151.732	-52.393	-50.838
5	5	0.5	0.2	-108.197	-84.818	-81.633	15	20	0.6	0.7	-639.383	-63.361	-61.91
5	5	0.5	0.5	9.274	-162.483	-163.005	15	30	0.2	0.2	-200.772	-65.025	-62.464
5	5	0.5	0.7	-526.117	-210.689	-210.87	15	30	0.2	0.5	-167.176	-104.229	-100.129
5	5	0.6	0.2	-173.361	-90.041	-87.431	15	30	0.2	0.7	18.683	-137.57	-133.024
5	5	0.6	0.5	49.158	-135.665	-134.478	15	30	0.5	0.2	161.756	-47.165	-45.048
5	5	0.6	0.7	-699.459	-178.635	-179.595	15	30	0.5	0.5	47.962	-62.547	-60.248
5	10	0.2	0.2	1064.461	-128.863	-127.003	15	30	0.5	0.7	-75.302	-76.234	-73.687
5	10	0.2	0.5	203.735	-207.413	-208.761	15	30	0.6	0.2	67.323	-43.167	-41.534
5	10	0.2	0.7	-94.878	-297.133	-296.062	15	30	0.6	0.5	-19.332	-59.943	-58.139
5	10	0.5	0.2	-93.408	-77.668	-76.457	15	30	0.6	0.7	-17.13	-63.18	-61.173
5	10	0.5	0.5	-654.471	-102.57	-103.688	15	40	0.2	0.2	36.969	-72.628	-69.546
5	10	0.5	0.7	332.931	-144.312	-146.785	15	40	0.2	0.5	-15.178	-101.716	-98.562
5	10	0.6	0.2	-169.053	-59.41	-59.602	15	40	0.2	0.7	-103.122	-122.674	-119.056
5	10	0.6	0.5	63.505	-109.252	-108.353	15	40	0.5	0.2	54.952	-56.233	-53.458
5	10	0.6	0.7	-1353.22	-122.199	-123.637	15	40	0.5	0.5	14.505	-71.001	-68.306
5	20	0.2	0.2	313.298	-94.42	-90.214	15	40	0.5	0.7	31.637	-74.519	-71.875
5	20	0.2	0.5	1435.496	-175.674	-172.445	15	40	0.6	0.2	3.904	-55.525	-53.487
5	20	0.2	0.7	-104.246	-207.484	-206.478	15	40	0.6	0.5	15.277	-66.395	-64.375
5	20	0.5	0.2	1695.914	-54.03	-54.491	15	40	0.6	0.7	-28.682	-71.796	-69.65
5	20	0.5	0.5	-43.633	-84.151	-84.021	30	5	0.2	0.2	-1394.99	-57.697	-57.462
5	20	0.5	0.7	321.686	-101.773	-101.497	30	5	0.2	0.5	-6438.42	-92.174	-91.853
5	20	0.6	0.2	-216.689	-54.383	-55.371	30	5	0.2	0.7	6188.805	-107.343	-107.263
5	20	0.6	0.5	-51.553	-75.3	-74.366	30	5	0.5	0.2	-177.108	-33.585	-32.785
5	20	0.6	0.7	-26.749	-96.432	-96.959	30	5	0.5	0.5	-128.246	-48.822	-49.195
5	30	0.2	0.2	21.324	-87.631	-87.895	30	5	0.5	0.7	-37.675	-63.07	-63.138
5	30	0.2	0.5	-83.263	-133.207	-128.895	30	5	0.6	0.2	-192.766	-26.28	-26.152
5	30	0.2	0.7	1292.766	-168.866	-164.89	30	5	0.6	0.5	-226.727	-49.149	-49.239
5	30	0.5	0.2	-87.437	-48.346	-48.156	30	5	0.6	0.7	244.26	-50.056	-50.014
5	30	0.5	0.5	114.345	-75.708	-73.846	30	10	0.2	0.2	887.138	-40.248	-38.253
5	30	0.5	0.7	129.019	-89.891	-88.584	30	10	0.2	0.5	91.978	-90.081	-86.919
5	30	0.6	0.2	203.25	-47.662	-47.134	30	10	0.2	0.7	-48.618	-107.919	-104.33
5	30	0.6	0.5	26.482	-68.369	-67.769	30	10	0.5	0.2	43.356	-35.66	-34.49
5	30	0.6	0.7	-48.218	-79.373	-79.442	30	10	0.5	0.5	134.341	-48.188	-46.45
5	40	0.2	0.2	-28.181	-70.438	-71.177	30	10	0.5	0.7	-572.786	-59.396	-57.443

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5	40	0.2	0.5	-47.091	-141.974	-137.019	30	10	0.6	0.2	180.012	-30.446	-29.172
5	40	0.2	0.7	169.648	-166.692	-159.894	30	10	0.6	0.5	262.174	-42.491	-41.197
5	40	0.5	0.2	232.466	-41.92	-41.522	30	10	0.6	0.7	-221.584	-49.66	-48.066
5	40	0.5	0.5	-50.815	-71.833	-70.561	30	20	0.2	0.2	17.57	-70.202	-68.437
5	40	0.5	0.7	-65.781	-88.633	-87.188	30	20	0.2	0.5	-76.648	-87.522	-83.764
5	40	0.6	0.2	95.755	-41.248	-41.131	30	20	0.2	0.7	-102.031	-105.512	-102.991
5	40	0.6	0.5	-337.021	-62.363	-61.601	30	20	0.5	0.2	-25.786	-64.208	-61.509
5	40	0.6	0.7	137.401	-75.084	-73.927	30	20	0.5	0.5	-195.755	-68.995	-65.967
15	5	0.2	0.2	1382.23	-66.236	-65.608	30	20	0.5	0.7	-12.808	-75.91	-72.546
15	5	0.2	0.5	-3114.31	-123.3	-124.703	30	20	0.6	0.2	36.709	-60.133	-57.464
15	5	0.2	0.7	-762.222	-180.02	-177.251	30	20	0.6	0.5	-28.893	-68.769	-66.26
15	5	0.5	0.2	312.494	-45.271	-45.203	30	20	0.6	0.7	-8.648	-74.745	-72.346
15	5	0.5	0.5	-158.832	-73.883	-73.922	30	30	0.2	0.2	-16.632	-83.612	-80.43
15	5	0.5	0.7	-176.129	-89.463	-88.718	30	30	0.2	0.5	-4.256	-84.067	-81.135
15	5	0.6	0.2	-47.277	-48.753	-50.276	30	30	0.2	0.7	-22.266	-88.912	-85.758
15	5	0.6	0.5	-285.822	-61.179	-61.678	30	30	0.5	0.2	12.282	-75.513	-72.03
15	5	0.6	0.7	-40.838	-83.226	-82.861	30	30	0.5	0.5	-12.309	-79.911	-76.956
15	10	0.2	0.2	1266.762	-41.67	-41.652	30	30	0.5	0.7	-2.186	-80.325	-77.784
15	10	0.2	0.5	-21970.6	-106.428	-104.369	30	30	0.6	0.2	9.385	-74.856	-71.836
15	10	0.2	0.7	330.852	-127.641	-125.988	30	30	0.6	0.5	1.032	-78.497	-75.826
15	10	0.5	0.2	61.867	-30.545	-30.122	30	30	0.6	0.7	-8.66	-80.323	-77.907
15	10	0.5	0.5	-108.936	-50.674	-49.646	30	40	0.2	0.2	1.012	-83.597	-79.347
15	10	0.5	0.7	-55.614	-64.946	-64.667	30	40	0.2	0.5	-35.467	-86.099	-83.287
15	10	0.6	0.2	311.448	-32.714	-32.54	30	40	0.2	0.7	-22.23	-89.364	-86.915
15	10	0.6	0.5	-19.852	-49.811	-49.947	30	40	0.5	0.2	2.428	-80.059	-76.881
15	10	0.6	0.7	-67.793	-60.111	-59.945	30	40	0.5	0.5	-0.828	-82.002	-79.127
15	20	0.2	0.2	-354.039	-57.176	-54.656	30	40	0.5	0.7	-1.092	-81.837	-79.248
15	20	0.2	0.5	-1217.28	-109.289	-105.68	30	40	0.6	0.2	2.463	-78.273	-75.699
15	20	0.2	0.7	163.248	-142.24	-136.652	30	40	0.6	0.5	1.436	-79.898	-77.626
15	20	0.5	0.2	-61.296	-38.271	-36.857	30	40	0.6	0.7	-1.31	-80.875	-78.575
15	20	0.5	0.5	-205.614	-57.65	-55.776							

Table 7

Relative bias in % of the ML ($Bias_{ML}$) and the Bayesian estimators ($Bias_{Bay}$ represents the Bayesian with β_b and $Bias_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.1$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$	n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$
5	5	0.2	0.2	-180.425	-97.729	-97.848	15	20	0.5	0.7	-1.779	-68.523	-65.365
5	5	0.2	0.5	-4085.79	-296.848	-296.841	15	20	0.6	0.2	-44.621	-55.605	-52.813
5	5	0.2	0.7	-1867.08	-353.484	-349.919	15	20	0.6	0.5	3.034	-60.238	-57.617
5	5	0.5	0.2	50.618	-49.194	-50.365	15	20	0.6	0.7	5.119	-64.958	-62.425

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5	5	0.5	0.5	-112.359	-99.353	-100.261	15	30	0.2	0.2	6.011	-74.512	-69.163
5	5	0.5	0.7	20.548	-149.938	-151.061	15	30	0.2	0.5	-14.988	-81.933	-77.68
5	5	0.6	0.2	-20.156	-42.548	-41.06	15	30	0.2	0.7	-21.005	-88.347	-85.029
5	5	0.6	0.5	738.784	-86.14	-86.637	15	30	0.5	0.2	2.785	-68.067	-64.702
5	5	0.6	0.7	209.891	-137.535	-137.512	15	30	0.5	0.5	7.305	-70.079	-66.955
5	10	0.2	0.2	702.919	-82.189	-81.101	15	30	0.5	0.7	-6.319	-72.935	-70.103
5	10	0.2	0.5	209.136	-182.931	-184.313	15	30	0.6	0.2	8.127	-65.341	-62.491
5	10	0.2	0.7	396.677	-250.645	-246.332	15	30	0.6	0.5	5.803	-67.464	-65.019
5	10	0.5	0.2	-213.951	-31.923	-32.773	15	30	0.6	0.7	-1.144	-69.318	-66.917
5	10	0.5	0.5	-608.719	-70.512	-72.082	15	40	0.2	0.2	1.56	-76.63	-71.004
5	10	0.5	0.7	292.342	-107.28	-106.708	15	40	0.2	0.5	-27.766	-80.567	-76.371
5	10	0.6	0.2	-114.348	-31.499	-33.501	15	40	0.2	0.7	-132.798	-82.327	-77.86
5	10	0.6	0.5	37.868	-60.978	-60.269	15	40	0.5	0.2	6.269	-70.345	-67.259
5	10	0.6	0.7	322.257	-86.07	-84.107	15	40	0.5	0.5	0.227	-71.222	-68.422
5	20	0.2	0.2	5140.523	-74.28	-73.683	15	40	0.5	0.7	-2.364	-72.428	-69.788
5	20	0.2	0.5	-738.356	-136.74	-131.187	15	40	0.6	0.2	3.306	-68.78	-66.52
5	20	0.2	0.7	-973.423	-193.073	-183.85	15	40	0.6	0.5	-0.237	-70.115	-68.227
5	20	0.5	0.2	33.357	-26.839	-26.188	15	40	0.6	0.7	-1.657	-70.571	-68.814
5	20	0.5	0.5	-29.743	-56.85	-55.812	30	5	0.2	0.2	-71.264	-30.015	-30.151
5	20	0.5	0.7	-293.616	-80.078	-78.315	30	5	0.2	0.5	203.093	-47.948	-48.552
5	20	0.6	0.2	3064.897	-23.056	-22.691	30	5	0.2	0.7	-162.961	-57.815	-57.962
5	20	0.6	0.5	-141.048	-49.363	-48.477	30	5	0.5	0.2	-65.295	-11.699	-11.46
5	20	0.6	0.7	-183.525	-64.047	-63.43	30	5	0.5	0.5	33.614	-27.163	-27.075
5	30	0.2	0.2	-63.151	-54.592	-50.382	30	5	0.5	0.7	-52.018	-27.074	-27.23
5	30	0.2	0.5	-88.897	-115.98	-108.684	30	5	0.6	0.2	-296.281	-7.696	-7.805
5	30	0.2	0.7	-74.222	-156.856	-147.408	30	5	0.6	0.5	-20.843	-17.36	-17.523
5	30	0.5	0.2	-5.832	-28.231	-27.638	30	5	0.6	0.7	67.838	-27.974	-27.849
5	30	0.5	0.5	-90.982	-51.166	-49.514	30	10	0.2	0.2	15.486	-56.606	-52.52
5	30	0.5	0.7	56.225	-65.307	-63.523	30	10	0.2	0.5	-25.161	-77.947	-74.029
5	30	0.6	0.2	25.728	-26.956	-26.756	30	10	0.2	0.7	-287.394	-97.207	-92.745
5	30	0.6	0.5	-76.56	-47.217	-46.01	30	10	0.5	0.2	23.314	-49.55	-45.933
5	30	0.6	0.7	-18.263	-57.962	-56.192	30	10	0.5	0.5	64.459	-55.552	-52.107
5	40	0.2	0.2	59.795	-57.455	-52.962	30	10	0.5	0.7	4.539	-62.899	-59.47
5	40	0.2	0.5	-104.315	-101.762	-94.837	30	10	0.6	0.2	20.208	-48.932	-46.516
5	40	0.2	0.7	-2423.67	-146.467	-137.357	30	10	0.6	0.5	23.429	-57.004	-54.44
5	40	0.5	0.2	71.194	-34.808	-33.115	30	10	0.6	0.7	20.733	-61.414	-58.784
5	40	0.5	0.5	17.511	-56.792	-54.364	30	20	0.2	0.2	3.856	-77.33	-71.309
5	40	0.5	0.7	-9.612	-68.39	-66.062	30	20	0.2	0.5	-9.562	-79.292	-74.84
5	40	0.6	0.2	42.379	-30.542	-29.469	30	20	0.2	0.7	-16.742	-82.423	-78.538
5	40	0.6	0.5	-7.197	-48.516	-46.937	30	20	0.5	0.2	0.587	-72.727	-69.369
5	40	0.6	0.7	80.748	-59.465	-57.386	30	20	0.5	0.5	0.296	-73.087	-69.506
15	5	0.2	0.2	-490.349	-40.138	-39.193	30	20	0.5	0.7	-1.527	-73.836	-70.474
15	5	0.2	0.5	-2803.13	-93.016	-95.135	30	20	0.6	0.2	-40.58	-70.874	-68.019

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15	5	0.2	0.7	-94231.7	-118.795	-118.882	30	20	0.6	0.5	2.45	-71.272	-68.618
15	5	0.5	0.2	-446.312	-20.598	-20.48	30	20	0.6	0.7	-0.147	-72.023	-69.408
15	5	0.5	0.5	-8.249	-39.589	-39.418	30	30	0.2	0.2	-3.493	-77.961	-72.732
15	5	0.5	0.7	-686.354	-50.31	-50.278	30	30	0.2	0.5	-1.877	-77.558	-72.005
15	5	0.6	0.2	-472.988	-14.685	-15.161	30	30	0.2	0.7	-6.558	-78.139	-73.528
15	5	0.6	0.5	187.689	-26.937	-27.048	30	30	0.5	0.2	1.787	-72.357	-69.404
15	5	0.6	0.7	27.892	-51.789	-51.997	30	30	0.5	0.5	-1.615	-72.959	-70.207
15	10	0.2	0.2	230.013	-35.213	-31.963	30	30	0.5	0.7	-2.106	-72.922	-70.443
15	10	0.2	0.5	-344.476	-82.708	-78.286	30	30	0.6	0.2	3.399	-70.612	-68.31
15	10	0.2	0.7	15.321	-111.873	-106.684	30	30	0.6	0.5	0.584	-70.95	-68.985
15	10	0.5	0.2	-22.65	-28.692	-27.291	30	30	0.6	0.7	-0.895	-71.066	-69.293
15	10	0.5	0.5	64.502	-41.437	-39.689	30	40	0.2	0.2	-3.362	-77.598	-72.447
15	10	0.5	0.7	-2.906	-54.824	-52.782	30	40	0.2	0.5	0.231	-76.673	-71.081
15	10	0.6	0.2	-39.196	-23.171	-22.183	30	40	0.2	0.7	-7.209	-77.56	-72.785
15	10	0.6	0.5	44.949	-39.477	-38.189	30	40	0.5	0.2	2.468	-70.765	-68.271
15	10	0.6	0.7	-32.648	-43.632	-41.748	30	40	0.5	0.5	-0.038	-70.983	-68.802
15	20	0.2	0.2	-39.603	-64.419	-58.788	30	40	0.5	0.7	-0.553	-71.21	-69.095
15	20	0.2	0.5	-114.491	-90.242	-86.207	30	40	0.6	0.2	0.738	-69.428	-68.01
15	20	0.2	0.7	-81.633	-97.36	-92.204	30	40	0.6	0.5	1.48	-69.411	-67.777
15	20	0.5	0.2	11.16	-57.031	-53.31	30	40	0.6	0.7	-0.532	-69.658	-68.31
15	20	0.5	0.5	-4.442	-63.884	-60.681							

Table 8

Relative bias in % of the ML ($Bias_{ML}$) and the Bayesian estimators ($Bias_{Bay}$ represents the Bayesian with β_b and $Bias_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.3$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$	n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$
5	5	0.2	0.2	-380.903	-7.31	-5.4	15	20	0.5	0.7	-0.998	-46.918	-45.141
5	5	0.2	0.5	-297.1	-95.193	-90.933	15	20	0.6	0.2	1.699	-44.539	-43.227
5	5	0.2	0.7	150.267	-143.281	-141.121	15	20	0.6	0.5	0.883	-44.362	-43.218
5	5	0.5	0.2	-56.277	46.127	44.998	15	20	0.6	0.7	-0.51	-44.749	-43.928
5	5	0.5	0.5	432.617	0.206	0.453	15	30	0.2	0.2	1.117	-56.168	-47.821
5	5	0.5	0.7	61.206	-27.888	-27.918	15	30	0.2	0.5	-2.359	-57.306	-50.176
5	5	0.6	0.2	-13.767	43.107	40.977	15	30	0.2	0.7	-2.578	-56.83	-49.285
5	5	0.6	0.5	-485.798	10.824	10.598	15	30	0.5	0.2	0.554	-43.555	-42.309
5	5	0.6	0.7	-247.168	-1.55	-1.634	15	30	0.5	0.5	-0.128	-43.808	-42.703
5	10	0.2	0.2	-131.468	-35.023	-27.847	15	30	0.5	0.7	0.43	-43.752	-42.498
5	10	0.2	0.5	185.621	-89.478	-77.998	15	30	0.6	0.2	1.457	-41.903	-41.005
5	10	0.2	0.7	-248.119	-108.161	-97.15	15	30	0.6	0.5	0.287	-41.945	-41.364
5	10	0.5	0.2	-31.098	-11.137	-8.976	15	30	0.6	0.7	-0.033	-42.031	-41.507
5	10	0.5	0.5	-26.568	-27.84	-25.328	15	40	0.2	0.2	1.409	-54.164	-46.953

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5	10	0.5	0.7	-50.187	-40.606	-37.555	15	40	0.2	0.5	-3.353	-55.342	-49.136
5	10	0.6	0.2	-50.377	-4.481	-4.542	15	40	0.2	0.7	-4.992	-55.617	-49.763
5	10	0.6	0.5	25.482	-23.963	-22.808	15	40	0.5	0.2	-0.053	-42.134	-41.392
5	10	0.6	0.7	25.378	-35.246	-33.298	15	40	0.5	0.5	0.659	-42.045	-41.048
5	20	0.2	0.2	10.828	-51.717	-41.437	15	40	0.5	0.7	-0.279	-42.07	-41.399
5	20	0.2	0.5	-21.449	-68.64	-61.552	15	40	0.6	0.2	-0.027	-40.787	-40.45
5	20	0.2	0.7	-63.852	-83.751	-76.923	15	40	0.6	0.5	0.108	-40.644	-40.275
5	20	0.5	0.2	8.852	-40.682	-36.675	15	40	0.6	0.7	-0.034	-40.829	-40.452
5	20	0.5	0.5	0.077	-46.392	-43.033	30	5	0.2	0.2	-4.067	-0.005	0
5	20	0.5	0.7	-4.033	-50.636	-47.69	30	5	0.2	0.5	6.026	-8.355	-8.332
5	20	0.6	0.2	7.13	-37.517	-35.116	30	5	0.2	0.7	-381.519	-18.351	-18.252
5	20	0.6	0.5	2.671	-41.089	-38.967	30	5	0.5	0.2	4.964	5.906	5.869
5	20	0.6	0.7	-11.56	-45.502	-43.638	30	5	0.5	0.5	-7.468	3.082	3.014
5	30	0.2	0.2	13.341	-59.329	-49.672	30	5	0.5	0.7	-15.179	-0.386	-0.457
5	30	0.2	0.5	-17.994	-64.693	-58.182	30	5	0.6	0.2	-4.802	8.471	8.463
5	30	0.2	0.7	-18	-65.289	-58.525	30	5	0.6	0.5	3.508	4.313	4.291
5	30	0.5	0.2	5.034	-45.253	-41.916	30	5	0.6	0.7	-289.118	2.066	2.117
5	30	0.5	0.5	-1.161	-47.406	-44.776	30	10	0.2	0.2	4.031	-60.499	-50.403
5	30	0.5	0.7	-3.027	-48.702	-46.545	30	10	0.2	0.5	-5.741	-64.16	-56.034
5	30	0.6	0.2	2.359	-42.974	-41.05	30	10	0.2	0.7	-6.347	-62.99	-55.608
5	30	0.6	0.5	-4.503	-44.644	-43.32	30	10	0.5	0.2	1.843	-51.672	-48.263
5	30	0.6	0.7	-1.451	-45.244	-43.908	30	10	0.5	0.5	1.684	-51.568	-48.275
5	40	0.2	0.2	-0.984	-58.17	-49.658	30	10	0.5	0.7	-2.506	-53.068	-50.129
5	40	0.2	0.5	-7.897	-59.238	-51.948	30	10	0.6	0.2	1.695	-49.025	-47.103
5	40	0.2	0.7	-9.097	-60.041	-53.542	30	10	0.6	0.5	3.728	-49.251	-47.491
5	40	0.5	0.2	1.26	-45.072	-42.743	30	10	0.6	0.7	-0.072	-49.626	-48.056
5	40	0.5	0.5	-0.797	-46.11	-44.267	30	20	0.2	0.2	1.105	-58.467	-49.686
5	40	0.5	0.7	-1.807	-46.459	-44.731	30	20	0.2	0.5	-1.151	-58.733	-50.616
5	40	0.6	0.2	2.136	-43.252	-41.836	30	20	0.2	0.7	-3.356	-59.335	-51.794
5	40	0.6	0.5	0.334	-43.83	-42.83	30	20	0.5	0.2	1.165	-45.931	-44.21
5	40	0.6	0.7	-1.045	-44.088	-43.199	30	20	0.5	0.5	0.869	-45.997	-44.415
15	5	0.2	0.2	-74.931	4.84	5.798	30	20	0.5	0.7	-0.619	-46.582	-45.336
15	5	0.2	0.5	323.87	-15.23	-15.118	30	20	0.6	0.2	0.58	-44.207	-43.462
15	5	0.2	0.7	59.503	-40.264	-40.218	30	20	0.6	0.5	0.216	-44.112	-43.436
15	5	0.5	0.2	7.849	15.234	15.098	30	20	0.6	0.7	-0.147	-44.297	-43.668
15	5	0.5	0.5	120.525	6.303	6.248	30	30	0.2	0.2	-0.07	-56.129	-48.75
15	5	0.5	0.7	-12.943	-2.154	-2.174	30	30	0.2	0.5	-0.044	-56.139	-48.759
15	5	0.6	0.2	52.143	18.306	18.276	30	30	0.2	0.7	-0.794	-55.821	-48.806
15	5	0.6	0.5	8.054	9.334	9.369	30	30	0.5	0.2	-0.122	-43.627	-42.777
15	5	0.6	0.7	-2.296	3.754	3.712	30	30	0.5	0.5	0.392	-43.329	-42.388
15	10	0.2	0.2	4.432	-59.604	-51.105	30	30	0.5	0.7	-0.284	-43.495	-42.701
15	10	0.2	0.5	-9.53	-63.083	-55.401	30	30	0.6	0.2	0.448	-41.879	-41.398
15	10	0.2	0.7	-22.137	-68.075	-60.897	30	30	0.6	0.5	-0.18	-42.106	-41.816

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15	10	0.5	0.2	8.999	-46.317	-42.314	30	30	0.6	0.7	0.22	-42.095	-41.672
15	10	0.5	0.5	-0.416	-49.214	-45.6	30	40	0.2	0.2	0.946	-53.592	-47.014
15	10	0.5	0.7	2.784	-51.402	-48.068	30	40	0.2	0.5	-2.535	-54.666	-48.894
15	10	0.6	0.2	33.381	-44.3	-42.08	30	40	0.2	0.7	-0.731	-53.616	-47.618
15	10	0.6	0.5	6.384	-45.365	-43.426	30	40	0.5	0.2	0.3	-41.79	-41.174
15	10	0.6	0.7	-6.388	-47.913	-45.969	30	40	0.5	0.5	0.022	-41.822	-41.29
15	20	0.2	0.2	2.87	-58.865	-49.359	30	40	0.5	0.7	-0.688	-42.035	-41.65
15	20	0.2	0.5	-3.719	-58.974	-50.509	30	40	0.6	0.2	0.717	-40.913	-40.456
15	20	0.2	0.7	-6.431	-60.174	-52.351	30	40	0.6	0.5	0.548	-40.653	-40.245
15	20	0.5	0.2	0.911	-46.634	-44.514	30	40	0.6	0.7	0.013	-40.976	-40.722
15	20	0.5	0.5	0.735	-46.989	-44.938							

Table 9

Relative bias in % of the ML ($Bias_{ML}$) and the Bayesian estimators ($Bias_{Bay}$ represents the Bayesian with β_b and $Bias_{BML}$ represents the Bayesian with $\hat{\beta}_b$) for $ICC_X = 0.5$ and different values of n, J, β_b, and β_w

n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$	n	J	β_b	β_w	$Bias_{ML}$	$Bias_{Bay}$	$Bias_{BML}$
5	5	0.2	0.2	-164.985	30.853	30.064	15	20	0.5	0.7	-0.22	-27.013	-26.284
5	5	0.2	0.5	-91.398	-2.688	-0.009	15	20	0.6	0.2	0.573	-24.739	-24.365
5	5	0.2	0.7	-11249.1	-39.249	-35.45	15	20	0.6	0.5	0.168	-24.833	-24.536
5	5	0.5	0.2	96.408	56.089	56.06	15	20	0.6	0.7	-0.221	-25.053	-24.896
5	5	0.5	0.5	22.581	36.746	36.707	15	30	0.2	0.2	-0.123	-38.818	-31.516
5	5	0.5	0.7	-18.976	28.04	28.119	15	30	0.2	0.5	-0.124	-38.316	-31.104
5	5	0.6	0.2	-45.493	54.128	53.045	15	30	0.2	0.7	-1.024	-38.757	-31.716
5	5	0.6	0.5	85.638	38.771	37.987	15	30	0.5	0.2	0.389	-23.985	-23.405
5	5	0.6	0.7	34.784	31.698	32.092	15	30	0.5	0.5	-0.225	-24.225	-23.839
5	10	0.2	0.2	3.087	-44.882	-34.243	15	30	0.5	0.7	-0.651	-24.334	-24.132
5	10	0.2	0.5	-24.56	-54.504	-45.425	15	30	0.6	0.2	0.225	-22.859	-22.637
5	10	0.2	0.7	264.16	-65.759	-56.246	15	30	0.6	0.5	0.318	-22.779	-22.525
5	10	0.5	0.2	4.413	-25.214	-21.39	15	30	0.6	0.7	0.038	-22.842	-22.671
5	10	0.5	0.5	-1.164	-30.273	-27.212	15	40	0.2	0.2	0.72	-35.116	-29.072
5	10	0.5	0.7	25.381	-35.121	-32.221	15	40	0.2	0.5	0.026	-35.599	-29.691
5	10	0.6	0.2	431.493	-22.177	-20.11	15	40	0.2	0.7	-0.6	-35.434	-29.604
5	10	0.6	0.5	0.409	-26.339	-24.864	15	40	0.5	0.2	0.182	-22.687	-22.291
5	10	0.6	0.7	-5.418	-29.331	-27.934	15	40	0.5	0.5	-0.056	-22.669	-22.337
5	20	0.2	0.2	2.174	-45.461	-33.726	15	40	0.5	0.7	-0.256	-22.893	-22.621
5	20	0.2	0.5	-5.226	-47.175	-37.32	15	40	0.6	0.2	0.251	-21.605	-21.404
5	20	0.2	0.7	-13.597	-50.043	-42.321	15	40	0.6	0.5	-0.315	-21.726	-21.707
5	20	0.5	0.2	2.893	-29.868	-27.276	15	40	0.6	0.7	0.027	-21.818	-21.687
5	20	0.5	0.5	0.702	-30.204	-28.086	30	5	0.2	0.2	32.27	12.072	12.191
5	20	0.5	0.7	-1.059	-31.021	-29.332	30	5	0.2	0.5	-13.866	-2.04	-2.062

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5	20	0.6	0.2	2.823	-27.324	-25.747	30	5	0.2	0.7	-2.588	-3.011	-2.939
5	20	0.6	0.5	0.358	-27.984	-27.006	30	5	0.5	0.2	6.004	8.522	8.553
5	20	0.6	0.7	-0.622	-28.591	-27.808	30	5	0.5	0.5	-1.864	6.028	6.046
5	30	0.2	0.2	-1.118	-43.94	-34.55	30	5	0.5	0.7	-2.62	3.931	3.91
5	30	0.2	0.5	-3.38	-43.802	-35.125	30	5	0.6	0.2	7.629	7.837	7.829
5	30	0.2	0.7	-5.405	-44.419	-36.288	30	5	0.6	0.5	0.18	7.132	7.124
5	30	0.5	0.2	1.063	-27.096	-25.587	30	5	0.6	0.7	1.126	6.588	6.579
5	30	0.5	0.5	-0.26	-27.139	-26.052	30	10	0.2	0.2	-1.672	-50.254	-40.083
5	30	0.5	0.7	-0.879	-27.124	-26.169	30	10	0.2	0.5	-2.743	-49.396	-39.691
5	30	0.6	0.2	1.631	-24.597	-23.649	30	10	0.2	0.7	-3.732	-50.118	-40.716
5	30	0.6	0.5	0.801	-24.669	-23.933	30	10	0.5	0.2	0.676	-33.464	-31.932
5	30	0.6	0.7	-0.025	-25.092	-24.539	30	10	0.5	0.5	-0.757	-33.783	-32.615
5	40	0.2	0.2	-0.286	-40.655	-32.185	30	10	0.5	0.7	-0.746	-33.947	-32.733
5	40	0.2	0.5	-3.114	-41.373	-33.552	30	10	0.6	0.2	1.361	-30.333	-29.705
5	40	0.2	0.7	-4.552	-41.592	-34.315	30	10	0.6	0.5	0.315	-30.791	-30.359
5	40	0.5	0.2	1.022	-24.608	-23.518	30	10	0.6	0.7	-0.227	-31.065	-30.802
5	40	0.5	0.5	-0.904	-25.201	-24.685	30	20	0.2	0.2	-0.194	-42.537	-34.137
5	40	0.5	0.7	-0.611	-25.064	-24.452	30	20	0.2	0.5	0.387	-42.302	-33.956
5	40	0.6	0.2	1.223	-23.036	-22.307	30	20	0.2	0.7	-0.726	-43.307	-34.829
5	40	0.6	0.5	0.086	-23.63	-23.293	30	20	0.5	0.2	-0.087	-26.456	-25.927
5	40	0.6	0.7	0.067	-23.325	-22.934	30	20	0.5	0.5	0.069	-26.524	-25.956
15	5	0.2	0.2	-49.017	5.309	5.33	30	20	0.5	0.7	-0.521	-27.079	-26.677
15	5	0.2	0.5	9.856	5.713	5.599	30	20	0.6	0.2	0.183	-25.035	-24.861
15	5	0.2	0.7	-10.242	-4.097	-3.765	30	20	0.6	0.5	-0.256	-25.187	-25.141
15	5	0.5	0.2	3.634	16.041	16.119	30	20	0.6	0.7	0.1	-25.282	-25.079
15	5	0.5	0.5	-1.772	11.103	10.99	30	30	0.2	0.2	0.905	-37.67	-30.916
15	5	0.5	0.7	-10.997	8.54	8.541	30	30	0.2	0.5	-0.626	-38.138	-31.72
15	5	0.6	0.2	9.453	15.968	16.075	30	30	0.2	0.7	-1.574	-38.38	-32.473
15	5	0.6	0.5	5.909	11.696	11.82	30	30	0.5	0.2	0.045	-24.436	-24.089
15	5	0.6	0.7	9.654	9.399	9.34	30	30	0.5	0.5	-0.509	-24.798	-24.617
15	10	0.2	0.2	2.597	-48.549	-36.968	30	30	0.5	0.7	-0.194	-24.503	-24.226
15	10	0.2	0.5	-6.183	-50.589	-41.012	30	30	0.6	0.2	0.457	-23.322	-23.08
15	10	0.2	0.7	-3.402	-49.337	-39.467	30	30	0.6	0.5	0.064	-23.224	-23.114
15	10	0.5	0.2	2.188	-32.991	-30.545	30	30	0.6	0.7	-0.076	-23.545	-23.49
15	10	0.5	0.5	0.793	-33.708	-31.654	30	40	0.2	0.2	1.383	-34.375	-28.764
15	10	0.5	0.7	0.025	-34.526	-32.589	30	40	0.2	0.5	-0.421	-35.044	-30.064
15	10	0.6	0.2	1.822	-30.649	-29.834	30	40	0.2	0.7	-1.567	-35.718	-31.128
15	10	0.6	0.5	1.872	-31.295	-30.632	30	40	0.5	0.2	-0.076	-23.337	-23.095
15	10	0.6	0.7	4.392	-31.398	-30.911	30	40	0.5	0.5	-0.053	-23.412	-23.18
15	20	0.2	0.2	0.533	-42.985	-33.471	30	40	0.5	0.7	-0.066	-23.021	-22.781
15	20	0.2	0.5	-3.369	-44.04	-35.625	30	40	0.6	0.2	0.231	-22.45	-22.297
15	20	0.2	0.7	0.616	-42.699	-33.427	30	40	0.6	0.5	0.028	-22.813	-22.734
15	20	0.5	0.2	0.972	-26.057	-24.997	30	40	0.6	0.7	-0.202	-22.456	-22.436

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15	20	0.5	0.5	-0.437	-26.877	-26.243								

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