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RESEARCH ARTICLE

On the kernel of the (κ, a)-Generalized fourier transform

Dmitry Gorbachev [1], Valerii Ivanov [2] and Sergey Tikhonov [3,4,5]

[1]Department of Applied Mathematics and Computer Science, Tula State University, 300012 Tula, Russia;
E-mail: dvgmail@mail.ru.
[2]Department of Applied Mathematics and Computer Science, Tula State University, 300012 Tula, Russia;
E-mail: ivaleryi@mail.ru.
[3]Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, 08193 Bellaterra, Barcelona, Spain;
E-mail: stikhonov@crm.cat.
[4]Catalan Institution for Research and Advanced Studies, Passeig de Lluís Companys 23, 08010 Barcelona, Spain;
E-mail: stikhonov@crm.cat.
[5]Department of Mathematics, Universitat Autònoma de Barcelona, Building C Science Faculty, 08193 Bellaterra, Barcelona, Spain; E-mail: stikhonov@crm.cat.

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Abstract

For the kernel $B_{\kappa,a}(x,y)$ of the (κ,a)-generalized Fourier transform $\mathcal{F}_{\kappa,a}$, acting in $L^2(\mathbb{R}^d)$ with the weight $|x|^{a-2}v_\kappa(x)$, where v_κ is the Dunkl weight, we study the important question of when $\|B_{\kappa,a}\|_\infty = B_{\kappa,a}(0,0) = 1$. The positive answer was known for $d \ge 2$ and $\frac{2}{a} \in \mathbb{N}$. We investigate the case $d = 1$ and $\frac{2}{a} \in \mathbb{N}$. Moreover, we give sufficient conditions on parameters for $\|B_{\kappa,a}\|_\infty > 1$ to hold with $d \ge 1$ and any a.

We also study the image of the Schwartz space under the $\mathcal{F}_{\kappa,a}$ transform. In particular, we obtain that $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R}^d)) = \mathcal{S}(\mathbb{R}^d)$ only if $a = 2$. Finally, extending the Dunkl transform, we introduce nondeformed transforms generated by $\mathcal{F}_{\kappa,a}$ and study their main properties.

1. Introduction

Let as usual Δ be the Laplacian operator in $\mathbb{R}^d$. For the Fourier transform

$$\mathcal{F}(f)(y) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-i\langle x,y\rangle}\,dx,$$

Howe [16] obtained the following spectral decomposition of $\mathcal{F}$ using the harmonic oscillator $-(\Delta - |x|^2)/2$ and its eigenfunctions forming the basis in $L^2(\mathbb{R}^d)$:

$$\mathcal{F} = \exp\Big(\frac{i\pi d}{4}\Big)\exp\Big(\frac{i\pi}{4}\,(\Delta - |x|^2)\Big).$$

Among other applications, this decomposition is useful to define the fractional power of the Fourier transform (see [4, 19]).

During the last 30 years, a lot of attention has been given to various generalizations of the Fourier transform. As an important example, to develop harmonic analysis on weighted spaces, the Dunkl transform was introduced in [12]. The Dunkl transform $\mathcal{F}_\kappa$ is defined with the help of a root system $\Omega \subset \mathbb{R}^d$, a reflection group $G \subset O(d)$, and multiplicity function $\kappa\colon \Omega \to \mathbb{R}_+$, such that κ is G-invariant.

Here, G is generated by reflections $\{\sigma_\alpha \colon \alpha \in \Omega\}$, where σ_α is a reflection with respect to hyperplane $(\alpha, x) = 0$.

The differential-difference Dunkl Laplacian operator Δ_κ plays the role of the classical Laplacian [20]. If $\kappa \equiv 0$, we have $\Delta_\kappa = \Delta$. Dunkl Laplacian allows us to define the Dunkl harmonic oscillator $\Delta_\kappa - |x|^2$ and the Dunkl transform

$$\mathcal{F}_\kappa = \exp\Big(\frac{i\pi}{2}\Big(\frac{d}{2} + \langle\kappa\rangle\Big)\Big) \exp\Big(\frac{i\pi}{4}(\Delta_\kappa - |x|^2)\Big),$$

where

$$\langle\kappa\rangle = \frac{1}{2}\sum_{\alpha\in\Omega}\kappa(\alpha).$$

Further extensions of Fourier and Dunkl transforms were obtained by Ben Saïd, Kobayashi, and Ørsted in [4]. They defined the a-deformed Dunkl harmonic oscillator

$$\Delta_{\kappa,a} = |x|^{2-a}\Delta_\kappa - |x|^a, \quad a > 0,$$

and the (κ, a)-generalized Fourier transform

$$\mathcal{F}_{\kappa,a} = \exp\Big(\frac{i\pi}{2}(\lambda_{\kappa,a} + 1)\Big) \exp\Big(\frac{i\pi}{2a}\Delta_{\kappa,a}\Big), \tag{1.1}$$

which is a two-parameter family of unitary operators in $L^2(\mathbb{R}^d, d\mu_{\kappa,a})$ equipped with the norm

$$\|f\|_{2,d\mu_{\kappa,a}} = \Big(\int_{\mathbb{R}^d}|f(x)|^2\,d\mu_{\kappa,a}(x)\Big)^{1/2}.$$

Here

$$\lambda_{\kappa,a} = \frac{2\lambda_\kappa}{a}, \quad \lambda_\kappa = \langle\kappa\rangle + \frac{d-2}{2}, \quad d\mu_{\kappa,a}(x) = c_{\kappa,a}v_{\kappa,a}(x)\,dx, \quad v_{\kappa,a}(x) = |x|^{a-2}v_\kappa(x),$$

$$v_\kappa(x) = \prod_{\alpha\in\Omega}|\langle\alpha, x\rangle|^{\kappa(\alpha)}, \quad c_{\kappa,a}^{-1} = \int_{\mathbb{R}^d} e^{-|x|^a/a}v_{\kappa,a}(x)\,dx.$$

Throughout the paper, we assume that $d + 2\langle\kappa\rangle + a - 2 = 2\lambda_\kappa + a > 0$ or, equivalently, $\lambda_{\kappa,a} > -1$. Note that under this condition, the weight function $v_{\kappa,a}$ is locally integrable.

For $a = 2$, (1.1) reduces to the Dunkl transform, while if $a = 2$ and $\kappa \equiv 0$, then (1.1) is the classical Fourier transform. For $a \neq 2$, we arrive at deformed Dunkl and Fourier transforms, which have various applications. In particular, for $a = 1$ and $\kappa \equiv 0$, the deformed Dunkl transform is the unitary inversion operator of the Schrödinger model of minimal representation of the group $O(N + 1, 2)$ [19].

The unitary operator $\mathcal{F}_{\kappa,a}$ on $L^2(\mathbb{R}^d, d\mu_{\kappa,a})$ can be written as the integral transform [4, (5.8)]

$$\mathcal{F}_{\kappa,a}(f)(y) = \int_{\mathbb{R}^d} B_{\kappa,a}(x, y)f(x)\,d\mu_{\kappa,a}(x)$$

with the continuous symmetric kernel $B_{\kappa,a}(x, y)$ satisfying $B_{\kappa,a}(0, y) = 1$. In particular, $B_{0,2}(x, y) = e^{-i\langle x,y\rangle}$. One of the fundamental questions in the theory of deformed transforms is to investigate basic properties of the kernel $B_{\kappa,a}(x, y)$, in particular, to know when it is uniformly bounded. To illustrate the importance of this property, note that the condition $|B_{\kappa,a}(x, y)| \le M$ implies the Hausdorff-Young inequality

$$\big\|\mathcal{F}_{\kappa,a}(f)\big\|_{p',d\mu_{\kappa,a}} \le M^{2/p-1}\big\|f\big\|_{p,d\mu_{\kappa,a}}, \quad 1 \le p \le 2, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

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A more important problem is to describe parameters so that there holds

$$\|B_{\kappa,a}\|_\infty = \sup_{x,y\in\mathbb{R}^d} |B_{\kappa,a}(x,y)| = B_{\kappa,a}(0,0) = 1. \tag{1.2}$$

In this case, the Hausdorff-Young inequality holds with the constant 1 and one can define the generalized translation operator $\tau^y f(x)$ in $L^2(\mathbb{R}^d, d\mu_{k,a})$ by

$$\mathcal{F}_{\kappa,a}(\tau^y f)(z) = B_{\kappa,a}(y,z)\mathcal{F}_{\kappa,a}(f)(z)$$

(see [15]), and moreover, its norm equals 1.

Let us list the known cases when (1.2) holds:

- for $a = 2$ [22] ($\mathcal{F}_{\kappa,2}$ reduces to the Dunkl transform);
- for $a = 1$ and $d = 1$, $\langle\kappa\rangle \geq 1/2$ [15, Section 6] [1] ;
- for $a = 1$ and $d \geq 2$, $\langle\kappa\rangle \geq 0$ [4, Theorem 5.11];
- for $\frac{2}{a} \in \mathbb{N}$ and $d \geq 2$, $\langle\kappa\rangle \geq 0$ [5, 10] [2] .

In this paper, we continue to study the case

$$\frac{2}{a} \in \mathbb{N}.$$

Its importance was discussed in [4], [7], and [10]. For example, in this case, one has a simple inversion formula for the (κ, a)-generalized Fourier transform [4, Theorem 5.3] and an explicit expression for the generalized translation operator τ^y [7]. Moreover, the kernel $B_{\kappa,2/n}$ with $n \in \mathbb{N}$ appears for dihedral groups G [10].

Our first goal in this paper is, on the one hand, to extend the list of parameters for which (1.2) holds for $\frac{2}{a} \in \mathbb{N}$; on the other hand, to point out the cases when (1.2) does not hold. The following theorem describes positive results, where, for completeness, we include all known cases.

Theorem 1.1 (see [10, $d \geq 2$]). *Let $0 < a \leq 1$, $\frac{2}{a} \in \mathbb{N}$. If $d = 1$, $\langle\kappa\rangle \geq \frac{1}{2}$ or $d \geq 2$, $\langle\kappa\rangle \geq 0$, then equality (1.2) is true.*

Our proof of Theorem 1.1 for $d = 1$ is based on an integral representation of $B_{\kappa,a}$ with the special kernel and a study of positiveness of this kernel. This approach is closely related to the theory of positive definite functions.

In the general case $d \geq 1$, we give the proof based on the approach developed in the papers [6, 10, 11], see Section 4.1, and the alternative proof based on representation with positive kernels, see Subsection 4.3.

With regard to negative results, we obtain the following theorem, where we specify parameters when Theorem 1.1 does not hold.

Theorem 1.2. *In either of the following cases:*

- *$d = 1$, $0 < a \leq 1$, and $\langle\kappa\rangle = \frac{1}{2} - \frac{a}{4}$, or*
- *$d \geq 1$, $a \in (1,2) \cup (2,\infty)$ and $\langle\kappa\rangle \geq 0$,*

we have

$$\|B_{\kappa,a}\|_\infty > 1. \tag{1.3}$$

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 in the case $d = 1$. In Section 3, we study the properties of the one-dimensional kernel $B_{\kappa,a}$ for $\lambda_{\kappa,a} < 0$. In particular, in Section 3.1, we investigate positive definiteness of kernels of the integral transforms generated by $\mathcal{F}_{\kappa,a}$. In Section 4, we prove Theorem 1.1 in full generality as well as Theorem 1.2 (Section 4.3).

[1] The case $d = 1$ was first considered in [4]. The assertion [4, Theorem 5.11] was corrected in [15].
[2] The case $\langle\kappa\rangle > 0$ was announced in [10, Remark 3]. The proof is similar to the one of [10, Theorem 9].

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In Section 5, we study the question of how the $\mathcal{F}_{\kappa,a}$ transform acts on Schwartz functions. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under the classical Fourier transform $\mathcal{F}_{0,2}$ and the Dunkl $\mathcal{F}_{\kappa,2}$ (see [8]), but the case of deformed transforms is more complicated. In fact, we show that $\mathcal{S}(\mathbb{R}^d)$ is *not invariant* under $\mathcal{F}_{\kappa,a}$ for $a \neq 2$, which contradicts a widely used statement in [18] (see Remark 5.3). If $\frac{a}{2} \notin \mathbb{N}$, then the generalized Fourier transform may not be infinitely differentiable, and if $\frac{2}{a} \notin \mathbb{N}$, then it may not be rapidly decreasing at infinity. For $d = 1$ and $\frac{2}{a} \in \mathbb{N}$, the generalized Fourier transform of $f \in \mathcal{S}(\mathbb{R})$ is rapidly decreasing due to the representation $\mathcal{F}_{\kappa,a}(f)(y) = F_1\big(|y|^{a/2}\big) + yF_2\big(|y|^{a/2}\big)$, where the even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$ (see Proposition 5.4).

Finally, in Section 6, we study one-dimensional nondeformed unitary transforms generated by $\mathcal{F}_{\kappa,a}$:

$$\mathcal{F}_r^\lambda(g)(v) = \int_{-\infty}^{\infty} e_{2r+1}(uv,\lambda)g(u)\,\frac{|u|^{2\lambda+1}\,du}{2^{\lambda+1}\Gamma(\lambda+1)},$$

where $r \in \mathbb{Z}_+$, $\lambda \geq -1/2$, and the kernel

$$e_{2r+1}(uv,\lambda) = j_\lambda(uv) + i(-1)^{r+1}\,\frac{(uv)^{2r+1}}{2^{2r+1}(\lambda+1)_{2r+1}}\,j_{\lambda+2r+1}(uv)$$

is an eigenfunction of the differential-difference operator

$$\delta_\lambda g(u) = \Delta_{\lambda+1/2}g(u) - 2r(\lambda+r+1)\,\frac{g(u)-g(-u)}{u^2}.$$

Here, $\Delta_{\lambda+1/2}$ is the one-dimensional Dunkl Laplacian for $\langle\kappa\rangle = \lambda + \frac{1}{2}$ and

$$(a)_0 = 1, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad n \geq 1,$$

is the Pochhammer symbol. Note that such unitary transforms give new examples of an important class of Bessel-Hankel type transforms with the kernel $k(uv)$ (see, e.g. [24, Chapter VIII]). In particular, they generalize the one-dimensional Dunkl transform ($r = 0$).

2. Proof of Theorem 1.1 in the one-dimensional case

In what follows, we assume that

$$d = 1, \quad a > 0, \quad \kappa = \langle\kappa\rangle \geq 0, \quad \lambda_\kappa = \kappa - \frac{1}{2}, \quad 2\lambda_\kappa + a > 0, \quad \lambda = \lambda_{\kappa,a} = \frac{2\lambda_\kappa}{a},$$

$$v_{\kappa,a}(x) = |x|^{2\kappa+a-2}, \quad d\mu_{\kappa,a}(x) = c_{\kappa,a}v_{\kappa,a}(x)\,dx, \quad c_{\kappa,a} = \frac{1}{2a^\lambda\Gamma(\lambda+1)},$$

and $\mathcal{F}_{\kappa,a}$ is the (κ,a)-generalized Fourier transform (1.1) on the real line. Firstly, let us investigate when the kernel of $\mathcal{F}_{\kappa,a}$ is uniformly bounded. Using [4, Section 5], we can write the kernel as

$$B_{\kappa,a}(x,y) = j_\lambda\Big(\frac{2}{a}\,|xy|^{a/2}\Big) + \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+2/a)}\,\frac{xy}{(ai)^{2/a}}\,j_{\lambda+\frac{2}{a}}\Big(\frac{2}{a}\,|xy|^{a/2}\Big), \tag{2.1}$$

where $j_\lambda(x) = 2^\lambda\Gamma(\lambda+1)x^{-\lambda}J_\lambda(x)$ is the normalized Bessel function and $J_\lambda(x)$ is the classical Bessel function. Then the asymptotic behavior of $J_\lambda(x)$ (see [25, Chapter VII, 7.1]) immediately allows us to derive the following

Proposition 2.1. *Let $d = 1$. The conditions*

$$0 < a \leq 2, \quad \kappa \geq \frac{1}{2} - \frac{a}{4}, \quad \textit{or} \quad a \geq 2, \quad \kappa \geq 0, \tag{2.2}$$

are necessary and sufficient for boundedness of the kernel $B_{\kappa,a}(x,y)$.

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The main goal of this section is to prove Theorem 1.1 for $d = 1$.

Proof. Note that $B_{\kappa,a}(x,y) = b_{\kappa,a}(xy)$, where

$$b_{\kappa,a}(x) = j_\lambda\Big(\frac{2}{a}\,|x|^{a/2}\Big) + \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+2/a)}\,\frac{x}{(ai)^{2/a}}\,j_{\lambda+\frac{2}{a}}\Big(\frac{2}{a}\,|x|^{a/2}\Big). \tag{2.3}$$

Therefore, under the conditions of Theorem 1.1, it suffices to establish the inequality $|b_{\kappa,a}(x)| \le 1$ for $x \in \mathbb{R}$.

Let $a = \frac{2}{R}$, $R \in \mathbb{N}$. Equality (2.3) can be written as

$$b_{\kappa,a}(x) = j_\lambda\Big(R|x|^{1/R}\Big) + \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+R)}\Big(\frac{R}{2}\Big)^R(-i)^R\,x j_{\lambda+R}\Big(R|x|^{1/R}\Big).$$

Let $x \in \mathbb{R}$. In the case $R = 2r+1$, $a = \frac{2}{2r+1}$, $r \in \mathbb{Z}_+$, and [3] $v = (2r+1)x^{\frac{1}{2r+1}}$, $x = \big(\frac{v}{2r+1}\big)^{2r+1}$, there holds

$$e_{2r+1}(v,\lambda) = b_{\kappa,a}\Big(\Big(\frac{v}{2r+1}\Big)^{2r+1}\Big) = j_\lambda(v) + i(-1)^{r+1}\,\frac{v^{2r+1}}{2^{2r+1}(\lambda+1)_{2r+1}}\,j_{\lambda+2r+1}(v). \tag{2.4}$$

In the case $R = 2r$, $a = \frac{1}{r}$, $r \in \mathbb{N}$, and $v = 2r|x|^{\frac{1}{2r}}\operatorname{sign} x$, $x = \big(\frac{v}{2r}\big)^{2r}\operatorname{sign} v$, we have

$$e_{2r}(v,\lambda) = b_{\kappa,a}\Big(\Big(\frac{v}{2r}\Big)^{2r}\operatorname{sign} v\Big) = j_\lambda(v) + (-1)^r\,\frac{v^{2r}}{2^{2r}(\lambda+1)_{2r}}\,j_{\lambda+2r}(v)\operatorname{sign} v. \tag{2.5}$$

In order to see that $|e_{2r+1}(v,\lambda)|, |e_{2r}(v,\lambda)| \le 1$, we will need several auxiliary results. We start with the following identity

$$\frac{v^2}{4(\lambda+1)(\lambda+2)}\,j_{\lambda+2}(v) = j_{\lambda+1}(v) - j_\lambda(v), \tag{2.6}$$

which follows from the recurrence relation for the Bessel function $J_\lambda(v)$ (see [25, Chapter III, 3.2]). Then, by induction, we establish

Lemma 2.2. *If $r \in \mathbb{N}$, then*

$$\frac{v^{2r}}{2^{2r}(\lambda+1)_{2r}}\,j_{\lambda+2r}(v) = (-1)^r j_\lambda(v) + \sum_{s=1}^{r-1}(-1)^{s+r}\binom{r}{s}\frac{(\lambda+r)_s}{(\lambda+1)_s}\,j_{\lambda+s}(v) + \frac{(\lambda+r+1)_{r-1}}{(\lambda+1)_{r-1}}\,j_{\lambda+r}(v). \tag{2.7}$$

Proof. For $r = 1$, the needed formula coincides with (2.6). Assume that (2.7) is valid for every $k \le r-1$ and λ. Denote by $a_s^r(\lambda)$, $s = 0, 1, \dots, r$, the coefficients by $j_{\lambda+s}(v)$ in the decomposition (2.7). Taking into account (2.6) and the inductive assumption, we derive that

$$\begin{aligned}\frac{v^{2r}}{2^{2r}(\lambda+1)_{2r}}\,j_{\lambda+2r}(v) &= \frac{v^{2r-2}}{2^{2r-2}(\lambda+1)_{2r-2}}\,\frac{v^2}{4(\lambda+2r-1)(\lambda+2r)}\,j_{(\lambda+2r-2)+2}(v)\\ &= \frac{v^{2r-2}}{2^{2r-2}(\lambda+1)_{2r-2}}\,\{j_{(\lambda+1)+2r-2}(v) - j_{\lambda+2r-2}(v)\}\\ &= \frac{\lambda+2r-1}{\lambda+1}\sum_{s=1}^{r} a_{s-1}^{r-1}(\lambda+1)j_{\lambda+s}(v) - \sum_{s=0}^{r-1} a_s^{r-1}(\lambda)j_{\lambda+s}(v).\end{aligned}$$

[3] As usual, $x^{\frac{1}{2r+1}} = -|x|^{\frac{1}{2r+1}}$ for $x < 0$.

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It is enough to show that

$$a_0^r(\lambda) = -a_0^{r-1}(\lambda), \quad a_r^r(\lambda) = \frac{\lambda+2r-1}{\lambda+1}\, a_{r-1}^{r-1}(\lambda+1),$$

$$a_s^r(\lambda) = \frac{\lambda+2r-1}{\lambda+1}\, a_{s-1}^{r-1}(\lambda+1) - a_s^{r-1}(\lambda), \quad s = 1,\dots,r-1.$$

Indeed, using the induction step, we have that

$$a_0^r(\lambda) = (-1)^r, \quad a_r^r(\lambda) = \frac{\lambda+2r-1}{\lambda+1}\,\frac{(\lambda+r+1)_{r-2}}{(\lambda+2)_{r-2}} = \frac{(\lambda+r+1)_{r-1}}{(\lambda+1)_{r-1}},$$

and, for $s = 1,\dots,r-1$,

$$\begin{aligned} a_s^r(\lambda) &= (-1)^{s+r}\left\{\frac{\lambda+2r-1}{\lambda+1}\binom{r-1}{s-1}\frac{(\lambda+r)_{s-1}}{(\lambda+2)_{s-1}} + \binom{r-1}{s}\frac{(\lambda+r-1)_s}{(\lambda+1)_s}\right\} \\ &= (-1)^{s+r}\binom{r}{s}\frac{(\lambda+r)_s}{(\lambda+1)_s}\left\{\frac{\lambda+2r-1}{\lambda+r+s-1}\,\frac{s}{r} + \frac{\lambda+r-1}{\lambda+r+s-1}\,\frac{r-s}{r}\right\} \\ &= (-1)^{s+r}\binom{r}{s}\frac{(\lambda+r)_s}{(\lambda+1)_s}, \end{aligned}$$

which completes the proof. □

Lemma 2.3. *For $r \in \mathbb{Z}_+$, we have*

$$\frac{v^{2r+1}}{2^{2r+1}(\lambda+1)_{2r+1}}\, j_{\lambda+2r+1}(v) = (-1)^{r+1}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{(\lambda+r+1)_s}{(\lambda+1)_s}\, j'_{\lambda+s}(v).$$

Proof. Using Lemma 2.2 and the equality

$$j'_\lambda(v) = -\frac{v}{2(\lambda+1)}\, j_{\lambda+1}(v),$$

we derive that

$$\begin{aligned} &\frac{v^{2r+1}}{2^{2r+1}(\lambda+1)_{2r+1}}\, j_{\lambda+2r+1}(v) = \frac{v}{2(\lambda+1)}\,\frac{v^{2r}}{2^{2r}(\lambda+2)_{2r}}\, j_{(\lambda+1)+2r}(v) \\ &= \frac{v}{2(\lambda+1)}\left\{(-1)^r j_{\lambda+1}(v) + \sum_{s=1}^{r-1}(-1)^{s+r}\binom{r}{s}\frac{(\lambda+r+1)_s}{(\lambda+2)_s}\, j_{\lambda+1+s}(v) + \frac{(\lambda+r+2)_{r-1}}{(\lambda+2)_{r-1}}\, j_{\lambda+1+r}(v)\right\} \\ &= (-1)^{r+1} j'_\lambda(v) + \sum_{s=1}^{r-1}(-1)^{s+r+1}\binom{r}{s}\frac{(\lambda+r+1)_s}{(\lambda+1)_s}\, j'_{\lambda+s}(v) - \frac{(\lambda+r+1)_r}{(\lambda+1)_r}\, j'_{\lambda+r}(v). \end{aligned}$$

□

Taking into account (2.4) and (2.5), Lemmas 2.2 and 2.3, and

$$j_\lambda(v) = c_\lambda \int_{-1}^{1}(1-t^2)^{\lambda-1/2}e^{-ivt}\,dt, \quad j'_\lambda(v) = -ic_\lambda\int_{-1}^{1}(1-t^2)^{\lambda-1/2}te^{-ivt}\,dt \tag{2.8}$$

with $c_\lambda = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda+1/2)}$, $\lambda > -1/2$ (see [25, Chapter III, 3.3], we arrive at the following integral representations of the functions $e_{2r+1}(v,\lambda)$, and $e_{2r}(v,\lambda)$.

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Lemma 2.4. *If $r \in \mathbb{Z}_+$, $\lambda > -1/2$, then*

$$e_{2r+1}(v,\lambda) = c_\lambda \int_{-1}^{1} (1-t^2)^{\lambda-1/2} q_{2r+1}(t,\lambda) e^{-ivt}\, dt, \tag{2.9}$$

where $q_{2r+1}(t,\lambda)$ is a polynomial of degree $2r+1$ with respect to t given by

$$q_{2r+1}(t,\lambda) = 1 + t\sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{(\lambda+r+1)_s}{(\lambda+1/2)_s} (1-t^2)^s.$$

Lemma 2.5. *If $r \in \mathbb{N}$, $\lambda > -1/2$, then*

$$e_{2r}(v,\lambda) = c_\lambda \int_{-1}^{1} (1-t^2)^{\lambda-1/2} q_{2r}(t,\lambda) e^{-ivt}\, dt,$$

where $q_{2r}(t,\lambda)$ is a polynomial of degree $2r$ with respect to t given by

$$q_{2r}(t,\lambda) = q_{2r}(t,v,\lambda) = 1 + \operatorname{sign} v \left\{ \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{(\lambda+r)_s}{(\lambda+1/2)_s} (1-t^2)^s \right\}.$$

For our further analysis, it is important to know for which λ the polynomials $q_{2r}(t,\lambda)$, $q_{2r+1}(t,\lambda)$ are nonnegative on $[-1,1]$. If $r = 0$, $q_1(t,\lambda) = 1-t \geq 0$ on $[-1,1]$, that is, q_1 does not depend on λ. This special case corresponds to parameters $a = 2$, $\kappa \geq 0$, and hence (2.9) is a well-known integral representation of the kernel of the one-dimensional Dunkl transform. In other cases, positivity conditions for $q_{2r}(t,\lambda)$ and $q_{2r+1}(t,\lambda)$ depend on λ. We will see that there holds

$$q_{2r}(t,\lambda) = 1 + \operatorname{sign} v\, P_{2r}^{(\lambda-1/2)}(t), \quad q_{2r+1}(t,\lambda) = 1 + P_{2r+1}^{(\lambda-1/2)}(t), \tag{2.10}$$

where $\{P_n^{(\alpha)}(t)\}_{n=0}^{\infty}$ is the system of Gegenbauer (ultraspherical) polynomials, that is, the family of polynomials orthogonal on $[-1,1]$ with respect to the weight function $(1-t^2)^{\alpha}$, $\alpha > -1$, normalized by $P_n^{(\alpha)}(1) = 1$. Note that, $P_n^{(-1/2)}(t) = \cos(n \arccos t)$ are the Chebyshev polynomials. For $\lambda > -1/2$, we have

$$C_n^{\lambda}(t) = \frac{\Gamma(2\lambda+n)}{n!\,\Gamma(2\lambda)} P_n^{(\lambda-1/2)}(t), \quad \lambda \neq 0, \quad \text{and} \quad \lim_{\lambda\to 0} \frac{1}{\lambda} C_n^{\lambda}(t) = \frac{2}{n} P_n^{(-1/2)}(t), \tag{2.11}$$

where $C_n^{\lambda}(t)$ are the Gegenbauer polynomials given in [1, Chapter X, 10.9].

Lemma 2.6. *For $\lambda > -1/2$ and $r \geq 0$, there hold*

$$P_{2r}^{(\lambda-1/2)}(t) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{(\lambda+r)_s}{(\lambda+1/2)_s} (1-t^2)^s$$

and

$$P_{2r+1}^{(\lambda-1/2)}(t) = t\sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{(\lambda+r+1)_s}{(\lambda+1/2)_s} (1-t^2)^s.$$

Therefore, (2.10) holds.

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Proof. Since $\frac{(-r)_s}{s!} = (-1)^s\binom{r}{s}$, taking into account (2.11), [1, Chapter X, 10.9(21)], we get

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{(\lambda+r)_s}{(\lambda+1/2)_s}(1-t^2)^s = \sum_{s=0}^{r}\frac{(-r)_s(\lambda+r)_s}{s!\,(\lambda+1/2)_s}(1-t^2)^s$$
$$= {}_2F_1(-r,\lambda+r;\lambda+1/2;1-t^2) = \frac{(2r)!\,\Gamma(2\lambda+1)}{2\Gamma(2\lambda+2r)}\frac{1}{\lambda}C_{2r}^{\lambda}(t) = P_{2r}^{(\lambda-1/2)}(t).$$

Similarly, using (2.11), [1, Chapter X, 10.9(22)], we arrive at

$$t\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{(\lambda+r+1)_s}{(\lambda+1/2)_s}(1-t^2)^s = t\sum_{s=0}^{r}\frac{(-r)_s(\lambda+r+1)_s}{s!\,(\lambda+1/2)_s}(1-t^2)^s$$
$$= t\,{}_2F_1(-r,\lambda+r+1;\lambda+1/2;1-t^2) = \frac{(2r+1)!\,\Gamma(2\lambda+1)}{2\Gamma(2\lambda+2r+1)}\frac{1}{\lambda}C_{2r+1}^{\lambda}(t) = P_{2r+1}^{(\lambda-1/2)}(t).$$

□

We complete the proof of Theorem 1.1 noting that $|P_n^{(\lambda-1/2)}(t)| \le 1$ for $t \in [-1,1]$ under the condition $\lambda \ge 0$ or, equivalently, $\kappa \ge 1/2$ (see [23, Chapter VII, 7.32.2]). Then in light of Remark 3.2 below and Lemma 2.6, the polynomials $q_{2r}(t,\lambda)$ and $q_{2r+1}(t,\lambda)$ are nonnegative on $[-1,1]$ and therefore Lemmas 2.4 and 2.5, together with (2.4) and (2.5), yield the statement of Theorem 1.1. To illustrate this, for $a = \frac{2}{2r+1}$, $v = (2r+1)x^{\frac{1}{2r+1}}$, we have

$$|b_{\kappa,a}(x)| = |e_{2r+1}(v,\lambda)| \le e_{2r+1}(0,\lambda) = 1.$$

□

The following integral representations of the kernel $B_{\kappa,a}(x,y)$ follow from the lemmas above.

Corollary 2.7. *If $\lambda = (2\kappa-1)/a$ and $\kappa > \frac{1}{2} - \frac{a}{4}$, then for $x \in \mathbb{R}$*

$$B_{\kappa,a}(x,y) = c_\lambda\int_{-1}^{1}(1-t^2)^{\lambda-1/2}(1+P_{2r+1}^{(\lambda-1/2)}(t))e^{-i(2r+1)(xy)^{\frac{1}{2r+1}}t}\,dt, \quad a = \frac{2}{2r+1}, \quad r\in\mathbb{Z}_+,$$

and

$$B_{\kappa,a}(x,y) = c_\lambda\int_{-1}^{1}(1-t^2)^{\lambda-1/2}(1+\operatorname{sign}(xy)\,P_{2r}^{(\lambda-1/2)}(t))e^{-i(2r|xy|^{\frac{1}{2r}}\operatorname{sign}(xy))t}\,dt, \quad a = \frac{1}{r}, \quad r\in\mathbb{N}.$$

Remark 2.8. The representations of $B_{\kappa,a}(x,y)$ given in Corollary 2.7 can be also obtained from

$$n!\int_0^{\pi} e^{iz\cos\theta}C_n^{\lambda}(\cos\theta)(\sin\theta)^{2\lambda}\,d\theta = 2^{\lambda}\sqrt{\pi}\,\Gamma(\lambda+1/2)(2\lambda)_n\,i^n z^{-\lambda}J_{\lambda+n}(z)$$

(see [1, Chapter X, 10.9(38)]) without applying Lemmas 2.2 and 2.3. These lemmas are of independent interest.

Remark 2.9. The positiveness of the polynomials $q_{2r}(t,\lambda)$, $q_{2r+1}(t,\lambda)$ is sufficient, but not necessary, for the estimate $|B_{\kappa,a}(x,y)| \le 1$ to hold. In [15], it was mentioned that for $a=1$, this estimate holds also for $1/4 < \kappa_0 \le \kappa < 1/2$, $\kappa_0 \approx 0.44$.

Note that the case $\kappa < 1/2$ corresponds to $\lambda < 0$, which we study in detail in the next section.

3. The case $d = 1$ and $\lambda_{\kappa,a} < 0$

Let us investigate whether the polynomials $q_{2r}(t,\lambda)$ and $q_{2r+1}(t,\lambda)$ are nonnegative for $\lambda = \lambda_{\kappa,a} = (2\kappa-1)/a < 0$. In order to do this, we decompose them by the polynomials $q_{2r}(t,0)$ and $q_{2r+1}(t,0)$. First, we decompose the Gegenbauer polynomials in terms of the Chebyshev polynomials $P_n^{(-1/2)}(\cos t) = \cos nt$

using the well-known result (see [1, 10.9(17)])

$$C_n^\lambda(\cos\theta) = \sum_{s=0}^{n} \frac{(\lambda)_s(\lambda)_{n-s}}{s!\,(n-s)!}\cos(n-2s)\theta. \tag{3.1}$$

Let us start with the case $R = 2r + 1$.

Lemma 3.1. *For any $r \in \mathbb{Z}_+$ and $\lambda > -1/2$, there holds*

$$P_{2r+1}^{(\lambda-1/2)}(t) = \sum_{s=0}^{r} b_s^r(\lambda) P_{2r+1-2s}^{(-1/2)}(t), \tag{3.2}$$

where

$$b_s^r(\lambda) = \frac{(2r+2-s)_s(\lambda)_s(\lambda+r+1)_{r-s}}{4^r(\lambda+1/2)_r\,s!}, \qquad s = 0,1,\dots,r. \tag{3.3}$$

Proof. Taking into account (2.11) and (3.1), we obtain

$$\frac{\Gamma(2\lambda+2r+1)}{(2r+1)!\,\Gamma(2\lambda)} P_{2r+1}^{(\lambda-1/2)}(t) = C_{2r+1}^\lambda(t) = 2\sum_{s=0}^{r} \frac{(\lambda)_s(\lambda)_{2r+1-s}}{s!\,(2r+1-s)!} P_{2r+1-2s}^{(-1/2)}(t).$$

Hence, the duplication formula for the gamma function implies (3.3). □

Remark 3.2. We see that in the decomposition (3.2), the zero coefficient is positive, and all other coefficients are also positive for $\lambda > 0$ and negative for $\lambda < 0$. Note that the normalization of the Gegenbauer polynomials $P_n^{(\alpha)}(1) = 1$ implies the equality

$$\sum_{s=0}^{r} b_s^r(\lambda) = 1,$$

and for $\lambda > 0$, the Gegenbauer polynomials $P_{2r+1}^{(\lambda-1/2)}(t)$ are the convex hull of the Chebyshev polynomials $P_{2r+1-2s}^{(-1/2)}(t)$, $s = 0,1,\dots,r$. In particular, one has $|P_{2r+1}^{(\lambda-1/2)}(t)| \le 1$, $t \in [-1,1]$, $\lambda \ge 0$.

Thus, we are in a position to state the required result on the decomposition for the polynomial $q_{2r+1}(t,\lambda)$.

Corollary 3.3. *For any $r \in \mathbb{Z}_+$ and $\lambda > -1/2$, there holds*

$$q_{2r+1}(t,\lambda) = \sum_{s=0}^{r} b_s^r(\lambda) q_{2r-2s+1}(t,0).$$

Using (2.10) and the properties of the coefficients $b_s^r(\lambda)$, $s = 0,1,\dots,r$, from Remark 3.2, we establish the following result.

Corollary 3.4. *For any $r \in \mathbb{N}$ and $\lambda \in (-1/2, 0)$, the polynomial $q_{2r+1}(t,\lambda)$ is negative at the points of local minimum of the Chebyshev polynomial $P_{2r+1}^{(-1/2)}(t) = \cos((2r+1)\arccos t)$.*

In the case $R = 2r$, similarly, one can obtain the decomposition of the polynomial $q_{2r}(t,\lambda)$.

Lemma 3.5. *For any $r \in \mathbb{N}$ and $\lambda > -1/2$, there holds*

$$P_{2r}^{(\lambda-1/2)}(t) = \sum_{s=0}^{r} d_s^r(\lambda) P_{2r-2s}^{(-1/2)}(t),$$

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where

$$d_s^r(\lambda) = \frac{2(2r+1-s)_s(\lambda)_s(\lambda+r)_{r-s}}{4^r(\lambda+1/2)_r s!}, \quad s = 0, 1, \ldots, r-1, \quad d_r^r(\lambda) = \frac{(r+1)_r(\lambda)_r}{4^r(\lambda+1/2)_r r!}.$$

Proof. In light of (2.11) and (3.1), we have

$$\frac{\Gamma(2\lambda+2r)}{(2r)!\,\Gamma(2\lambda)} P_{2r}^{(\lambda-1/2)}(t) = C_{2r}^{\lambda}(t) = 2\sum_{s=0}^{r-1} \frac{(\lambda)_s(\lambda)_{2r-s}}{s!\,(2r-s)!} P_{2r-2s}^{(-1/2)}(t) + \frac{((\lambda)_r)^2}{(r!)^2} P_0^{(-1/2)}(t).$$

Then, using the duplication formula for gamma function, we obtain for $s = 0, 1, \ldots, r-1$

$$d_s^r(\lambda) = \frac{2(2r)!\,(\lambda)_s\Gamma(2\lambda)\Gamma(\lambda+2r-s)}{(2r-s)!\,s!\,\Gamma(\lambda)\Gamma(2\lambda+2r)} = \frac{2(2r+1-s)_s(\lambda)_s(\lambda+r)_{r-s}}{4^r(\lambda+1/2)_r s!},$$

while for $s = r$,

$$d_r^r(\lambda) = \frac{(2r)!\,(\lambda)_r\Gamma(2\lambda)\Gamma(\lambda+r)}{(r!)^2\Gamma(\lambda)\Gamma(2\lambda+2r)} = \frac{(r+1)_r(\lambda)_r}{4^r(\lambda+1/2)_r r!}.$$

□

Corollary 3.6. *For any $r \in \mathbb{N}$ and $\lambda > -1/2$, the following decomposition holds*

$$q_{2r}(t,\lambda) = \sum_{s=0}^{r} d_s^r(\lambda) q_{2r-2s}(t,0).$$

Since $d_0^r(\lambda) > 0$ and $\operatorname{sign} d_s^r(\lambda) = \operatorname{sign}\lambda$, $s = 1, \ldots, r$, for $\lambda > -1/2$, taking into account (2.10), we arrive at the following result.

Corollary 3.7. *If $r \in \mathbb{N}$ and $\lambda \in (-1/2, 0)$, then the polynomial $q_{2r}(t,\lambda)$ is negative at the points of local extremum of the Chebyshev polynomial $P_{2r}^{(-1/2)}(t) = \cos(2r \arccos t)$.*

Corollaries 3.4 and 3.7 show that the polynomials $q_{2r}(t,\lambda)$ and $q_{2r+1}(t,\lambda)$ for $r \geq 1$ change sign and, therefore, the method of the proof of the estimate $|B_{\kappa,a}(x,y)| \leq 1$ used in Theorem 1.1 cannot be applied when $\lambda < 0$. In this case, the problem remains open.

3.1. On positive definiteness

Recall that a continuous function f on $\mathbb{R}$ is positive definite if for any $\{v_s\}_{s=1}^n \subset \mathbb{R}$ and $\{z_s\}_{s=1}^n \subset \mathbb{C}$ there holds

$$\sum_{s,l=1}^{n} z_s \overline{z_l} f(v_s - v_l) \geq 0.$$

Bochner's theorem states that any continuous positive definite function $f(v)$, $f(0) = 1$, is the Fourier transform of a probability measure.

Recall that the function $e_{2r+1}(v,\lambda)$ is given in (2.4). By (2.9) and (2.10),

$$e_{2r+1}(v,\lambda) = c_\lambda \int_{-1}^{1} (1-t^2)^{\lambda-1/2}(1 + P_{2r+1}^{(\lambda-1/2)}(t))e^{-ivt}\,dt.$$

Then, by Remark 3.2 and Corollaries 3.3 and 3.4, we deduce the following

Corollary 3.8. *The function $e_{2r+1}(v,\lambda)$ is positive definite if and only if $\lambda \geq 0$.*

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4. The kernel of the generalized Fourier transform in the multivariate case

4.1. When the kernel is bounded by one

In this section, for the sake of completeness, we give the proof of Theorem 1.1 in the general case, following ideas from the paper [10]. We stress that the proof below also allows one to deal with the case $d = 1$.

Let $d \in \mathbb{N}$, $a > 0$, $\langle\kappa\rangle \geq 0$, and $\eta = \lambda_\kappa = \langle\kappa\rangle + \frac{d-2}{2} > 0$. For $w \geq 0$ and $\tau \in [-1, 1]$, define

$$\Psi_a^d(w, \tau) = 2^{2\eta/a}\Gamma\Big(\frac{2\eta + a}{a}\Big)\sum_{j=0}^{\infty} e^{-\frac{i\pi j}{a}}\,\frac{\eta + j}{\eta}\,w^{-2\eta/a}J_{\frac{2(\eta+j)}{a}}(w)C_j^{\eta}(\tau). \tag{4.1}$$

Putting in (4.1)

$$x, y \in \mathbb{R}^d, \quad x = |x|x', \quad y = |y|y', \quad x', y' \in \mathbb{S}^{d-1}, \quad w = \frac{2}{a}\,(|x||y|)^{a/2}, \quad \tau = \langle x', y'\rangle, \tag{4.2}$$

we obtain the function

$$K_a^d(x, y) = a^{2\eta/a}\Gamma\Big(\frac{2\eta + a}{a}\Big)(|x||y|)^{-\eta}\sum_{j=0}^{\infty} e^{-\frac{i\pi j}{a}}\,\frac{\eta + j}{\eta}\,J_{\frac{2(\eta+j)}{a}}\Big(\frac{2}{a}(|x||y|)^{a/2}\Big)C_j^{\eta}(\langle x', y'\rangle).$$

Note that $K_a^d(x, y) = K_a^d(y, x)$ and $\Psi_a^d(0, \tau) = K_a^d(0, y) = 1$.

Let $V_\kappa f(x)$ be the intertwining operator in the Dunkl harmonic analysis [20], which is a positive operator satisfying

$$V_\kappa f(x) = \int_{\mathbb{R}^d} f(\xi)\, d\mu_x^\kappa(\xi).$$

The representing measures $\mu_x^\kappa(\xi)$ are compactly supported probability measures with $\operatorname{supp}\mu_x^\kappa(\xi) \subset \operatorname{co}\{gx : g \in G\}$ [22].

The kernel of the generalized Fourier transform $\mathcal{F}_{\kappa,a}$ is given by [4, Chapters 4, 5]

$$\begin{aligned} B_{\kappa,a}(x, y) &= V_\kappa K_a^d(x, |y|\cdot)(y') \\ &= a^{2\eta/a}\Gamma\Big(\frac{2\eta + a}{a}\Big)(|x||y|)^{-\eta}\sum_{j=0}^{\infty} e^{-\frac{i\pi j}{a}}\,\frac{\eta + j}{\eta}\,J_{\frac{2(\eta+j)}{a}}\Big(\frac{2}{a}(|x||y|)^{a/2}\Big)V_\kappa C_j^{\eta}(\langle x', \cdot\rangle)(y'). \end{aligned} \tag{4.3}$$

If $\langle\kappa\rangle = 0$, that is, $\eta = (d-2)/2$, then $V_\kappa = \mathrm{I}$ is the identity operator and $K_a^d(x, y) = B_{0,a}(x, y)$.

Since the operator V_κ is positive and $V_\kappa 1 = 1$, the proof of equality (1.2) can proceed as follows: If $|\Psi_a^d(w, \tau)| \leq 1$, then $|B_{\kappa,a}(x, y)| \leq 1$.

Let further $\frac{2}{a} \in \mathbb{N}$, $a = \frac{2}{R}$. The inequality $|\Psi_a^d(w, \tau)| \leq 1$ was proved in [10] under the conditions $\eta > 0$ and $\eta = (d-2)/2$. We note that the proof of this estimate in [10] also holds for $\eta = \langle\kappa\rangle + (d-2)/2 > 0$. Hence, (1.2) is valid for $\langle\kappa\rangle + (d-2)/2 > 0$. If $\eta = 0$, then $d = 1$, $\kappa = 1/2$ or $d = 2$, $\kappa \equiv 0$, and in these cases, (1.2) holds as well (see Theorem 1.1 and [5]). This completes the proof of Theorem 1.1.

4.2. Positive definiteness of Ψ_a^d. Another proof of Theorem 1.1

In [10], to evaluate the kernel $B_{0,a}(x, y)$, the authors used the Laplace transform of the function Ψ_a^d given by (4.1). In the general case $d \geq 1$, we give another proof of the estimate $|\Psi_a^d| \leq 1$ based on the approach from the papers [6, 11], which allows one to show positive definiteness of the function

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$\Psi_a^d(w,\tau)$ with respect to w. Note also that for $a=1,2$, the functions

$$\Psi_1^d(w,\tau)=j_{\lambda_\kappa-1/2}(w\sqrt{(1+\tau)/2}),\quad \Psi_2^d(w,\tau)=e^{-iw\tau},$$

are positive definite (see [4, Example 4.18], [10]).

Let

$$I_\eta(b)=\Big(\frac{b}{2}\Big)^\eta\sum_{j=0}^\infty\frac{1}{j!\,\Gamma(j+\eta+1)}\Big(\frac{b^2}{4}\Big)^j$$

be the modified Bessel function of the first kind (see [1, Chapter 7, 7.2.2]) and

$$\Phi_2^{(m)}(\beta_1,\dots,\beta_m;\gamma;x_1,\dots,x_m)=\sum_{j_1,\dots,j_m\geq 0}\frac{(\beta_1)_{j_1}\cdots(\beta_m)_{j_m}}{(\gamma)_{j_1+\dots+j_m}}\frac{x_1^{j_1}}{j_1!}\cdots\frac{x_m^{j_m}}{j_m!}$$

be the second Humbert function of m variables (see [14, Chapter 2, 2.1.1.2]). In the case when $\gamma-\sum_{j=1}^m\beta_j>0$ and $\beta_j>0$, $j=1,\dots,m$, the hypergeometric function $\Phi_2^{(m)}$ admits the following integral representation

$$\Phi_2^{(m)}(\beta_1,\dots,\beta_m;\gamma;x_1,\dots,x_m)=C_\beta^{(\gamma)}\int_{T^m}e^{\sum_{j=1}^m x_jt_j}\Big(1-\sum_{j=1}^m t_j\Big)^{\gamma-\sum_{j=1}^m\beta_j-1}\prod_{j=1}^m t_j^{\beta_j-1}\,dt_1\cdots dt_m,\qquad(4.4)$$

where

$$C_\beta^{(\gamma)}=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\sum_{j=1}^m\beta_j)\prod_{j=1}^m\Gamma(\beta_j)}$$

and

$$T^m=\Big\{(t_1,\dots,t_m)\colon t_j\geq 0,\ j=1,\dots,m,\ \sum_{j=1}^m t_j\leq 1\Big\}$$

is the unit simplex in $\mathbb{R}^m$ ([9, 17]).

Taking into account the results from [6, 11], we obtain the following proposition, which gives an alternative proof of Theorem 1.1.

Proposition 4.1. *Let $d\in\mathbb{N}$, $a=\frac{2}{R}$, $R\in\mathbb{N}$, $R\geq 2$, $\eta=\langle\kappa\rangle+\frac{d-2}{2}>0$, $\tau\in[-1,1]$, and $q=\arccos\tau$. The function $w\mapsto\Psi_a^d(w,\tau)$ is a positive definite entire function of exponential type*

$$\theta(a,\tau)=\begin{cases}\cos\frac{q}{R}, & R=2r \text{ and } 0\leq q\leq\pi \text{ or } R=2r+1 \text{ and } 0\leq q\leq\pi/2,\\ \cos\frac{\pi-q}{R}, & R=2r+1 \text{ and } \pi/2\leq q\leq\pi.\end{cases}\qquad(4.5)$$

Proof. Let us consider the function

$$f_{R,\eta}(b,\tau)=\Gamma(R\eta+1)\Big(\frac{b}{2}\Big)^\eta\sum_{j=0}^\infty\frac{j+\eta}{\eta}I_{R(j+\eta)}(b)C_j^\eta(\tau),\qquad R\in\mathbb{N}.$$

Let $\tau=\langle x',y'\rangle$, $b=-iw$, $R=\frac{2}{a}$, $R\geq 2$. Since $I_\eta(-iw)=e^{-i\frac{\pi\eta}{2}}J_\eta(w)$ [1, Chapter 7, 7.2.2], then $f_{R,\eta}(b,\tau)=\Psi_a^d(w,\tau)$. It is known (see [6, 11]) that

$$f_{R,\eta}(b,\tau)=b_0\Phi_2^{(R-1)}(\eta,\dots,\eta;R\eta;b_1-b_0,\dots,b_{R-1}-b_0),$$

where $b_j = b\cos\left(\frac{q-2\pi j}{R}\right)$, $j = 0, \dots, R-1$. In light of (4.4) and using $\eta > 0$, we get

$$\begin{aligned}\Psi_a^d(w,\tau) &= f_{R,\eta}(-iw,\tau)\\ &= C_\eta \int_{T^{R-1}} e^{-iw\left\{\cos\left(\frac{q}{R}\right)+\sum_{j=1}^{R-1}\left(\cos\left(\frac{q-2\pi j}{R}\right)-\cos\left(\frac{q}{R}\right)\right)t_j\right\}}\Bigl(1-\sum_{j=1}^{R-1} t_j\Bigr)^{\eta-1}\prod_{j=1}^{R-1} t_j^{\eta-1}\,dt_1\cdots dt_{R-1},\end{aligned} \tag{4.6}$$

where $C_\eta = \Gamma(R\eta)/(\Gamma(\eta))^R$. For $\alpha > 0$ and $\beta \in \mathbb{R}$, the function $\alpha e^{i\beta w}$ is positive definite, hence the function $\Psi_a^d(w,\tau)$ is also positive definite with respect to w.

The representation (4.6) implies that $\Psi_a^d(w,\tau)$ is an entire function of exponential type with respect to w. Let us calculate its type. Since for $j = 1, \dots, R-1$, $q = \arccos\tau$, $\tau \in [-1,1]$, $\cos\frac{q}{R} - \cos\frac{q-2\pi j}{R} = 2\sin\frac{\pi i}{R}\,\sin\frac{\pi j - q}{R} \geq 0$, then for any $(t_1, \dots, t_{R-1}) \in T^{R-1}$,

$$\min_{1\leq j\leq R-1}\cos\frac{q-2\pi j}{R} \leq \cos\frac{q}{R} + \sum_{j=1}^{R-1}\Bigl(\cos\frac{q-2\pi j}{R} - \cos\frac{q}{R}\Bigr)t_j \leq \cos\frac{q}{R}.$$

Since

$$\min_{1\leq j\leq R-1}\cos\frac{q-2\pi j}{R} = \begin{cases} -\cos\frac{q}{R}, & R = 2r \text{ or } R = 2r+1 \text{ and } 0 \leq q \leq \pi/2,\\ -\cos\frac{\pi-q}{R}, & R = 2r+1 \text{ and } \pi/2 \leq q \leq \pi,\end{cases}$$

the type of the function $\Psi_a^d(w,\tau)$ is given by (4.5). □

Remark 4.2. Let $t, \tau \in (-1,1)$, $\lambda = 2\eta/a$, $a = 1/r$, $r \in \mathbb{N}$. Then the function $\Psi_a^d(w,\tau)$ has the following integral representation

$$\Psi_a^d(w,\tau) = c_\lambda \int_{-1}^{1} (1-t^2)^{\lambda-1/2} \sum_{j=0}^{\infty} \frac{\eta+j}{\eta} P_{2j/a}^{(\lambda-1/2)}(t) C_j^\eta(\tau) e^{-iwt}\,dt. \tag{4.7}$$

Indeed, using (2.8) and (4.1) and Lemmas 2.2 and 2.6, we get

$$\begin{aligned}\Psi_a^d(w,\tau) &= \sum_{j=0}^{\infty}(-1)^{rj}\frac{\eta+j}{\eta}\frac{w^{2rj}}{2^{2rj}(\lambda+1)_{2rj}}\, j_{\lambda+2rj}(w) C_j^\eta(\tau)\\ &= \sum_{j=0}^{\infty}\frac{\eta+j}{\eta}\Bigl\{j_\lambda(w) + \sum_{s=1}^{rj-1}(-1)^s\binom{rj}{s}\frac{(\lambda+rj)_s}{(\lambda+1)_s}\, j_{\lambda+s}(w) + (-1)^{rj}\frac{(\lambda+rj+1)_{rj-1}}{(\lambda+1)_{rj-1}}\, j_{\lambda+rj}(w)\Bigr\}C_j^\eta(\tau)\\ &= c_\lambda\int_{-1}^{1}(1-t^2)^{\lambda-1/2}\sum_{j=0}^{\infty}\frac{\eta+j}{\eta}P_{2rj}^{(\lambda-1/2)}(t)C_j^\eta(\tau)e^{-iwt}\,dt.\end{aligned}$$

The integral representation (4.7) implies that

$$\psi_a^d(t,\tau) = \sum_{j=0}^{\infty}\frac{\eta+j}{\eta}P_{2rj}^{(\lambda-1/2)}(t)C_j^\eta(\tau) \geq 0, \quad t,\ \tau \in (-1,1),$$

is equivalent to the positive definiteness of the function $\Psi_a^d(w,\tau)$. Proposition 4.1 shows that the function $\psi_a^d(t,\tau)$ is positive and its support as a function of t lies on the interval $I_a = [-\theta(a,\tau), \theta(a,\tau)]$.

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In the case $a = 1$, one can easily write an explicit formula for $\psi_1^d(t,\tau)$. In more detail, we have $r = 1$, $\lambda = 2\eta$, and $\theta(a,\tau) = \cos\frac{q}{2} = \sqrt{(1+\tau)/2}$. Then using [2, Chapter VIII, 8.7(31)], we obtain

$$\begin{aligned}\psi_1^d(t,\tau) &= (2\pi c_\lambda)^{-1}(1-t^2)^{1/2-2\eta}\int_{-\infty}^{\infty} j_{\eta-1/2}(w\sqrt{(1+\tau)/2})e^{iwt}\,dw\\ &= c(\eta)(1-t^2)^{1/2-2\eta}(1+\tau)^{1/2-\eta}(1+\tau-2t^2)^{\eta-1}\chi_{I_1}(t)\\ &= c(\eta)(1-t^2)^{1/2-2\eta}(1+\tau)^{1/2-\eta}(\tau-P_2^{(-1/2)}(t))^{\eta-1}\chi_{I_1}(t),\end{aligned}$$

where $\chi_{I_1}(t)$ is the characteristic function of the interval I_1 and $P_2^{(-1/2)}(t) = \cos(2\arccos t)$.

4.3. Parameters when the kernel is not bounded by one

We will show that in some cases the uniform norm of the kernel $B_{\kappa,a}(x,y)$ is not bounded by 1 and can be either finite or infinite. We mention that the conditions when the norm is not finite are known only in the one-dimensional case. In particular, if $d = 1$, $0 < a < 2$, and $\kappa < \frac{1}{2} - \frac{a}{4}$, then $\|B_{\kappa,a}\|_\infty = \infty$ (see Proposition 2.1).

Proof of Theorem 1.2. Let first $d = 1$ and $\lambda = \lambda_{\kappa,a} = \frac{2\kappa-1}{a}$. Changing variables $x = (\frac{a}{2}|v|)^{2/a}\operatorname{sign} v$ in (2.3), we get

$$\begin{aligned}e_{\kappa,a}(v,\lambda) = b_{\kappa,a}\Big(\Big(\frac{a}{2}|v|\Big)^{2/a}\operatorname{sign} v\Big) &= j_\lambda(v) + \frac{\Gamma(\lambda+1)\cos\frac{\pi}{a}}{2^{2/a}\Gamma(\lambda+1+2/a)}|v|^{2/a}j_{\lambda+\frac{2}{a}}(v)\operatorname{sign} v\\ &\quad - i\,\frac{\Gamma(\lambda+1)\sin\frac{\pi}{a}}{2^{2/a}\Gamma(\lambda+1+2/a)}|v|^{2/a}j_{\lambda+\frac{2}{a}}(v)\operatorname{sign} v.\end{aligned} \tag{4.8}$$

If $0 < a \le 1$ and $\kappa = \frac{1}{2} - \frac{a}{4}$, then $\lambda = -1/2$, $j_{-1/2}(v) = \cos v$, $j_{1/2}(v) = \frac{\sin v}{v}$.

Assume first that $\cos\frac{\pi}{a} = 0$ or $a = \frac{2}{2r+1}$, $r \in \mathbb{N}$. Since $\sin\frac{\pi}{a} \ne 0$, $j_{-1/2}(2\pi) = 1$, $j_{1/2}(2\pi) = 0$, and $j_{2r+1/2}(2\pi) \ne 0$ (see [25, Chapter 15, 15.28]), then from (4.8), we get $|e_{\kappa,a}(2\pi,-1/2)| = |e_{2r+1}(2\pi,-1/2)| > 1$.

Let now $\cos(\pi/a) \ne 0$. In light of (4.8), there holds

$$|e_{\kappa,a}(v,\lambda)| \ge \Big|\cos v + \frac{\Gamma(\frac{1}{2})\cos\frac{\pi}{a}}{2^{2/a}\Gamma(\frac{2}{a}+\frac{1}{2})}|v|^{2/a}j_{\frac{2}{a}-\frac{1}{2}}(v)\operatorname{sign} v\Big|.$$

Since

$$(2\pi s)^{2/a}j_{\frac{2}{a}-\frac{1}{2}}(2\pi s) = \frac{2^{2/a}\Gamma(\frac{2}{a}+\frac{1}{2})}{\Gamma(\frac{1}{2})}\Big\{\cos\frac{\pi}{a} + O\Big(\frac{1}{s}\Big)\Big\}$$

as $s \to +\infty$ (see [25, Chapter VII, 7.1]), we deduce that for sufficiently large s

$$|e_{\kappa,a}(2\pi s,\lambda)| \ge 1 + \cos^2\frac{\pi}{a} + O\Big(\frac{1}{s}\Big) > 1,$$

which completes the proof of (1.3). Note that similar arguments also imply the proof of (1.3) for $1 < a < 2$ and $\kappa = \frac{1}{2} - \frac{a}{4}$.

Now, we consider the multivariate case. Suppose that $d \ge 1$, $a \in (1,2)\cup(2,\infty)$, $\langle\kappa\rangle \ge 0$, and $\eta = \lambda_\kappa$, $\lambda = \frac{2\eta}{a} = \frac{2\langle\kappa\rangle+d-2}{a}$. Recall that x', y', w are defined in (4.2). In view of (2.2) and the proof for the case $d = 1$, we may assume that $\eta, \lambda > -1/2$ and $\cos\frac{\pi}{a} \ne 0$.

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Note that (4.3) can be equivalently written as

$$B_{\kappa,a}(x,y)=\sum_{j=0}^{\infty}e^{-\frac{i\pi j}{a}}\,\frac{\eta+j}{\eta}\,\frac{\Gamma(\lambda+1)}{2^{2j/a}\,\Gamma(\lambda+1+2j/a)}\,w^{2j/a}j_{\lambda+\frac{2j}{a}}(w)V_{\kappa}C_j^{\eta}(\langle x',\cdot\rangle)(y').$$

For Gegenbauer polynomials (2.11), the following estimate holds

$$\sup_{-1\le t\le 1}\left|\frac{1}{\eta}\,C_j^{\eta}(t)\right|\le \widetilde{c}(\eta)j^{2\eta-1},\quad j\ge 1,$$

[4, Lemma 4.9]. Using (2.11), we get

$$\left|\frac{\eta+j}{\eta}\,V_{\kappa}C_j^{\eta}(\langle x',\cdot\rangle)(y')\right|\le c(\eta)(j^{2\eta}+1).$$

Since $|j_\lambda(w)|\le 1$, $\frac{1}{\eta}\,C_1^{\eta}(t)=2t$, then for $0\le w\le 1$, we have

$$\left|\sum_{j=2}^{\infty}e^{-\frac{i\pi j}{a}}\,\frac{\eta+j}{\eta}\,\frac{\Gamma(\lambda+1)}{2^{2j/a}\Gamma(\lambda+1+2j/a)}\,w^{2j/a}j_{\lambda+\frac{2j}{a}}(w)V_{\kappa}C_j^{\eta}(\langle x',\cdot\rangle)(y')\right|$$
$$\le c(\eta)w^{4/a}\sum_{j=2}^{\infty}\frac{\Gamma(\lambda+1)(j^{2\eta}+1)}{2^{2j/a}\Gamma(\lambda+1+2j/a)}\le c(a,\kappa)w^{4/a}$$

and

$$\begin{aligned}B_{\kappa,a}(x,y)=j_\lambda(w)&+\frac{2(\eta+1)\Gamma(\lambda+1)\cos\frac{\pi}{a}}{2^{2/a}\Gamma(\lambda+1+2/a)}\,w^{2/a}j_{\lambda+\frac{2}{a}}(w)V_{\kappa}(\langle x',\cdot\rangle)(y')\\&-i\,\frac{2(\eta+1)\Gamma(\lambda+1)\sin\frac{\pi}{a}}{2^{2/a}\Gamma(\lambda+1+2/a)}\,w^{2/a}j_{\lambda+\frac{2}{a}}(w)V_{\kappa}(\langle x',\cdot\rangle)(y')+O(w^{4/a}),\quad w\to 0.\end{aligned}$$

Therefore,

$$|B_{\kappa,a}(x,y)|\ge\left|j_\lambda(w)+\frac{2(\eta+1)\Gamma(\lambda+1)\cos\frac{\pi}{a}}{2^{2/a}\Gamma(\lambda+1+2/a)}\,w^{2/a}j_{\lambda+\frac{2}{a}}(w)V_{\kappa}(\langle x',\cdot\rangle)(y')+O(w^{4/a})\right|. \tag{4.9}$$

Since the operator V_κ is an isomorphism on the space of homogeneous polynomials of degree 1, then for suitable $x',y'\in\mathbb{S}^{d-1}$, we deduce

$$V_{\kappa}(\langle x',\cdot\rangle)(y')=\int_{\mathbb{R}^d}\langle x',\xi\rangle\,d\mu_{y'}^{\kappa}(\xi)\neq 0,\quad \operatorname{sign}(V_{\kappa}(\langle x',\cdot\rangle)(y'))=\operatorname{sign}\,(\cos\,(\pi/a)). \tag{4.10}$$

We have

$$j_\lambda(w)=1+O(w^2),\quad j_{\lambda+\frac{2}{a}}(w)=1+O(w^2)$$

as $w\to 0$. This, (4.9) and (4.10) imply that, for given $x',y'\in\mathbb{S}^{d-1}$ as above and sufficiently small positive w,

$$|B_{\kappa,a}(x,y)|\ge 1+\frac{2(\eta+1)\Gamma(\lambda+1)|\cos\frac{\pi}{a}|}{2^{2/a}\Gamma(\lambda+1+2/a)}\,w^{2/a}|V_{\kappa}(\langle x',\cdot\rangle)(y')|+O(w^{4/a})+O(w^2)>1.\qquad\square$$

Remark 4.3.

(i) For $0<a\le 1$, $\frac{2}{a}\in\mathbb{N}$, equality (1.2) holds provided $\kappa\ge\kappa_0(a)$, where $\frac{1}{2}-\frac{a}{4}<\kappa_0(a)\le 1/2$. The problem of determining $\kappa_0(a)$ is open.

(ii) Theorem 1.2 shows that our conjecture in [15] asserting that (1.3) holds under the condition $2\langle\kappa\rangle + d + a \geq 3$ is not valid for $d \geq 1$ and $a \in (1, 2) \cup (2, \infty)$.

5. Image of the Schwartz space

In this section, for the (κ, a)-generalized Fourier transform, we study the image of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Denote by $\mathcal{S}(\mathbb{R}_+)$ the subspace of even functions from $\mathcal{S}(\mathbb{R})$.

5.1. The case of $a > 0$

Let $d\nu_\eta(u) = b_\eta u^{2\eta+1}\,du$, $b_\eta^{-1} = 2^\eta\Gamma(\eta+1)$, $d\nu_{\eta,a}(u) = b_{\eta,a}u^{2\eta+a-1}du$, $b_{\eta,a}^{-1} = a^{2\eta/a}\Gamma(2\eta/a+1)$,

$$H_\eta(f_0)(v) = \int_0^\infty f_0(u)j_\eta(uv)\,d\nu_\eta(u), \quad f_0 \in L^1(\mathbb{R}_+, d\nu_\eta), \quad \eta \geq -1/2,$$

be the Hankel transform and

$$H_{\eta,a}(f_0)(v) = \int_0^\infty f_0(u)j_{\frac{2\eta}{a}}\Big(\frac{2}{a}(vu)^{a/2}\Big)\,d\nu_{\eta,a}(u), \quad 2\eta + a \geq 1,$$

the a-deformed Hankel transform (see [4, 15]).

Recall that $\lambda_\kappa = \langle\kappa\rangle + (d-2)/2$, $\lambda = \lambda_{\kappa,a} = 2\lambda_\kappa/a$.

Example 5.1. Consider $f(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^d)$, $f(x) = f_0(\rho)$, $\rho = |x|$. If $|y| = v$, $\rho = (a/2)^{1/a}u^{2/a}$, then (see [4, 15])

$$\begin{aligned}\mathcal{F}_{\kappa,a}(f)(y) = H_{\lambda_\kappa,a}(f_0)(v) &= \int_0^\infty f_0(\rho)j_{\frac{2\lambda_\kappa}{a}}\Big(\frac{2}{a}(v\rho)^{a/2}\Big)\,d\nu_{\lambda_\kappa,a}(\rho)\\ &= \int_0^\infty g_a(u)j_\lambda\Big(\Big(\frac{2}{a}\Big)^{1/2}v^{a/2}u\Big)\,d\nu_\lambda(u),\end{aligned}$$

where

$$g_a(u) = \exp\Big(-\Big(\frac{a}{2}\Big)^{2/a}u^{4/a}\Big), \quad u \geq 0.$$

Assuming that $\mathcal{F}_{\kappa,a}(f)(y)$ is rapidly decreasing at infinity, the same is true for the function

$$G_a(v) = H_\lambda(g_a)(v) = \int_0^\infty g_a(u)j_\lambda(uv)\,d\nu_\lambda(u).$$

Considering even extensions of the functions g_a and G_a, we get the following representation via the Dunkl transform:

$$G_a(v) = \int_{-\infty}^\infty g_a(u)j_\lambda(uv)\,d\mu_{\kappa,2}(u) = \mathcal{F}_{\kappa,2}(g_a)(v), \quad v \in \mathbb{R}.$$

If $2/a$ is a noninteger, then either $4/a \notin \mathbb{N}$ or $4/a = 2s+1$, $s \in \mathbb{Z}_+$, and the function

$$g_a(u) = \exp\Big(-\Big(\frac{a}{2}\Big)^{2/a}|u|^{4/a}\Big), \quad u \in \mathbb{R},$$

has finite smoothness at the origin. On the other hand, since $g_a, G_a \in L^1(\mathbb{R}, d\mu_{\kappa,2})$, we obtain

$$g_a(u) = \mathcal{F}_{\kappa,2}(G_a)(u) = \int_{-\infty}^\infty G_a(v)j_\lambda(uv)\,d\mu_{\kappa,2}(v),$$

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and the right-hand side is infinitely differentiable at the origin if $G_a(v)$ is fast decreasing. This contradiction shows that, for $2/a \notin \mathbb{N}$, $\mathcal{F}_{\kappa,a}(f)(y)$ cannot rapidly decay at infinity and so it is not a Schwartz function.

Moreover, let us denote for convenience the derivative of f by ∂f. Since

$$(\partial_v^2 j_\lambda(uv))\big|_{v=0} = -\frac{u^2}{2(\lambda+1)}, \quad (\partial_v^4 j_\lambda(uv))\big|_{v=0} = \frac{3u^4}{4(\lambda+1)(\lambda+2)},$$

then

$$G_a(v) = \xi_0 + \xi_1 v^2 + O(v^4), \qquad v \to 0, \qquad \xi_1 \neq 0,$$

and

$$\mathcal{F}_{\kappa,a}(f)(y) = \xi_0 + \widetilde{\xi}_1 |y|^a + O(|y|^{2a}), \qquad y \to 0, \qquad \widetilde{\xi}_1 \neq 0.$$

If a is not even, then $\mathcal{F}_{\kappa,a}(f)(y)$ has finite smoothness at the origin.

The following statement follows from Example 5.1.

Proposition 5.2. *Let $d \in \mathbb{N}$.*

(i) *The condition $\frac{a}{2} \in \mathbb{N}$ is necessary for the embedding $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R}^d)) \subset C^\infty(\mathbb{R}^d)$ to hold.*
(ii) *The condition $\frac{2}{a} \in \mathbb{N}$ is necessary for the set $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R}^d))$ to consist of rapidly decreasing functions at infinity.*

Remark 5.3. We see that $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R}^d)) = \mathcal{S}(\mathbb{R}^d)$ only for $a = 2$. Hence, the claim in [18, Lemma 2.12] that the Schwartz space is invariant under the generalized Fourier transform is false for $a \neq 2$.

The conditions in Proposition 5.2 are also sufficient, at least in the one-dimensional case. Indeed, suppose $\lambda = (2\kappa - 1)/a \geq -1/2$ and denote the even and odd parts of f, as usual, by

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Then the one-dimensional generalized Fourier transform can be written as

$$\mathcal{F}_{\kappa,a}(f)(y) = 2\int_0^\infty (B_{\kappa,a}(\,\cdot\,,y))_e(x) f_e(x)\,d\mu_{\kappa,a}(x) + 2\int_0^\infty (B_{\kappa,a}(\,\cdot\,,y))_o(x) f_o(x)\,d\mu_{\kappa,a}(x). \quad (5.1)$$

Putting in (5.1)

$$\begin{gathered} x = x(u) = \left(\frac{a}{2}\right)^{\frac{1}{a}} u^{\frac{2}{a}}, \quad g(u) = Af(u) = f(x(u)) = f\left(\left(\frac{a}{2}\right)^{\frac{1}{a}} u^{\frac{2}{a}}\right), \\ g_e(u) = Af_e(u), \quad g_o(u) = Af_o(u), \quad x, u \in \mathbb{R}_+, \end{gathered} \quad (5.2)$$

and taking into account (2.1), we deduce

$$\begin{aligned} \mathcal{F}_{\kappa,a}(f)(y) = &\int_0^\infty Af_e(u) j_\lambda\left(\left(\frac{2}{a}\right)^{1/2} |y|^{a/2} u\right) d\nu_\lambda(u) \\ &+ e^{-\frac{\pi i}{a}} \left(\frac{2}{a}\right)^{1/a} y \int_0^\infty u^{-2/a} Af_o(u) j_{\lambda+\frac{2}{a}}\left(\left(\frac{2}{a}\right)^{1/2} |y|^{a/2} u\right) d\nu_{\lambda+\frac{2}{a}}(u) \\ &= H_\lambda(g_e)\left(\left(\frac{2}{a}\right)^{1/2} |y|^{a/2}\right) + e^{-\frac{\pi i}{a}} \left(\frac{2}{a}\right)^{1/a} y H_{\lambda+\frac{2}{a}}(u^{-2/a} g_o)\left(\left(\frac{2}{a}\right)^{1/2} |y|^{a/2}\right). \end{aligned} \quad (5.3)$$

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If $f \in \mathcal{S}(\mathbb{R})$, then $g_e(u)$, $u^{-2/a}g_o(u)$ decrease rapidly at infinity and there exist

$$\lim_{u\to 0+0} u^{-2/a}g_o(u) = \left(\frac{a}{2}\right)^{\frac{1}{a}} f'(0).$$

Therefore, the representation (5.3) shows that, for even a, we have the embedding $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$.

Further, if $\frac{2}{a} \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R})$, then the functions $g_e(u), u^{-2/a}g_o(u) \in \mathcal{S}(\mathbb{R}_+)$ as well as (see [20]) the even functions

$$H_\lambda(g_e)(v),\ H_{\lambda+\frac{2}{a}}(u^{-2/a}g_o)(v) \in \mathcal{S}(\mathbb{R}).$$

Hence, we prove the following

Proposition 5.4. *Suppose* $\frac{2}{a} \in \mathbb{N}$, $\lambda \geq -1/2$, $f \in \mathcal{S}(\mathbb{R})$, *then*

$$\mathcal{F}_{\kappa,a}(f)(y) = F_1\big(|y|^{a/2}\big) + yF_2\big(|y|^{a/2}\big), \tag{5.4}$$

where the even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$.

Thus, for $2/a \in \mathbb{N}$, the generalized Fourier transform decreases rapidly at infinity and we arrive at the following result.

Proposition 5.5. (i) *The embedding* $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$ *is valid if and only if* $\frac{a}{2} \in \mathbb{N}$.
(ii) *The set* $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R}))$ *consists of rapidly decreasing functions at infinity if and only if* $\frac{2}{a} \in \mathbb{N}$.

Let

$$\mathcal{S}_{org}(\mathbb{R}) = \{f \in \mathcal{S}(\mathbb{R}) \colon \partial^k f(0) = 0,\ k \in \mathbb{N}\}.$$

The class $\mathcal{S}_{org}(\mathbb{R})$ is dense in $L^2(\mathbb{R}, d\mu_{\kappa,a})$. Suppose $a > 0$, $f(x) \in \mathcal{S}_{org}(\mathbb{R})$, then the functions $g_e(u), u^{-2/a}g_o(u)$ belong to $\mathcal{S}(\mathbb{R}_+)$, cf. (5.3). Therefore, the even functions $H_\lambda(g_e)(v)$, $H_{\lambda+\frac{2}{a}}(u^{-2/a}g_o)(v)$ also belong to $\mathcal{S}(\mathbb{R})$.

Thus, we obtain the following

Proposition 5.6. *Suppose* $a > 0$, $\lambda \geq -1/2$, *and* $f \in \mathcal{S}_{org}(\mathbb{R})$; *then the generalized Fourier transform* $\mathcal{F}_{\kappa,a}(f)$ *enjoys the representation (5.4).*

5.2. The case of irrational a

We will show that if a is irrational, any nontrivial Schwartz function possesses similar properties to the Gaussian function in Example 5.1. To see this, we need auxiliary properties of the kernel of the generalized Fourier transform.

Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$, $x = \rho x'$, $\rho = |x| \in \mathbb{R}_+$, $x' \in \mathbb{S}^{d-1}$, and dx' be the Lebesgue measure on the sphere. If $a_\kappa^{-1} = \int_{\mathbb{S}^{d-1}} v_\kappa(x')\,dx'$, $d\sigma_\kappa(x') = a_\kappa v_\kappa(x')\,dx'$, then $d\mu_{\kappa,a}(x) = d\nu_{\lambda_\kappa,a}(\rho)\,d\sigma_\kappa(x')$ and $c_{\kappa,a} = b_{\kappa,a}a_\kappa$.

Denote by $\mathcal{H}_n^d(v_\kappa)$ the subspace of κ-spherical harmonics of degree $n \in \mathbb{Z}_+$ in $L^2(\mathbb{S}^{d-1}, d\sigma_\kappa)$ (see [13, Chapter 5]). Let $\mathcal{P}_n^d$ be the space of homogeneous polynomials of degree n in $\mathbb{R}^d$. Then $\mathcal{H}_n^d(v_\kappa)$ is the restriction of $\ker \Delta_\kappa \cap \mathcal{P}_n^d$ to the sphere $\mathbb{S}^{d-1}$.

If l_n is the dimension of $\mathcal{H}_n^d(v_\kappa)$, we denote by $\{Y_n^j \colon j = 1, \ldots, l_n\}$ the real-valued orthonormal basis of $\mathcal{H}_n^d(v_\kappa)$ in $L^2(\mathbb{S}^{d-1}, d\sigma_\kappa)$. A union of these bases forms an orthonormal basis in $L^2(\mathbb{S}^{d-1}, d\sigma_\kappa)$ consisting of k-spherical harmonics.

Let us rewrite (4.3) as follows

$$B_{\kappa,a}(x,y) = \sum_{j=0}^{\infty} e^{-\frac{i\pi j}{a}}\,\frac{\lambda_\kappa + j}{\lambda_\kappa}\,\frac{\Gamma(2\lambda_\kappa/a+1)(|x||y|)^j}{a^{2j/a}\,\Gamma(2(\lambda_\kappa+j)/a+1)}\,j_{\frac{2(\lambda_\kappa+j)}{a}}\Big(\frac{2}{a}\,(|x||y|)^{a/2}\Big)V_\kappa C_j^{\lambda_\kappa}(\langle x', \cdot\rangle)(y').$$

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Integrating this and using orthogonality of κ-spherical harmonics, as in the case of the Dunkl kernel (see [13, Theorem 5.3.4], [21, Corollary 2.5]), we obtain the following crucial property of the kernel of the generalized Fourier transform.

Proposition 5.7. *If $x, y \in \mathbb{R}^d$, $x = \rho x'$, $y = \nu y'$, then*

$$\int_{\mathbb{S}^{d-1}} B_{\kappa,a}(x, \nu y')Y_n^j(y')\, d\sigma_\kappa(y') = \frac{e^{-\frac{i\pi n}{a}}\Gamma(2\lambda_\kappa/a+1)}{a^{2n/a}\,\Gamma(2(\lambda_\kappa+n)/a+1)}\,\nu^n j_{\frac{2(\lambda_\kappa+n)}{a}}\Big(\frac{2}{a}(\rho\nu)^{a/2}\Big)Y_n^j(x).$$

Proposition 5.8. *For irrational a and a nontrivial function $f \in \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{F}_{\kappa,a}(f) \notin \mathcal{S}(\mathbb{R}^d)$.*

Proof. **1.** First, assume that $f(x) = \rho^n\psi(\rho)Y_n^j(x')$, where $n \in \mathbb{Z}_+$, $x = \rho x'$, $|x'| = 1$, and $\psi \in \mathcal{S}(\mathbb{R}_+)$. Since $\rho^n Y_n^j(x') = Y_n^j(x)$ is a homogeneous polynomial of degree n, then $f(x) = \psi(|x|)Y_n^j(x) \in \mathcal{S}(\mathbb{R}^d)$. If $y = \nu y'$, $\rho = (a/2)^{1/a}u^{2/a}$, then by [15]

$$\begin{aligned}\mathcal{F}_{\kappa,a}(f)(y) = e^{-\frac{i\pi n}{a}}Y_n^j(y)H_{\lambda_\kappa+n,a}(\psi)(\nu) &= e^{-\frac{i\pi n}{a}}Y_n^j(y)\int_0^\infty \psi(\rho)j_{\frac{2(\lambda_\kappa+n)}{a}}\Big(\frac{2}{a}(\nu\rho)^{a/2}\Big)\,d\nu_{\lambda_\kappa,a}(\rho)\\ &= e^{-\frac{i\pi n}{a}}Y_n^j(y)\int_0^\infty \psi\Big(\Big(\frac{a}{2}\Big)^{1/a}u^{2/a}\Big)j_{\frac{2(\lambda_\kappa+n)}{a}}\Big(\Big(\frac{2}{a}\Big)^{1/2}\nu^{a/2}u\Big)\,d\nu_{\frac{2(\lambda_\kappa+n)}{a}}(u).\end{aligned}$$

Moreover, following the ideas used in Example 5.1, we obtain that the function

$$g_a(\nu) = \int_0^\infty \psi\Big(\Big(\frac{a}{2}\Big)^{1/a}u^{2/a}\Big)j_{\frac{2(\lambda_\kappa+n)}{a}}\Big(\Big(\frac{2}{a}\Big)^{1/2}\nu^{a/2}u\Big)\,d\nu_{\frac{2(\lambda_\kappa+n)}{a}}(u)$$

with irrational a cannot decrease rapidly as $\nu \to \infty$, and it has finite smoothness at the origin. Therefore, it follows that $\mathcal{F}_{\kappa,a}(f) \notin \mathcal{S}(\mathbb{R}^d)$.

2. Let now $f \in \mathcal{S}(\mathbb{R}^d)$ be any nonzero function. Then its spherical κ-harmonic expansion is given by

$$f(\rho x') = \sum_{n=0}^\infty \sum_{j=1}^{l_n} f_{nj}(\rho)Y_n^j(x'), \quad f_{nj}(\rho) = \int_{\mathbb{S}^{d-1}} f(\rho x')Y_n^j(x')\, d\sigma_\kappa(x')$$

(see [15]). Since the subspaces $\mathcal{H}_n^d(\nu_\kappa)$ are orthogonal, then $f_{nj}^{(j)}(0) = 0$, $j = 0, 1, \dots, n-1$. Changing variables $x' \to -x'$ implies $f_{nj}(-\rho) = (-1)^n f_{nj}(\rho)$. Hence, setting a nonzero function $g_{nj}(f)(x) = f_{nj}(\rho)Y_n^j(x') \in \mathcal{S}(\mathbb{R}^d)$, we have $g_{nj}(x) = \rho^n\psi(\rho)Y_n^j(x')$ with some $\psi \in \mathcal{S}(\mathbb{R}_+)$.

In order to apply the results obtained in case 1, we need to show that

$$\mathcal{F}_{\kappa,a}(g_{nj}(f))(y) = g_{nj}(\mathcal{F}_{\kappa,a}(f))(y).$$

Indeed, we note that

$$\mathcal{F}_{\kappa,a}(g_{nj}(f))(y) = e^{-i\pi n/a}Y_n^j(y)H_{\lambda_\kappa+n,a}(\psi)(\nu).$$

On the other hand, in view of Proposition 5.7, we deduce that

$$\begin{aligned}g_{nj}(\mathcal{F}_{\kappa,a}(f))(y) &= Y_n^j(y')\int_{\mathbb{S}^{d-1}} \mathcal{F}_{\kappa,a}(f)(\nu y')Y_n^j(y')\, d\sigma_{\kappa,a}(y')\\ &= e^{-\frac{i\pi n}{a}}Y_n^j(y)\int_{\mathbb{R}^d} \frac{f(x)\Gamma(2\lambda_\kappa/a+1)}{a^{2n/a}\,\Gamma(2(\lambda_\kappa+n)/a+1)}j_{\frac{2(\lambda_\kappa+n)}{a}}\Big(\frac{2}{a}(\rho\nu)^{a/2}\Big)Y_n^j(x)\, d\mu_{\kappa,a}(x)\\ &= e^{-\frac{i\pi n}{a}}Y_n^j(y)\int_0^\infty \psi(\rho)j_{\frac{2(\lambda_\kappa+n)}{a}}\Big(\frac{2}{a}(\rho\nu)^{a/2}\Big)\,d\nu_{\lambda_\kappa+n,a}(\rho)\\ &= e^{-\frac{i\pi n}{a}}Y_n^j(y)H_{\lambda_\kappa+n,a}(\psi)(\nu).\end{aligned}$$

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Assuming here that $f, \mathcal{F}_{\kappa,a}(f) \in \mathcal{S}(\mathbb{R}^d)$ yields $g_{nj}(f), \mathcal{F}_{\kappa,a}(g_{nj}(f)) \in \mathcal{S}(\mathbb{R}^d)$, which contradicts the first case. □

Summarizing, the generalized Fourier transform for irrational a drastically deforms even very smooth functions. It was mentioned in [4, Chapter 5] that the generalized Fourier transform has a finite order only for rational a. Therefore, the case of irrational a is of little interest in harmonic analysis.

6. Nondeformed unitary transforms generated by $\mathcal{F}_{\kappa,a}$

Let $a = \frac{2}{2r+1}$ and $\lambda = \lambda_{\kappa,a} = (2\kappa - 1)/a \geq -1/2$. Recalling that A is given by (5.2), we may assume that $x, u \in \mathbb{R}$. Since

$$\int_{-\infty}^{\infty} |f(x)|^2 \, d\mu_{\kappa,a}(x) = \int_{-\infty}^{\infty} |Af(u)|^2 \, d\widetilde{\nu}_\lambda(u), \qquad d\widetilde{\nu}_\lambda(u) = \frac{|u|^{2\lambda+1}\, du}{2^{\lambda+1}\Gamma(\lambda+1)},$$

the linear operator $A\colon L^2(\mathbb{R}, d\mu_{\kappa,a}) \to L^2(\mathbb{R}, d\widetilde{\nu}_\lambda)$ is an isometric isomorphism and the inverse operator is given by $A^{-1}g(x) = g((2r+1)^{1/2} x^{1/(2r+1)})$.

In view of (2.4) and (5.2), $B_{\kappa,a}(x,y) = e_{2r+1}(uv, \lambda)$ and

$$A\mathcal{F}_{\kappa,a}(f)(v) = \int_{\mathbb{R}} e_{2r+1}(uv,\lambda) Af(u)\, d\widetilde{\nu}_\lambda(u).$$

This formula defines the *nondeformed* transform $\mathcal{F}_r^\lambda$, for $\lambda > -1/2$ and $r \in \mathbb{Z}_+$,

$$\begin{aligned}
\mathcal{F}_r^\lambda(g)(v) &= \int_{-\infty}^{\infty} e_{2r+1}(uv,\lambda) g(u)\, d\widetilde{\nu}_\lambda(u) \\
&= \int_{-\infty}^{\infty} \Big(j_\lambda(uv) + i(-1)^{r+1} \frac{(uv)^{2r+1}}{2^{2r+1}(\lambda+1)_{2r+1}} j_{\lambda+2r+1}(uv) \Big) g(u)\, d\widetilde{\nu}_\lambda(u) \\
&= c_\lambda \int_{-\infty}^{\infty} \int_{-1}^{1} (1-t^2)^{\lambda-1/2} (1 + P_{2r+1}^{(\lambda-1/2)}(t))\, e^{-iuvt}\, dt\, g(u)\, d\widetilde{\nu}_\lambda(u).
\end{aligned}$$

Moreover, its kernel satisfies the estimate $|e_{2r+1}(uv,\lambda)| \leq M_\lambda < \infty$ and, importantly, $M_\lambda = 1$ for $\lambda \geq 0$. If $r = 0$, we recover the one-dimensional Dunkl transform. Below, we study invariant subspaces ($\subset C^\infty$) of the $\mathcal{F}_r^\lambda$ transform. The Plancherel theorem for $\mathcal{F}_{\kappa,a}$ given by

$$\int_{-\infty}^{\infty} |\mathcal{F}_{\kappa,a}(f)(y)|^2 \, d\mu_{\kappa,a}(y) = \int_{-\infty}^{\infty} |f(x)|^2 \, d\mu_{\kappa,a}(x), \quad f \in L^2(\mathbb{R}, d\mu_{\kappa,a}),$$

implies that $\mathcal{F}_r^\lambda$ is a unitary operator in $L^2(\mathbb{R}, d\widetilde{\nu}_\lambda)$, that is

$$\int_{-\infty}^{\infty} |\mathcal{F}_r^\lambda(g)(v)|^2 \, d\widetilde{\nu}_\lambda(v) = \int_{-\infty}^{\infty} |g(u)|^2 \, d\widetilde{\nu}_\lambda(u).$$

Since the reverse operator satisfies $(\mathcal{F}_{\kappa,a})^{-1}(f)(x) = \mathcal{F}_{\kappa,a}(f)(-x)$ [4, Theorem 5.3], we have

$$(\mathcal{F}_r^\lambda)^{-1}(f)(u) = \int_{-\infty}^{\infty} \overline{e_{2r+1}(uv,\lambda)} f(v)\, d\widetilde{\nu}_\lambda(v).$$

If $g, \mathcal{F}_r^\lambda(g) \in L^1(\mathbb{R}, d\widetilde{\nu}_\lambda)$, then one may assume that $g, \mathcal{F}_r^\lambda(g) \in C_b(\mathbb{R})$. Moreover, the inversion formula

$$g(u) = \int_{-\infty}^{\infty} \overline{e_{2r+1}(uv,\lambda)} \mathcal{F}_r^\lambda(g)(v)\, d\widetilde{\nu}_\lambda(v) \tag{6.1}$$

holds not only in an L_2 sense but also pointwise.

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Considering the derivatives of the kernel $e_{2r+1}(uv,\lambda)$, we note that

$$\partial_v^n e_{2r+1}(uv,\lambda) = (iu)^n c_\lambda \int_{-1}^{1} t^n (1-t^2)^{\lambda-1/2}(1+P_{2r+1}^{(\lambda-1/2)}(t))\, e^{-iuvt}\, dt,$$

and so,

$$|\partial_v^n e_{2r+1}(uv,\lambda)| \le M_\lambda |u|^n.$$

Then, for $g \in \mathcal{S}(\mathbb{R})$, we have

$$|\partial^n \mathcal{F}_r^\lambda(g)(v)| = \left|\int_{-\infty}^{\infty} g(u)\partial_v^n e_{2r+1}(uv,\lambda)\, d\widetilde{\nu}_\lambda(u)\right| \le M_\lambda \int_{-\infty}^{\infty} |u|^n |g(u)|\, d\widetilde{\nu}_\lambda(u) < \infty. \tag{6.2}$$

Therefore, $\mathcal{F}_r^\lambda(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$. However, $\mathcal{F}_r^\lambda(\mathcal{S}(\mathbb{R})) \not\subset \mathcal{S}(\mathbb{R})$. Indeed, assuming $g, \mathcal{F}_r^\lambda(g) \in \mathcal{S}(\mathbb{R})$, by orthogonality of the Gegenbauer polynomials for $s = 0, 1, \ldots, r-1$,

$$\int_{-1}^{1} t^{2s+1}(1-t^2)^{\lambda-1/2}(1 \pm P_{2r+1}^{(\lambda-1/2)}(t,\lambda))\, dt = 0$$

and

$$\partial^{2s+1}\mathcal{F}_r^\lambda(g)(0) = 0, \quad \partial^{2s+1} g(0) = 0, \tag{6.3}$$

which is not true for arbitrary g.

Put for $n \in \mathbb{Z}_+$

$$\mathcal{S}_n(\mathbb{R}) = \{g \in \mathcal{S}(\mathbb{R}) \colon \partial^{2s+1} g(0) = 0, \quad s = 0, 1, \ldots, n-1\}, \quad \mathcal{S}_0(\mathbb{R}) = \mathcal{S}(\mathbb{R}).$$

The set $\mathcal{S}_n(\mathbb{R})$ is dense in $L^2(\mathbb{R}, d\mu_{\kappa,a})$ and in $L^2(\mathbb{R}, d\widetilde{\nu}_\lambda)$.

Example 6.1. Consider the function $g_{2s+1}(u) = u^{2s+1}e^{-u^2} \in \mathcal{S}_s(\mathbb{R})$, $s \in \mathbb{Z}_+$. By means of (2.1), (2.4), [1, Chapter VI, 6.1], and [2, Chapter VIII, 8.6(14)], we get

$$\begin{aligned}
\mathcal{F}_r^\lambda(g_{2s+1})(v) &= i(-1)^{r+1} v^{-\lambda} \int_0^\infty u^{\lambda+2s+2} e^{-u^2} J_{\lambda+2r+1}(uv)\, du \\
&= i c_{r,\lambda,s}\, v^{2r+1} \sum_{l=0}^{\infty} \frac{(\lambda+s+r+2)_l}{(\lambda+2r+2)_l}\Big(-\frac{v^2}{4}\Big)^l \\
&= i c_{r,\lambda,s}\, v^{2r+1} \Phi\Big(\lambda+s+r+2, \lambda+2r+2, -\frac{v^2}{4}\Big), \quad c_{r,\lambda,s} > 0.
\end{aligned}$$

We consider two cases. If $s = 0, 1, \ldots, r-1$, then asymptotics as $v \to \infty$ [3, Chapter VI,6.13.1] implies

$$\Phi\Big(\lambda+s+r+2, \lambda+2r+2, -\frac{v^2}{4}\Big) = \frac{\Gamma(\lambda+2r+2)}{\Gamma(r-s)}\Big(\frac{v^2}{4}\Big)^{-(\lambda+s+r+2)}\Big(1+O\Big(\frac{1}{v^2}\Big)\Big),$$

that is, $\mathcal{F}_r^\lambda(g_{2s+1}) \notin \mathcal{S}(\mathbb{R})$. If $s \ge r$, then applying the Kummer transform [3, Chapter VI,6.3.7)] gives us

$$\begin{aligned}
\mathcal{F}_r^\lambda(g_{2s+1})(v) &= i c_{r,\lambda,s}\, v^{2r+1} e^{-v^2/4} \Phi\Big(r-s, \lambda+2r+2, \frac{v^2}{4}\Big) \\
&= i c_{r,\lambda,s}\, v^{2r+1} e^{-v^2/4} \sum_{l=0}^{s-r} \frac{(r-s)_l}{(\lambda+2r+2)_l}\Big(\frac{v^2}{4}\Big)^l,
\end{aligned} \tag{6.4}$$

that is, $\mathcal{F}_r^\lambda(g_{2s+1})(v) \in \mathcal{S}_r(\mathbb{R})$.

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Since $g_{2r+1} \in \mathcal{S}_r(\mathbb{R})$ and $\mathcal{F}_r^\lambda(g_{2r+1})(v) \in \mathcal{S}_r(\mathbb{R})$, one can conjecture that $\mathcal{F}_r^\lambda(\mathcal{S}_r(\mathbb{R})) = \mathcal{S}_r(\mathbb{R})$. In order to show this (see Proposition 6.3), we will need some auxiliary results.

Recall that for the weight function $|x|^{2\lambda+1}$, the differential-difference Dunkl operator of the first and second order are given by

$$T_{\lambda+1/2}g(u) = \partial g(u) + (\lambda + 1/2)\,\frac{g(u) - g(-u)}{u},$$

$$\Delta_{\lambda+1/2}g(u) = T^2_{\lambda+1/2}g(u) = \partial^2 g(u) + \frac{2\lambda+1}{u}\,\partial g(u) - (\lambda+1/2)\,\frac{g(u)-g(-u)}{u^2}$$

(see [4, 20]). Let us define the operator

$$\delta_\lambda g(u) = T^2_{\lambda+1/2}g(u) - 2r(\lambda+r+1)\,\frac{g(u)-g(-u)}{u^2},$$

which is obtained by changing variables $x = x(u)$ as in (5.2) in the Dunkl Laplacian

$$\Delta_\kappa f(x) = \frac{(2r+1)^{2r-1}}{u^{4r}}\,\delta_\lambda g(u).$$

By direct calculations, we verify that the kernel $e_{2r+1}(uv,\lambda)$ is the eigenfunction of δ_λ:

$$(\delta_\lambda)_u e_{2r+1}(uv,\lambda) = -|v|^2 e_{2r+1}(uv,\lambda). \tag{6.5}$$

Using [20, Proposition 2.18] for $g \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} T^2_{\lambda+1/2}g(u)e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u) = \int_{-\infty}^{\infty} g(u)(T^2_{\lambda+1/2})_u e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u). \tag{6.6}$$

Suppose $g \in \mathcal{S}_1(\mathbb{R})$, then

$$\frac{g(u)-g(-u)}{u^2} = \frac{2g_o(u)}{u^2} \in \mathcal{S}(\mathbb{R})$$

and

$$\int_{-\infty}^{\infty} \frac{g(u)-g(-u)}{u^2}\,e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u) = \int_{-\infty}^{\infty} g(u)\,\frac{e_{2r+1}(uv,\lambda)-e_{2r+1}(-uv,\lambda)}{u^2}\,d\widetilde{\nu}_\lambda(u).$$

This, (6.5) and (6.6) for any $g \in \mathcal{S}_1(\mathbb{R})$ yield

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_\lambda g(u)e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u) &= \int_{-\infty}^{\infty} g(u)(\delta_\lambda)_u e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u)\\ &= -|v|^2\int_{-\infty}^{\infty} g(u)e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u).\end{aligned} \tag{6.7}$$

Applying (6.7) for $g \in \mathcal{S}_n(\mathbb{R})$, we get

$$\int_{-\infty}^{\infty} \delta^n_\lambda g(u)e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u) = (-1)^n|v|^{2n}\int_{-\infty}^{\infty} g(u)e_{2r+1}(uv,\lambda)\,d\widetilde{\nu}_\lambda(u). \tag{6.8}$$

Lemma 6.2. *Suppose $g \in \mathcal{S}(\mathbb{R})$, and*

$$a_n(g)(v) = \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{1}{(k-l)!}\,\frac{\partial^{2l+1}g(0)}{(2l+1)!}\,v^{2k+1}e^{-v^2},$$

then $g - a_n(g) \in \mathcal{S}_{n+1}(\mathbb{R})$.

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Proof. We have $g - a_n(g) \in \mathcal{S}(\mathbb{R})$. Applying the Leibniz rule for $s = 0, 1, \ldots, n$,

$$\begin{aligned}\frac{\partial^{2s+1} a_n(g)(0)}{(2s+1)!} &= \frac{1}{(2s+1)!} \sum_{k=0}^{s} \sum_{l=0}^{k} \frac{\partial^{2l+1} g(0)}{(k-l)!\,(2l+1)!} \binom{2s+1}{2k+1} (2k+1)! \partial^{2s-2k} (e^{-u^2})(0) \\ &= \sum_{k=0}^{s} \sum_{l=0}^{k} \frac{\partial^{2l+1} g(0)(-1)^{s+k}}{(k-l)!\,(2l+1)!\,(s-k)!} = \sum_{l=0}^{s} \frac{\partial^{2l+1} g(0)}{(2l+1)!} \sum_{m=0}^{s-l} \frac{(-1)^{s+l+m}}{m!\,(s-l-m)!} \\ &= \frac{1}{(s-l)!} \sum_{l=0}^{s} \frac{(-1)^{s+l} \partial^{2l+1} g(0)}{(s-l)!\,(2l+1)!} (1-1)^{s-l} = \frac{\partial^{2s+1} g(0)}{(2s+1)!}.\end{aligned}$$

□

Proposition 6.3. *Let $\lambda > -1/2$ and $r \in \mathbb{Z}_+$. We have $\mathcal{F}_r^\lambda(\mathcal{S}_r(\mathbb{R})) = \mathcal{S}_r(\mathbb{R})$.*

If $r = 0$, we recover the result by de Jeu [8] for the Dunkl transform.

Proof. Let $r \in \mathbb{N}$ and $g \in \mathcal{S}_r(\mathbb{R})$. It is enough to show that g satisfies the condition:

$$\partial^m (v^{2n} \mathcal{F}_r^\lambda(g)(v)) \text{ is bounded for any } m \in \mathbb{Z}_+,\ n \ge r+1. \tag{6.9}$$

We have

$$\partial^m (v^{2n} \mathcal{F}_r^\lambda(g)(v)) = \partial^m (v^{2n} \mathcal{F}_r^\lambda(g - a_{n-1}(g))(v)) + \partial^m (v^{2n} \mathcal{F}_r^\lambda(a_{n-1}(g))(v)).$$

Since $\partial^{2l+1} g(0) = 0$, $l = 0, 1, \ldots, r-1$, then

$$a_{n-1}(g)(v) = \sum_{k=r}^{n-1} \sum_{l=r}^{k} \frac{1}{(k-l)!} \frac{\partial^{2l+1} g(0)}{(2l+1)!} v^{2k+1} e^{-v^2} \in \mathcal{S}_r(\mathbb{R}).$$

By (6.4), condition (6.9) is valid for $a_{n-1}(g)$. By Lemma 6.2, $g - a_{n-1}(g) \in \mathcal{S}_n(\mathbb{R})$. In light of (6.8), $v^{2n} \mathcal{F}_r^\lambda(g - a_{n-1}(g))(v)$ is the $\mathcal{F}_r^\lambda$-transform of the function $(-1)^n \delta_\lambda^n (g - a_{n-1}(g)) \in \mathcal{S}(\mathbb{R})$. Applying for this transform inequality (6.2), we get the property (6.9) for $g - a_{n-1}(g)$. Therefore, $\mathcal{F}_r^\lambda(g) \in \mathcal{S}(\mathbb{R})$. By virtue of (6.1) and (6.3), $\mathcal{F}_r^\lambda(g) \in \mathcal{S}_r(\mathbb{R})$. □

Remark 6.4. An alternative proof of Proposition 6.3 reads as follows. Let $g \in \mathcal{S}_r(\mathbb{R})$. In light of (5.1)–(5.3), we have the following representation

$$\begin{aligned}\mathcal{F}_r^\lambda(g)(v) &= \int_{-\infty}^{\infty} j_\lambda(uv) g(u)\, d\widetilde{\nu}_\lambda(u) + \frac{i(-1)^{r+1} v^{2r+1}}{2^{2r+1} (\lambda+1)_{2r+1}} \int_{-\infty}^{\infty} u^{2r+1} j_{\lambda+2r+1}(uv) g(u)\, d\widetilde{\nu}_\lambda(u) \\ &= \int_0^\infty j_\lambda(uv) g_e(u)\, d\nu_\lambda(u) + i(-1)^{r+1} v^{2r+1} \int_0^\infty j_{\lambda+2r+1}(uv) u^{-(2r+1)} g_o(u)\, d\nu_{\lambda+2r+1}(u) \\ &= H_\lambda(g_e)(v) + +i(-1)^{r+1} v^{2r+1} H_{\lambda+2r+1}(u^{-(2r+1)} g_o)(v) = F_1(v) + v^{2r+1} F_2(v),\end{aligned}$$

where even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$. Therefore, $\mathcal{F}_r^\lambda(g) \in \mathcal{S}_r(\mathbb{R})$. The first proof was given to underline the important properties of $\mathcal{F}_r^\lambda$-transform and its kernel.

Since $f \in \mathcal{S}(\mathbb{R})$ implies $Af \in \mathcal{S}_r(\mathbb{R})$, Proposition 6.3 yields the following result.

Corollary 6.5. *Let $a = \frac{2}{2r+1}$ and $\kappa \ge \frac{r}{2r+1}$. If $f \in \mathcal{S}(\mathbb{R})$, then $\mathcal{F}_{\kappa,a}(f)((2r+1)^{-(r+1/2)} v^{2r+1}) \in \mathcal{S}_r(\mathbb{R})$ or, equivalently, $\mathcal{F}_{\kappa,a}(f)(v) = g((2r+1)^{1/2} v^{1/(2r+1)})$, $g \in \mathcal{S}_r(\mathbb{R})$; cf. (5.4).*

Now, we discuss the case $\lambda = -1/2$.

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Remark 6.6. If $\lambda = -1/2$, $r \in \mathbb{Z}_+$, then

$$\delta_{-1/2} g(u) = \partial^2 g(u) - r(2r+1)\,\frac{g(u)-g(-u)}{u^2}, \quad e_1(uv,-1/2) = e^{-iuv} \quad (r=0).$$

Taking into account (2.4), and passing to the limit in (2.4) as $\lambda \to -1/2$, we deduce that for $r \geq 1$

$$\begin{aligned} e_{2r+1}(uv,-1/2) &= \cos(uv) + i(-1)^{r+1}\frac{(uv)^{2r+1}}{2^{2r+1}(1/2)_{2r+1}} j_{2r+1/2}(uv) \\ &= e^{-iuv} - (r+1/2)\int_{-1}^{1}\sum_{s=0}^{r-1}(-1)^s\binom{r}{s+1}\frac{(r+3/2)_s}{s!}\,(1-t^2)^s t e^{-iuvt}\,dt. \end{aligned}$$

Taking this into account and analyzing the proofs above, we note that all mentioned results in this section for the transform $\mathcal{F}_r^\lambda$ in the case $\lambda > -1/2$ are also valid for $\lambda = -1/2$. In particular, $\mathcal{F}_r^{-1/2}$, $r \in \mathbb{Z}_+$, are the unitary transforms in the nonweighted $L^2(\mathbb{R}, dx)$, where $\mathcal{F}_0^{-1/2}$ corresponds to the classical Fourier transform.

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