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ON CONTINUITY OF DERIVATIONS AND EPIMORPHISMS ON SOME VECTOR-VALUED GROUP ALGEBRAS

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For a locally compact Abelian group G and a commutative Banach algebra B, let $L^1(G,B)$ be the Banach algebra of all Bochner integrable functions. We show that if G is compact and B is a *nonunital* Banach algebra without nontrivial zero divisors, then (i) all derivations on $L^1(G,B)$ are continuous if and only if all derivations on B are continuous, and (ii) each epimorphism from a Banach algebra X onto $L^1(G,B)$ is continuous provided every epimorphism from X onto B is continuous. If G is noncompact then every derivation on $L^1(G,B)$ and every epimorphism from a commutative Banach algebra onto $L^1(G,B)$ are continuous. Our results extend the results of Neumann and Velasco for *nonunital* Banach algebras.

1. INTRODUCTION

In [**10**] Neumann and Velasco gave some results concerning the automatic continuity of homomorphisms and derivations on $L^1(G,B)$, the convolution algebra of all Bochner integrable functions on a locally compact Abelian group G with values in a commutative *unital* Banach algebra B. They showed that if G is a locally compact noncompact Abelian group and B is any commutative *unital* Banach algebra, then all derivations on $L^1(G,B)$ and all epimorphisms from a Fréchet algebra onto $L^1(G,B)$ are continuous. It is also shown that if G is compact, then all derivations on $L^1(G,B)$ are continuous precisely when all derivations on B are continuous; and each epimorphism from a Fréchet algebra X onto $L^1(G,B)$ is continuous whenever each epimorphism from X onto B is continuous. The proofs given in [**10**] depend on the fact that B is *unital*. In this note we extend the above results to a *nonunital* Banach algebra B under the assumption that B has no nontrivial zero divisors. Our results extend the results of Neumann and Velasco to a larger class of Banach algebras that includes some radical Banach algebras.

2. PRELIMINARIES

Let B be a commutative Banach algebra (not necessarily *unital*), and let G be a locally compact Abelian group with Haar measure h and dual group Γ. We assume

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that if G is compact, the Haar measure h is normalised so that $h(G) = 1$. Let $L^1(G,B)$ denote the Banach algebra of all Bochner integrable functions from G into B, and let

$$(f * g)(s) := \int_G f(s-t)g(t)\,dh(t) \quad \text{for all } f,g \in L^1(G,B) \text{ and } s \in G$$

and

$$\|f\|_1 := \int_G \|f(t)\|\,dh(t) \quad \text{for all } f \in L^1(G,B).$$

It is easy to show that $L^1(G,B)$ is also a commutative Banach algebra.

For any $\gamma \in \Gamma$, let ϕ_γ denote the evaluation of the vector-valued Fourier transform at γ. That is, ϕ_γ is a mapping from $L^1(G,B)$ onto B given by

$$\phi_\gamma(f) := \widehat{f}(\gamma) := \int_G \overline{\gamma(t)}f(t)\,dh(t) \quad \text{for all } f \in L^1(G,B).$$

It can be shown that ϕ_γ is a continuous algebra homomorphism from $L^1(G,B)$ onto B for each $\gamma \in \Gamma$. Further, for any $\gamma \in \Gamma$, let

$$M_\gamma := \{f \in L^1(G,B) : \widehat{f}(\gamma) = \theta\},$$

where θ is the zero vector of B. Clearly M_γ is a closed ideal of $L^1(G,B)$. Notice that if B has no nontrivial zero divisors, then M_γ is a closed prime ideal of $L^1(G,B)$, where an ideal I of a commutative algebra is said to be prime if the product $xy \in I$ only if either $x \in I$ or $y \in I$. It is straightforward to prove that $\bigcap_{\gamma \in \Gamma} M_\gamma = \{0\}$. For relevant information on $L^1(G,B)$ and for related results in harmonic analysis on Abelian groups we refer to [**5, 8, 11**].

A closed ideal $\Im$ of a commutative Banach algebra A is said to be a separating ideal of A if, for every sequence $\{x_n\}$ in A, there is a positive integer m such that

$$\overline{x_1x_2\cdots x_n\Im} = \overline{x_1x_2\cdots x_m\Im} \quad \text{for all } n \geqslant m.$$

Recall that a linear operator $D : A \to A$ is called a derivation if $D(xy) = xDy + yDx$ for all x, y in A. An algebra homomorphism h from a Banach algebra B to A is called an epimorphism if h is onto. For any linear operator T from a Banach space X into a Banach space Y,

$$\Im(T) =: \{y \in Y \mid \exists x_n \to 0 \text{ in } X \text{ with } Tx_n \to y\}$$

is said to be the separating subspace of T. It is easy to see that $\Im(T)$ is a closed subspace of Y. By the closed graph theorem, T is continuous if and only if $\Im(T) = \{0\}$. For any derivation D on a commutative Banach algebra A or any epimorphism h from a Banach algebra B onto A, the separating subspace $\Im(D)$ or $\Im(h)$ is a separating ideal of A. Finally we note that there can be at most finitely many closed prime ideals not containing a separating ideal $\Im$ of a commutative Banach algebra. For a proof of the above statement we refer to [**6**].

The following properties of the separating subspace are very useful in some of the proofs of the main results of the paper.

Proposition 2.1. *Let S be a linear operator from a Banach space X into a Banach space Y, and R be a continuous linear operator from Y into a Banach space Z. Then*

(i) *$R \circ S$ is continuous if and only if $\Im(S)$ is contained in the kernel of R,*
(ii) $\overline{R(\Im(S))} = \Im(R \circ S)$.

Proof: See [**12**]. ∎

For relevant information on separating ideals, their relation to the prime ideals of the Banach algebra, and for related results on automatic continuity theory we refer to [**1, 2, 3, 4, 7, 9, 12**].

For each $\gamma \in \Gamma$ and for any $x \in B$, we let

$$(\gamma \otimes x)(s) := \gamma(s)x \qquad \text{for all } s \in G.$$

If G is a compact group, then $\gamma \otimes x \in L^1(G, B)$ and further $\|\gamma \otimes x\|_1 = \|x\|$. We define Φ_γ to be the evaluation of the vector-valued Fourier transform at γ in Γ. We recall some of the properties of the product $\gamma \otimes x$ for future reference. Since the proof of Proposition 2.2 is straightforward, it is left to the reader.

Proposition 2.2. *Let G be a compact Abelian group with dual group Γ, and B be a commutative Banach algebra. Let $x, y \in B$, $f \in L^1(G, B)$, and γ be a nontrivial character in Γ. Then*

(i) $\Phi_\gamma(\gamma \otimes x) = x$;
(ii) $\gamma \otimes (\alpha x + \beta y) = \alpha(\gamma \otimes x) + \beta(\gamma \otimes y)$ *for all scalars α and β;*
(iii) $(\gamma \otimes xy) = (\gamma \otimes x) * (\gamma \otimes y)$;
(iv) $\left(\gamma \otimes x\widehat{f}(\gamma)\right) = (\gamma \otimes x) * \left(\gamma \otimes \widehat{f}(\gamma)\right) = (\gamma \otimes x) * f$;
(v) *If B has multiplicative identity 1, then $\gamma \otimes 1$ is an idempotent in $L^1(G, B)$, and further $\left(\gamma \otimes \widehat{f}(\gamma)\right) = (\gamma \otimes 1) * f$.*

3. Main Results

The following lemma is the key to the continuity of derivations on $L^1(G, B)$.

Lemma 3.1. *Let G be a compact Abelian group, B be a commutative Banach algebra without nontrivial zero divisors, and γ be a character in Γ. Then*

(i) *for any derivation δ on $L^1(G,B)$, there exists a unique derivation $D = D(\gamma,\delta)$ on B such that the following diagram commutes:*

$$\begin{array}{ccc} L^1(G,B) & \xrightarrow{\ \delta\ } & L^1(G,B) \\ \big\downarrow{\scriptstyle \Phi_\gamma} & & \big\downarrow{\scriptstyle \Phi_\gamma} \\ B & \xrightarrow[D]{} & B \end{array}$$

(ii) *for any derivation D on B, and for a given nonzero element z in B, there exists a derivation $\delta = \delta(D,\gamma,z)$ on $L^1(G,B)$ such that the following diagram commutes:*

$$\begin{array}{ccc} L^1(G,B) & \xrightarrow{\ \delta\ } & L^1(G,B) \\ \big\downarrow{\scriptstyle \Phi_\gamma} & & \big\downarrow{\scriptstyle \Phi_\gamma} \\ B & \xrightarrow[D_z]{} & B \end{array}$$

where D_z is the derivation defined by $D_z(x) = zD(x)$ for all x in B.

Proof: (i) Let δ be a derivation on $L^1(G,B)$ and $\gamma \in \Gamma$. Define $D : B \to B$ by $D(x) = \Phi_\gamma(\delta(\gamma \otimes x))$ for all $x \in B$. By using the properties (i) and (ii) of Proposition 2.2, it is easy to show that D is a derivation on B. Now we show that the diagram commutes. Fix a nonzero element w in B. For any $f \in L^1(G,B)$, we have

$$\begin{aligned} D\Big(w\widehat{f}(\gamma)\Big) &= \Phi_\gamma\Big(\delta\Big(\gamma \otimes w\widehat{f}(\gamma)\Big)\Big) \\ &= \Phi_\gamma(\delta((\gamma \otimes w) * f)) \qquad \text{(by (iv) of Proposition 2.2)} \\ &= \Phi_\gamma[(\gamma \otimes w) * \delta f + \delta(\gamma \otimes w) * f], \end{aligned}$$

and so

$$D\Big(w\widehat{f}(\gamma)\Big) = w\Phi_\gamma(\delta f) + \Phi_\gamma(\delta(\gamma \otimes w))\widehat{f}(\gamma). \tag{1}$$

On the other hand,

$$D\Big(w\widehat{f}(\gamma)\Big) = wD\Big(\widehat{f}(\gamma)\Big) + D(w)\widehat{f}(\gamma). \tag{2}$$

Now the above equations (1) and (2) imply that

$$w[D\Big(\widehat{f}(\gamma)\Big) - \Phi_\gamma(\delta f)] = \theta.$$

Since w is a nonzero vector, and B has no nontrivial zero divisors, $D\left(\widehat{f}(\gamma)\right) = \Phi_\gamma(\delta f)$, as required.

If D_1 is any derivation on B such that the diagram is commutative, then for any x in B we have

$$\begin{aligned} D_1(x) &= D_1(\Phi_\gamma(\gamma \otimes x)) \qquad \text{(by (i) of Proposition 2.2)} \\ &= \Phi_\gamma(\delta(\gamma \otimes x)) = D(\Phi_\gamma(\gamma \otimes x)) = D(x). \end{aligned}$$

Thus D is unique.

(ii) Let D be a derivation on B and z be a nonzero element in B. Define the mapping $\delta : L^1(G,B) \to L^1(G,B)$ by $\delta(f) = \gamma \otimes zD\left(\widehat{f}(\gamma)\right)$. Then it is easy to show that δ is a derivation on $L^1(G,B)$. The rest of the proof is similar to the proof in part (i). ∎

Now we are ready for one of the main results.

Theorem 3.2. *Let G be a compact Abelian group and B be a commutative Banach algebra without nontrivial zero divisors. Then*

(i) *all derivations on $L^1(G,B)$ are continuous if and only if all derivations on B are continuous;*

(ii) *every epimorphism from a Banach algebra X onto $L^1(G,B)$ is continuous provided every epimorphism from X onto B is continuous.*

Proof: (i) We first suppose that all derivations on B are continuous. Let δ be any derivation on $L^1(G,B)$, and γ be a continuous character on G. By Lemma 3.1(i), there exists a derivation $D = D(\gamma, \delta)$ on B such that

$$D \circ \Phi_\gamma = \Phi_\gamma \circ \delta.$$

By assumption D is continuous, and so $\Phi_\gamma \circ \delta$ is continuous. Therefore by (i) of Proposition 2.1, the separating ideal $\Im(\delta)$ of δ is contained in M_γ. Since γ is an arbitrary continuous character and $\bigcap_{\gamma \in \Gamma} M_\gamma = \{0\}$, it follows that $\Im(\delta) = \{0\}$. Hence δ is continuous.

Conversely, suppose that all derivations on $L^1(G,B)$ are continuous. Let D be a derivation on B and z be a fixed nonzero element in B. By Lemma 3.1(ii), for each $\gamma \in \Gamma$, there exists a derivation δ on $L^1(G,B)$ such that $\Phi_\gamma \circ \delta = D_z \circ \Phi_\gamma$. We claim D_z is continuous. Let $x_n \to \theta$ in B, and suppose $D_z(x_n) \to y$ in B. Since $\|\gamma \otimes x_n\|_1 = \|x_n\|$, $\gamma \otimes x_n \to 0$ in $L^1(G,B)$. Because δ is continuous and $\Phi_\gamma \circ \delta = D_z \circ \Phi_\gamma$, it follows that $(D_z \circ \Phi_\gamma)(\gamma \otimes x_n) \to \theta$. However, $\Phi_\gamma(\gamma \otimes x_n) = x_n$. Hence $D_z(x_n) \to \theta$. Therefore $y = \theta$. This proves that D_z is continuous, and hence

$\Im(D_z) = \{\theta\}$. By Proposition 2.1(ii), $\overline{z\Im(D)} = \Im(D_z) = \{\theta\}$. Since B has no nontrivial zero divisors, and z is a nonzero element in B, $\Im(D) = \{\theta\}$. Hence D is continuous.

(ii) Suppose that every epimorphism from a Banach algebra X onto B is continuous. Let $h : X \to L^1(G,B)$ be an epimorphism and fix γ in Γ. Since $\Phi_\gamma : L^1(G,B) \to B$ is an epimorphism, $\Phi_\gamma \circ h : X \to B$ is also an epimorphism. By our assumption, $\Phi_\gamma \circ h$ is continuous and hence $\Im(h)$ is contained in M_γ. Since γ is arbitrary and $\bigcap_{\gamma\in\Gamma} M_\gamma = \{0\}$, it follows that h is continuous. □

Now we turn to the case where G is a noncompact locally compact Abelian group.

Theorem 3.3. *Let G be a noncompact locally compact Abelian group, and B be a commutative Banach algebra without nontrivial zero divisors. Then every derivation on $L^1(G,B)$ is continuous. Also, every epimorphism from a commutative Banach algebra onto $L^1(G,B)$ is continuous.*

Proof: First we show that every derivation on $L^1(G,B)$ is continuous. Let δ be a derivation on $L^1(G,B)$. To prove that δ is continuous, it suffices to show that $\Im(\delta) = \{0\}$. Since $\bigcap_{\gamma\in\Gamma} M_\gamma = \{0\}$, it is enough to show $\Im(\delta)$ is contained in M_γ for each $\gamma \in \Gamma$. Since any separating ideal of an algebra is contained in all but finitely many closed prime ideals of the algebra, and since each M_γ is a closed prime ideal in $L^1(G,B)$, there exists at most finitely many continuous characters $\gamma_1, \gamma_2, \dots, \gamma_n$ such that $\Im(\delta)$ is not contained in M_{γ_i} for $i = 1, 2, \dots, n$. Let $f \in \left(\Im(\delta) \cap \left(\bigcap_{i=2}^{n} M_{\gamma_i}\right)\right) \setminus M_{\gamma_1}$. Since M_{γ_1} is a prime ideal, such a function f exists. By the Hahn–Banach theorem, since $\widehat{f}(\gamma_1) \neq \theta$, there exists a continuous linear functional λ on B such that $\lambda\left(\widehat{f}(\gamma_1)\right) \neq 0$. Now consider the basic open set

$$N := \left\{\gamma \in \Gamma : \left|\lambda\left(\widehat{f}(\gamma)\right) - \lambda\left(\widehat{f}(\gamma_1)\right)\right| < \left|\lambda\left(\widehat{f}(\gamma_1)\right)\right|\right\}$$

of Γ containing γ_1. Since G is a noncompact Abelian group, the dual group Γ is not discrete. Hence γ_1 is not isolated in Γ. Obviously, the choice of f implies that the characters $\gamma_2, \gamma_3, \cdots, \gamma_n$ do not belong to N. Hence there is a character $\gamma_0 \in \Gamma - \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ such that $\gamma_0 \in N$. Since $\Im(\delta)$ is contained in M_{γ_0}, $\widehat{f}(\gamma_0) = \theta$. Hence

$$\left|\lambda\left(\widehat{f}(\gamma_1)\right)\right| = \left|\lambda\left(\widehat{f}(\gamma_1)\right) - \lambda\left(\widehat{f}(\gamma)\right)\right| < \left|\lambda\left(\widehat{f}(\gamma_1)\right)\right|.$$

This is a contradiction and so $\Im(\delta) \subseteq M_\gamma$ for all $\gamma \in \Gamma$, and δ must be a continuous derivation.

A similar argument shows that every epimorphism from a Banach algebra onto $L^1(G,B)$ is continuous. □

REMARK. Even though in the statement (ii) of Theorem 3.2 we assumed that X is a commutative Banach algebra, the proof can be modified if X is a Fréchet algebra. Similarly in Theorem 3.3, it can be shown that every epimorphism from a Fréchet algebra onto $L^1(G, B)$ is continuous.

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