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THE FIXED POINT PROPERTY IN BANACH SPACES WHOSE CHARACTERISTIC OF UNIFORM CONVEXITY IS LESS THAN 2

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Abstract

We prove that every Banach space X with characteristic of uniform convexity less than 2 has the fixed point property whenever X satisfies a certain orthogonality condition.

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Introduction

If B is a subset of a Banach space X, a map $T: B \to B$ is said to be a *nonexpansive mapping* when the inequality $\|Tx - Ty\| \le \|x - y\|$ holds for every $x, y \in B$. We say that X has the (*weak*) *fixed point property* (FPP) if every nonexpansive mapping on a nonempty weakly compact convex subset of X has a fixed point.

It is known that L_1 fails to have the FPP [1], but B. Maurey [7] has proved that all reflexive subspaces of L_1 and also c_0 have the FPP. In order to generalize these results Borwein and Sims [2] introduced the notion of weak orthogonality in Banach lattices. Later Sims [8] proved that every weakly orthogonal Banach lattice has the FPP and in this paper, he also introduces the notion of weak orthogonality in Banach spaces (WOTH) which is a generalization of weak orthogonality in Banach lattices. Sims also proved that the (WOTH) is related to the non-strict Opial condition, but it is not known

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whether spaces with WOTH enjoy the FPP.

An open question in fixed point theory is the following. If X is a superreflexive Banach space, does X have the FPP? In particular, if X is a Banach space with $\varepsilon_0(X) < 2$ (characteristic of uniform convexity), does X have the FPP? (See [3].)

On the other hand, we introduce in [4] a geometric property (AMCP) which is a sufficient condition for the FPP.

In this note, we prove that every Banach space (WOTH) so that $\varepsilon_0(X) < 2$ has the AMCP and hence the FPP.

I. Basic definitions and results

Let X be a Banach space. We denote by $[X]$ the quotient space $l_\infty(X)/c_0(X)$ endowed with the norm given by $\|[z_n]\| = \limsup_n \|z_n\|$ where $[z_n]$ denotes the equivalence class of $(z_n) + c_0(X)$. If K is a nonempty bounded subset of X, and x is a point of X we denote by $R(x, K)$ the number: $R(x, K) := \sup\{\|x - z\|: z \in K\}$. When K is also convex, and x, y are points in K, the set of quasi-midpoints of x and y in K is:

$$M(x, y) := \{z \in K: \max\{\|z - x\| \, \|z - y\|\} \leq \frac{1}{2}\|x - y\|\}.$$

We write $[K] := \{[z_n] \in [X]: z_n \in K, \ n = 1, 2, \ldots\}$.

If K is a closed, bounded and convex subset of X, then $[K]$ is a closed, bounded and convex subset of $[X]$.

Let $\mathcal{S}(\mathbb{N})$ be the set of all strictly increasing sequences of natural numbers. If β, γ belong to $\mathcal{S}(\mathbb{N})$ we write $\beta \neq \gamma$ whenever $\beta(n) \neq \gamma(n)$, $n = 1, 2, \ldots$.

Definition 1.1. Let M be a nonempty bounded convex subset of a Banach space X. A sequence (x_n) in M is said to be *equilateral in* M if for any $\beta, \gamma \in \mathcal{S}(\mathbb{N})$ such that $\beta \neq \gamma$ the following equality holds:

$$\|[x_{\beta(n)}\| = \|[x_{\beta(n)}] - [x_{\gamma(n)}]\| = \|[x_{\gamma(n)}]\| = D(x_n) = \operatorname{diam}(M)$$

where $D(x_n) := \limsup_n(\limsup_m \|x_n - x_m\|)$ is the asymptotic diameter of (x_n).

Definition 1.2. A bounded closed convex subset M of a Banach space X, with $0 \in M$ is said to have the AMC-property if for every weakly null sequence (x_n) which is equilateral in M there exist $\rho \in]0, 1[$, and $\beta, \gamma \in \mathcal{S}(\mathbb{N})$ with $\beta \neq \gamma$, such that the set

$$M_\rho((x_n), \beta, \gamma) := M([x_{\beta(n)}], [x_{\gamma(n)}]) \cap \{[z_n]: d([z_n], M) \leq \rho \operatorname{diam}(M)\}$$

is nonempty and $R(x, M_\rho((x_n), \beta, \gamma)) < \operatorname{diam}(M)$ for some x in M.

When every weakly compact convex of X in the above conditions has the AMC-property we say that X has the AMC-property.

LEMMA 1.3. *Let X be a Banach space. If X has the AMC-property then X has the* FPP. (*See* [4].)

As in [7] a Banach space X is said to have the WOTH if every weakly null sequence (x_n) of X satisfies:

$$\lim_{n\to\infty} | \|x_n + x\| - \|x_n - x\| | = 0 \quad \text{for any } x \in X.$$

If X is a Banach space, we write, as is usual, $\varepsilon_0(X) := \sup\{\varepsilon : \delta(\varepsilon) = 0\}$ where $\delta()$ is the modulus of convexity of X.

II. The main result

THEOREM 2.1. *Let X be a Banach space so that $\varepsilon_0(X) < 2$ and moreover X is* WOTH *then X has the* AMCP.

PROOF. Let K be a nonempty weakly compact convex subset of X such that $0 \in K$. Without loss of generality, we can assume that $\operatorname{diam}(K) = 1$.

Let us see that K has the AMC-property. Indeed, let (x_n) be a weakly null sequence equilateral in K. Since X is WOTH we can find $\beta, \gamma \in \mathscr{S}(\mathbb{N})$ with $\beta \neq \gamma$ such that:

$$\lim_{n\to\infty} | \|x_{\beta(n)} + x_{\gamma(n)}\| - \|x_{\beta(n)} - x_{\gamma(n)}\| | = 0.$$

Consequently $\| [x_{\beta(n)}] - [x_{\gamma(n)}] \| = \| [x_{\beta(n)}] + [x_{\gamma(n)}] \|$ and since (x_n) is equilateral in K, it is known that $\| [x_{\beta(n)}] - [x_{\gamma(n)}] \| = \operatorname{diam}(K) = 1$. Hence $\| [x_{\beta(n)}] + [x_{\gamma(n)}]/2 \| = 1/2$.

Consider the set $M_{1/2}((x_n), \beta, \gamma)$. This set is nonempty since

$$\frac{[x_{\beta(n)}] + [x_{\gamma(n)}]}{2} \in M_{1/2}((x_n), \beta, \gamma).$$

Moreover if $[z_n] \in M_{1/2}((x_n), \beta, \gamma)$ we know that

$$\| [z_n] - [x_{\beta(n)}] \| \leq \frac{1}{2} \quad \text{and} \quad \| [z_n] - [x_{\gamma(n)}] \| \leq \frac{1}{2}.$$

Hence using the definition of the quotient norm of $[X]$, we deduce that given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$\|z_n - x_{\beta(n)}\| \leq \frac{1}{2} + \varepsilon, \qquad \|z_n - z_{\beta(n)}\| \leq \frac{1}{2} + \varepsilon \qquad \text{and}$$

$$\|x_{\beta(n)} - x_{\gamma(n)}\| \geq \left(\frac{1}{2} + \varepsilon\right) 2 \frac{1 - \varepsilon}{1 + 2\varepsilon}.$$

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And then for every $n \geq n_0$

$$\| z_n - \frac{x_{\beta(n)} + x_{\gamma(n)}}{2} \| \leq \left(1 - \delta\left(2\frac{1-\varepsilon}{1+2\varepsilon}\right)\right)\frac{1}{2}.$$

Consequently

$$\| z_n - \frac{x_{\beta(n)} + x_{\gamma(n)}}{2} \| \leq \left(1 - \delta\left(2\frac{1-\varepsilon}{1+2\varepsilon}\right)\right)\frac{1}{2}.$$

As $\delta(\cdot)$ is a continuous function in $[0, 2[$ we have

$$\| z_n - \frac{x_{\beta(n)} + x_{\gamma(n)}}{2} \| \leq (1 - \delta(2^-))\frac{1}{2}.$$

Hence

$$\|[z_n]\| \leq \frac{1}{2}(1 - \delta(2^-)) + \frac{1}{2}.$$

But by hypothesis, it is known that $\varepsilon_0(X) < 2$ and consequently $\delta(2^-) \neq 0$; therefore we obtain that $R(0, M_\rho((x_n), \beta, \gamma)) < 1$ which completes the proof.

Corollary 2.2. *Let X be a weakly-orthogonal Banach lattice so that $\varepsilon_0(X) < 2$. Then X has the* AMCP.

Proof. It is known that every weakly-orthogonal Banach lattice is WOTH hence we can use the theorem (2.1).

Remark 2.3. In [8] is proved that every weakly-orthogonal Banach lattice has the FPP. Using the same technique it is easy to see that ever weakly-orthogonal Banach lattice has the AMCP.

On the other hand when X is a Banach space with a Schauder basis (e_n) we can define the following coefficients associated to (e_n). (See [4].)

$c := \max\{\sup\{\|P_n\|: n \in \mathbb{N}\}, \sup\{\|I - P_n\|: n \in \mathbb{N}\}$ where P_n is the natural projection on the segment $[1, n]\}$.

$$\eta := \sup\{\|I - P_F\|: F \text{ is any segment in } N\}.$$

It is easy to prove that if X is a Banach space with a Schauder basis (e_n) so that $2c + \eta < 4$ then X has the AMCP whenever X is WOTH.

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