CHARACTERIZATIONS OF RIGHT NAKAYAMA RINGS

by MANABU HARADA

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We have studied relationships between almost relative projectivity and Nakayama rings [**8**]. In this paper we shall further investigate certain characterizations of right Nakayama rings in terms of almost relative projectives (or injectives). We shall consider three conditions (A), (B) and (C) (see Section 1), which are always satisfied for the relative projective modules, but not for almost relative projectives in general. As an application of [**9**, Theorem] and [**10**, Theorem 2], we shall show that a right artinian ring is right Nakayama if and only if one of the above three conditions holds true for almost relative projectives (Corollary to Theorem 1). Moreover we shall give a characterization of two-sided Nakayama rings related to (C) and the dual ($C^{\#}$) (Theorem 2). Finally we shall investigate the transitivity of almost relative projectives, which is the converse of (B), and give some characterizations of right Nakayama rings related to the transitivity.

1. Preliminaries. In this paper we always assume that R is an associative ring with identity and every module M is a unitary right R-module. We shall denote the length, the socle and the Jacobson radical of M by $|M|$, $\mathrm{Soc}(M)$ and $J(M)$, respectively. In particular we denote $J(R)$ by J. We follow [**8**] and [**11**] for other terminology. We recall here the definition of almost relative projectivity [**8**]. Let M and N be R-modules. For any diagram with K a submodule of M:

$$\begin{tikzcd} M_1 \arrow[r, dashed, "\bar{h}"] \arrow[d, dashed, "\cap"] & N \arrow[d, "h"] \arrow[dl, dashed, "\tilde{h}"'] & \\ M \arrow[r, "\nu"] & M/K \arrow[r] & 0 \text{ (exact)}, \end{tikzcd}$$

if either there exists $\tilde{h} : N \to M$ with $\nu\tilde{h} = h$ or there exist a non-zero direct summand M_1 of M and $\bar{h} : M_1 \to N$ with $h\bar{h} = \nu \mid M_1$, then N is called *almost M-projective* [**8**] (if we always obtain the first case, we say that N is *M-projective* [**2**]).

Let $\{M_i, N_j\}_{i=1}^{m}{}_{j=1}^{n}$ be any set of finitely generated R-modules such that M_i is almost N_j-projective for any pair i and j. We consider the following property:

(A) *$\sum_i^{\oplus} M_i$ is always almost $\sum_j^{\oplus} N_j$-projective for any set $\{M_i, N_j\}$ as above.*

As is easily seen, the above property is equivalent to M_i being almost $\sum_j^{\oplus} N_j$-projective for all i (cf. [**5**, Lemma 2]). We note that $\sum^{\oplus} M_i$ is $\sum^{\oplus} N_j$-projective if M_i is N_j-projective for all i and j [**2**].

Let R be a perfect ring. Let M_0, M_1 and M_2 be finitely generated R-modules and M_1 indecomposable. Assume that M_0 is almost M_1-projective but not M_1-projective. Then M_1 is M_2-projective by [**9**, Proposition 1], if M_0 is M_2-projective. However if M_0 is almost M_2-projective, then M_1 is not almost M_2-projective in general.

By (B) we shall denote the above property:

(B) *For any indecomposable R-module M_1 and any finitely generated R-modules M_0*

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and M_2, *if* M_0 *is almost* M_1*-projective but not* M_1*-projective, and* M_0 *is almost* M_2*-projective, then* M_1 *is always almost* M_2*-projective.*

Finally we shall give one more property of relative projectives which is not satisfied for almost relative projectives. Let M and N be R-modules. Assume that M is N-projective. Then it is well known that M is N'-projective and N/N'-projective for any submodule N' of N. We study the same properties for almost relative projectives. We assume that M is almost N-projective. Take a diagram for a submodule K of N':

$$\begin{tikzcd} & M \arrow[d, "h"] & \\ N' \arrow[r, "v'"] & N'/K \arrow[r] & 0. \end{tikzcd}$$

From the above we can derive the diagram:

$$\begin{tikzcd} & M \arrow[d, "h"] & \\ N' \arrow[r] \arrow[d, phantom, "\cap"] & N'/K \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\ N \arrow[r, "v"] & N/K \arrow[r] & 0. \end{tikzcd}$$

Assume that there exist a non-zero direct summand N_1 of N and $\tilde{h} : N_1 \to M$ such that $h\tilde{h} = v \mid N_1$. Then $v(N_1) = (N_1 + K)/K \subset h\tilde{h}(N_1) \subset h(M) \subset N'/K$. Hence $N_1 \subset N'$, because v is the canonical epimorphism. Accordingly N_1 is also a direct summand of N', and $h\tilde{h} = v' \mid N_1$. If there exists $\tilde{h} : M \to N$ with $v\tilde{h} = h$, then $\tilde{h}(M) \subset N'$ as above. Therefore M is almost N'-projective. However M is not almost N/N'-projective in general. Hence we consider the following property:

(C) *If M is almost N-projective, then M is also almost N/N'-projective for any submodule N' of N.*

Related to (C) we can consider the following condition: If M is almost N-projective, then M' (or M/M') is also almost N-projective, where M' is any submodule of M. We shall study this property in the forthcoming paper **[12]**.

2. Right Nakayama rings. We assume that R is right artinian. If eR is a uniserial module for each primitive idempotent e, we call R a *right Nakayama ring.* We shall give several characterizations of right Nakayama rings with respect to the above properties (A), (B) and (C). We recall here the definitions of LPSM and lifting modules (**[9]** and **[13]**). Let $\{e_i\}_{i=1}^n$ be a set of primitive idempotents and A_i a submodule of e_iR for each i. If every element f in $\mathrm{Hom}_R(e_iR/e_iJ, e_jR/e_jJ)$ for any pair (i, j) (or f^{-1}) is lifted to an element in $\mathrm{Hom}_R(e_iR/A_i, e_jR/A_j)$ (or in $\mathrm{Hom}_R(e_jR/A_j, e_iR/A_i)$), then we say that $\sum_i^{\oplus} e_iR/A_i$ has the *lifting property of simple modules*, briefly LPSM. Next, let M be an R-module. If, for any submodule N of M, there exists a direct decomposition $M = M_1 \oplus M_2$ such that $N \supset M_1$ and $N \cap M_2$ is a small submodule of M_2, then we call M a *lifting module.*

We note that if e_1R/A_1 is almost e_2R/A_2-projective or $e_1R \not\approx e_2R$, then $e_1R/A_1 \oplus e_2R/A_2$ has LPSM by definition.

We frequently use the following.

LEMMA 1([**6**, Theorem 5]). *Assume that R is a semi-perfect ring. Let A_1 and A_2 be submodules of eR such that $eJ^{i+1} \subset A_j \subset eJ^i$ for $j=1,2$. If $eR/A_1 \oplus eR/A_2$ has* LPSM, *then $A_1 \subset A_2$ or $A_2 \subset A_1$. In particular R is a right Nakayama ring if and only if R is right artinian and every (two) finite direct sum of local modules eR/A_j has* LPSM *for each primitive idempotent e.*

Let M_1 be indecomposable. If M_0 is almost M_1-projective and M_1 is not a local module, then M_0 is M_1-projective by [**11**, Theorem 1]. Furthermore if a local module eR/A is almost fR/B-projective ($eR \not\approx fR$), then eR/A is fR/B-projective by definition. On the other hand (A), (B) and (C) always hold for relative projectives. From this observation we may study essentially (A), (B) and (C) on local modules eR/A for a fixed primitive idempotent e.

THEOREM 1. *Let R be a perfect ring and e a fixed primitive idempotent. Then the following conditions are equivalent*:

(1) *eR is a uniserial module (and hence $|eR| < \infty$),*

(2) (A) *holds on local modules eR/A,*

(3) (B) *holds whenever M_0, M_1 and M_2 are local modules of the form eR/A,*

(4) (C) *holds on local modules of the form eR/A.*

Proof. We first remark that $|eR| < \infty$ if eR is uniserial. Since R is perfect, $eR \supset eJ \supset eJ^2 \supset \cdots$ is a composition series of eR. Then $eJ^i = a_iR$ for some a_i in $eJ^i - eJ^{i+1}$ and $a_{i+1}R = eJ^{i+1} = a_iJ$, i.e., $a_{i+1} = a_ij_i$ for some $j_i \in J$. Therefore $eJ^n = 0$ for some n.

(1)$\Rightarrow$(2). Assume that eR/A_0 is almost eR/A_i-projective for $1 \leqq i \leqq m$. Since $eR/A_i \oplus eR/A_j$ has LPSM for $i \neq j$ by (1) and Lemma 1, eR/A_0 is almost $\sum^{\oplus} eR/A_i$-projective by [**9**, Theorem].

(2)$\Rightarrow$(1). Let A_1 and A_2 be the modules in Lemma 1 and $eJ^i \neq 0$. Since eJ^i is characteristic and $eJeeJ^i \subset eJ^{i+1} \subset A_j \subset eJ^i$, eR/eJ^i and eR/A_k are mutually almost relative projective for $k=1,2$ by [**5**, Proposition 2], but eR/eJ^i is neither eR/A_1 nor eR/A_2-projective by [**1**, p. 22, Exercise 4]. Then eR/eJ^i being almost $eR/A_1 \oplus eR/A_2$-projective by (2), $eR/A_1 \oplus eR/A_2$ has LPSM by [**9**, Theorem]. Hence $A_1 \subset A_2$ or $A_2 \subset A_1$ by Lemma 1. As a consequence eJ^i/eJ^{i+1} is simple, and so eR is uniserial.

(3)$\Rightarrow$(1). Let A_j be as above. Then eR/eJ^i is almost eR/A_j-projective, but not eR/A_j-projective as in (2)$\Rightarrow$(1) for $j=1,2$. Hence eR/A_1 is almost eR/A_2-projective by (3), and so $eR/A_1 \oplus eR/A_2$ has LPSM from the remark after the definition of LPSM. Therefore eR is uniserial as above.

(1)$\Rightarrow$(3). Let B_0, B_1 and B_2 be submodules of eR. Assume that eR/B_0 is almost eR/B_1-projective, but not eR/B_1-projective. Then $B_0 \supset B_1$ by (1). If eR/B_0 is almost eR/B_2-projective, then $eJeB_0 \subset B_2$ by [**11**, Proposition 2], and hence $eJeB_1 \subset eJeB_0 \subset B_2$. Therefore eR/B_1 is almost eR/B_2-projective by [**11**, Proposition 2].

(1)$\Rightarrow$(4). Assume that eR is uniserial and $M(=eR/A)$ is almost N $(=eR/B)$-projective. Then $eJeA \subset B$. Let N' be a submodule of N and $N/N' = eR/C$. Since $C \supset B$, $C \supset eJeA$. Further $eR/C \oplus eR/A$ has LPSM by assumption. Therefore M is almost N/N'-projective by [**11**, Proposition 2].

(4)$\Rightarrow$(1). Let A_i be as in (2)$\Rightarrow$(1). Then as is shown in (2)$\Rightarrow$(1), eR/A_1 is almost eR/eJ^{i+1}-projective. Hence eR/A_1 is almost eR/A_2-projective by (4). Accordingly eR is uniserial as before.

COROLLARY. *Let R be a right artinian ring. Then the following are equivalent*:
(1) *R is right Nakayama*,
(2) (A) *holds*,
(3) (B) *holds*,
(4) (C) *holds for local modules M and N.*

Proof. (1)$\Rightarrow$(2). Let R be right Nakayama. Let M_0, M_1 and M_2 be finitely generated R-modules. Assume that M_0 is almost M_i-projective for $i=1,2$. We take a direct decomposition of M_i into indecomposable modules T_{ij} ($i=1,2$; $j=1,\dots,n(i)$). Then M_0 is clearly almost T_{ij}-projective for all i, j. We may assume that there exists an integer $m(i)$ such that M_0 is almost T_{ik}-projective but not T_{ik}-projective for all $k \geqq m(i)$ and M_0 is $T_{ik'}$-projective for all $k' < m(i)$ $(i=1,2)$. Then T_{ik} is a local module by [**11**, Theorem 1] for $k \geqq m(i)$. Further, $\sum_{\substack{i=1,2 \\ k\geqq m(i)}}^{\oplus} T_{ik}$ has LPSM by Lemma 1 and the remark after the definition of LPSM. Hence M_0 is almost $M_1 \oplus M_2$-projective by [**9**, Theorem].

(2)$\Rightarrow$(3). We may assume by [**11**, Theorem 1] that M_1 is a local module. Let $M_2 = \sum^{\oplus} T_{2i}$ be the direct decomposition of M_2 as in (1)$\Rightarrow$(2). Then there exists an integer m such that M_0 is T_{2j}-projective for all $j<m$ and M_0 is almost $T_{2j'}$-projective but not $T_{2j'}$-projective for all $j' \geqq m$. Further since M_0 is almost $M_1 \oplus \sum_{j'\geqq m}^{\oplus} T_{2j'}$-projective by (2) and M_0 is not M_1-projective, $M_1 \oplus \sum_{j'\geqq m}^{\oplus} T_{2j'}$ is a lifting module by [**9**, Theorem]. Hence M_1 is almost $\sum_{j'\geqq m}^{\oplus} T_{2j'}$-projective by [**10**, Theorem 1]. Moreover M_1 is T_{2j}-projective for $j<m$ by [**9**, Proposition 1]. Hence M_1 is almost M_2-projective by (2).

The remaining implications are clear from Theorem 1 and the observation before Theorem 1.

From the above we know that (A) is equivalent to

(A′) (A) *holds for local modules M_i and N_j*

and (B) is equivalent to

(B′) (B) *holds for local modules M_0, M_1 and M_2.*

However we do not have the same result for (C) (see Theorem 2 below).

Next we shall study a dual result to Theorem 1. If every indecomposable injective module is uniserial, we say that R is *right co-Nakayama* ([**3**] and [**6**]). We shall give some characterizations of right co-Nakayama rings, which are dual to the Corollary. We refer to [**4**] for the definition of almost relative injectives. First we define properties ($A^{\#}$), ($B^{\#}$) and ($C^{\#}$) dual to (A), (B) and (C), respectively.

Let U_0, U_1 and U_2 be finitely generated R-modules.

($A^{\#}$) *If U_0 is almost U_i-injective for $i=1,2$, then U_0 is almost $U_1 \oplus U_2$-injective.*

Assume that U_1 is indecomposable.

($B^{\#}$) *If U_0 is almost U_1-injective but not U_1-injective, and U_0 is almost U_2-injective, then U_1 is always almost U_2-injective.*

($C^{\#}$) *If U_0 is almost U_2-injective, then U_0 is always almost U'-injective for any submodule U' of U_2.*

As the dual to the Corollary we obtain together with results in [**10**]

COROLLARY$^{\#}$. *Let R be a right artinian ring. Then the following conditions are equivalent*:
(1) *R is right co-Nakayama*,
(2) ($A^{\#}$) *holds*,
(3) ($B^{\#}$) *holds*,
(4) ($C^{\#}$) *holds whenever U_0 and U_2 are finitely generated and uniform.*

The implications (1)$\Leftrightarrow$(2) are given in [**4**].

Finally we shall give a characterization of two-sided Nakayama rings. Let $\{M_i\}_{i=1}^{n}$ be a set of indecomposable R-modules and $M=\sum_i^{\oplus} M_i$. Take a submodule N of M. If there exists a suitable direct decomposition $M=\sum_i^{\oplus} M_i'$ such that $M_i \approx M_i'$ for all i and $N=\sum_i^{\oplus} N\cap M_i'$, then we call N a *standard submodule.* The following lemma is well known.

LEMMA 2. *Let R be a two-sided Nakayama ring. Then any submodule T of $P=\sum_{i=1}^{n\,\oplus} e_iR$ is a standard submodule, where the e_i are primitive idempotents.*

Proof. See [**15**], [**16**, Section 55] or [**8**, Lemma 5].

If eRe is a local ring for an idempotent e, e is called a *local idempotent.* In this case eJ is the unique maximal submodule in eR.

LEMMA 3. *Let R be any ring and e_1, e_2 local idempotents. Let B be a submodule in $e_1J\oplus e_2J \subset e_1R\oplus e_2R$ and C a submodule in e_1J such that $B \not\supset C\oplus 0$. If e_1R/C is almost $(e_1R\oplus e_2R)/B$-projective, then $(e_1R\oplus e_2R)/B$ is decomposable.*

Proof. Since $M=(e_1R\oplus e_2R)/B$ is not local, e_1R/C is M-projective by [**11**, Theorem 1], provided M is indecomposable. However since e_1R is a projective cover of e_1R/C, and further there exists a natural homomorphism h of e_1R into M such that $h(C)\neq 0$ by assumption, e_1R/C is not M-projective by [**1**, p. 22, Exercise 4] (cf. the proof of [**5**, Lemma 6]). Therefore M is decomposable.

From [**1**, Proposition 2.5] and the dual result to [**11**, Theorem 1] we obtain dually to the above:

LEMMA $3^{\#}$. *Let U_1 and U_2 be indecomposable injective modules and B an essential submodule of $U_1\oplus U_2$. If A is almost B-injective for $0\neq A\subset U_1$ such that $\pi_1(B)\not\subset A$, then B is decomposable, where $\pi_1: U_1\oplus U_2\to U_1$ is the projection.*

Let e be a local idempotent. By $M(e)$ we denote the set of finitely generated R-modules M such that $M/J(M)=(eR/eJ)^{(n(M))}$, the direct sum of $n(M)$-copies of eR/eJ.

THEOREM 2. *Let R be a perfect ring and e a fixed primitive idempotent. Then the following conditions are equivalent*:

(1) *eR is uniserial (and hence $|eR|<\infty$) and every submodule in $eR \oplus eR$ is standard*,

(2) (C) *holds whenever M and N are any members in $\boldsymbol{M}(e)$*.

Compare [**7**, Theorem 5].

Proof. (1)$\Rightarrow$(2). In this proof A, A_j and A_j' mean submodules of eR. N is a direct sum of local modules, $N=\sum^{\oplus} eR/A_j$ by [**8**, Lemma 5]. Assume $M=M_1 \oplus M_2$. Then it is clear from the definition that M is almost N-projective if and only if M_i is so for $i=1,2$. Hence we may assume $M=eR/A$. Now $N/N' \approx \sum^{\oplus} eR/A_j'$ by [**8**, Lemma 5] for a submodule N' of N. Then each eR/A_j' is an epimorphic image of some eR/A_i. Hence we can assume that any A_j' contains some A_i. Since $A_i \supset eJeA$ by [**11**, Proposition 2], M is almost eR/A_j'-projective by the same proposition. As a consequence M is almost N/N'-projective by Theorem 1.

(2)$\Rightarrow$(1). Put $P=eR\oplus eR$ and take submodules $A_1 \subset B_1$, $A_2 \subset B_2$ in eR such that $h: B_1/A_1 \approx B_2/A_2$. We shall show that h (or h^{-1}) is induced from an element in $eRe=\mathrm{Hom}_R(eR, eR)$. If $B_i=eR$ for $i=1$ or 2, then this is clear. Hence we assume $B_i \subset eJ$ for $i=1,2$. Since eR is a uniserial module of finite length by Theorem 1, we may assume $A_1=eJ^{n_1}$, $B_1=eJ^{n_1-a}$, $A_2=eJ^{n_2}$ and $B_2=eJ^{n_2-a}$ $(n_1 \geqq n_2)$. Then eR/eJ^{n_1-1} is almost eR/eJ^{n_1}-projective and almost eR/eJ^{n_2}-projective by [**11**, Proposition 2]. Hence eR/eJ^{n_1-1} is almost $eR/eJ^{n_1} \oplus eR/eJ^{n_2}$-projective by Theorem 1. Put $C=(B_1/A_1)(h)$, the graph in $eRe/J^{n_1} \oplus eR/eJ^{n_2}$. Let $\tilde{C}$ be the submodule of P such that $\tilde{C} \supset eJ^{n_1} \oplus eJ^{n_2}$ and $C=\tilde{C}/(eJ^{n_1} \oplus eJ^{n_2})$. Then eR/eJ^{n_1-1} is almost $(P/(eJ^{n_1} \oplus eJ^{n_2}))/C=P/\tilde{C}\,(=\bar{P})$-projective by (2). Since $B_i \subset eJ$, $\bar{P}$ is decomposable by Lemma 3. Hence h is liftable to an element $\tilde{h}$ in $\mathrm{Hom}_R(eR/eJ^{n_1}, eR/eJ^{n_2})$ or in $\mathrm{Hom}_R(eR/eJ^{n_2}, eR/eJ^{n_1})$ by [**14**, Lemma 2.1] (cf. [**7**, p. 526, Remark]). $\tilde{h}$ is clearly liftable to an element in eRe. Therefore we obtain (1) by [**8**, Lemma 5].

COROLLARY. *Let R be a two-sided artinian ring. Then the following are equivalent*:

(1) *R is two-sided Nakayama*,

(2) (C) *holds whenever M and N are finitely generated R-modules*,

(3) ($\mathrm{C}^{\#}$) *holds whenever U_0 and U_2 are finitely generated R-modules.*

Proof. (1)$\Rightarrow$(2). Assume that R is two-sided Nakayama. Then every finitely generated R-module N is a direct sum of local modules, $N=\sum^{\oplus} e_jR/A_j$ (by Lemma 2). Hence we can use the same argument as in the proof of Theorem 2.

(2)$\Rightarrow$(1). We assume (C) for any finitely generated modules M and N. Then R is right Nakayama from the Corollary to Theorem 1. We may assume that R is a basic ring with $J^2=0$. Then eJ is simple or zero for any primitive idempotent e. Assume $h: e_1J \approx e_2J$ for two primitive idempotents e_1 and e_2. Then in the same manner as in the proof of Theorem 2 we can show that h is liftable to an element in e_1Re_2 or in e_2Re_1. Hence R is left Nakayama by [**14**, Lemma 4.3].

(1)$\Rightarrow$(3). If R is two-sided Nakayama, then every finitely generated R-module is a direct sum of uniserial modules. Hence R is right co-Nakayama and we may assume that U_0 is uniform and $U_2=\sum_i^{\oplus} V_i$; the V_i are unform. Since U_0 is almost U_2-injective, U_0 is almost V_i-injective for all i. Let U' be any submodule of U_2 and $U'=\sum_j^{\oplus} W_j$; the W_j are

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uniform. Then every W_j is monomorphic to some V_i, since W_j is uniform, and hence U_0 is almost W_j-injective from Corollary$^{\#}$. Therefore U_0 is almost U'-injective from Corollary$^{\#}$ ($(1)\Rightarrow(2)$). Hence ($\mathrm{C}^{\#}$) holds.

$(3)\Rightarrow(1)$. We assume ($\mathrm{C}^{\#}$) for any finitely generated R-modules U_0 and U. Then R is right co-Nakayama from Corollary$^{\#}$. In order to show that R is two-sided Nakayama, we may assume $J^2=0$. Let e be any primitive idempotent. We shall show that eR is uniserial i.e., eJ is simple, provided $eJ\neq 0$. Assume $eJ\neq 0$ and $eJ=A_1\oplus A_2\oplus B$, where A_1 is simple, so that $A_1\neq 0$, and similarly for A_2. Put $U_1=eR/(A_2\oplus B)$, $U_2=eR/(A_1\oplus B)$ and $U=U_1\oplus U_2$. Then U_i is a uniserial module with $|U_i|=2$ for $i=1,2$. Since R is a right co-Nakayama ring with $J^2=0$, U_i is injective. Put $U_0=A_1$ and $E=E(U_0)$ $(=U_1)$. Since $|U_i|=2$ and U_i is injective, U_0 is almost U_i-injective, by [**5**, Proposition 5]. As a consequence U_0 is almost U-injective from Corollary$^{\#}$. We take the submodule U' of U such that (i) $U'\supset J(U)=J(U_1)\oplus J(U_2)$, (ii) $U'/J(U)=\{\bar{x}+\bar{x}\mid \bar{x}\in eR/eJ\}\subset eR/eJ\oplus eR/eJ=U/J(U)$. Then U_0 is almost U'-injective by ($\mathrm{C}^{\#}$). Hence U' is decomposable by (ii) and Lemma $3^{\#}$. Since $|\mathrm{Soc}(U')|=2$ and $|U'|=3$, $U'=W_1\oplus W_2$ with $|W_1|=2$, $|W_2|=1$. From (i) and (ii) we know that W_1 is uniserial and $\pi_i\mid W_1$ is an isomorphism, where $\pi_i: U\to U_i$ is the projection for $i=1,2$. Hence U_1 is isomorphic to U_2 by $t=(\pi_2\mid W_1)(\pi_1\mid W_1)^{-1}$, which induces the identity mapping of eR/eJ by (ii). Therefore there exists j in eJe such that the left sided multiplication of $(e+j)$ gives t, i.e., $(e+j)(A_2\oplus B)=A_1\oplus B$. Thus $A_1\oplus B=A_2\oplus B$ for $J^2=0$. As a consequence eJ is simple, i.e., R is right Nakayama. Therefore R is two-sided Nakayama by [**3**, Theorem 5.4].

We shall give a right Nakayama ring where (B) does not hold if M_1 is not indecomposable. Let $R=\sum_i^{\oplus} e_iR$ be a right Nakayama ring with the following structure ($\{e_i\}_{i=1}^4$ is a set of mutually orthogonal primitive idempotents with $1=\sum e_i$):

$$\begin{array}{cccc}
e_1R & e_2R & e_3R & e_4R \\
| & | & | & | \\
(13)R & (22)R & (34)R & (42)R \\
| & | & | & | \\
(13)(34)R & (22)(22)R & 0 & 0 \\
| & | & & \\
0 & 0 & &
\end{array}$$

and $R=\sum^{\oplus} e_iK\oplus\sum^{\oplus}(ij)K\oplus\sum^{\oplus}(ij)(jk)K$, $(ij)=e_i(ij)e_j$ and other products among (ij) are zero except as in the above diagram, where K is a field. Put $A_0=(13)R$, $A_1=(13)(34)R$, $B_0=(22)(22)R$ and $B_1=(22)R$. Then $M_0=e_1R/A_0\oplus e_2R/B_0$ is almost M_1 $(=e_1R/A_1\oplus e_2R/B_1)$-projective, but not M_1-projective and M_0 is almost M_2 $(=e_2R)$-projective. However M_1 is not almost M_2-projective, since $(22)(22)\neq 0$. Similarly M_0' $(=e_1R/A_0)$ is almost M_1-projective, but not M_1-projective and M_0' is M_2-projective. However M_1 is not M_2-projective (cf. [**9**, Proposition 1]).

3. Transitivity on relative projectives. In this section we shall investigate the transitivity of relative projectives: if M_0 is M_1-projective and M_1 is M_2-projective, is M_0 then M_2-projective? The similar property on almost relative projectives is in some sense the converse of (B).

PROPOSITION 1. *Let R be a perfect ring. Then transitivity of relative projectives on $\boldsymbol{M}(e)$ always holds. If transitivity on $\boldsymbol{M}(e_1)\cup\boldsymbol{M}(e_2)$ holds, then $e_1J=e_2J=0$, where $e_1R\approx e_2R$.*

Proof. Let M_1, M_2 be in $\boldsymbol{M}(e)$. Assume that M_1 is M_2-projective and take a projective cover P_i of M_i, $i=1,2$: $P_i=\sum_{j=1}^{n_i}{}^{\oplus} eR_{ij}$; $eR_{ij}=eR$, and $M_i=P_i/A_i$. Let f be an element in $\mathrm{Hom}_R(P_1,P_2)$; $f=\sum r_{ij}e_{ij}(P_1\to P_2)$, where $e_{ij}(P_1\to P_2)=1_{eR}$: $eR_{1i}\to eR_{2j}$, $r_{ij}\in eRe$. Since $r_{ij}e_{ij}\in\mathrm{Hom}_R(P_1,P_2)$, $(r_{ij}e_{ij})A_1\subset A_2$ by [**1**, p. 22, Exercise 4]. Assume that M_2 is M_3-projective and P_3 is a projective cover of M_3 as above. Take any element g in $\mathrm{Hom}_R(P_1,P_3)$ and $g=(g_{ij})$. Then $e_{ij}(P_1\to P_3)=e_{ij}(P_2\to P_3)e_{i1}(P_1\to P_2)$ and $g_{ij}e_{ij}(P_1\to P_3)A_1=g_{ij}e_{1j}(P_2\to P_3)e_{i1}(P_1\to P_2)A_1\subset g_{ij}e_{1j}(P_2\to P_3)A_2\subset A_3$. Hence $g(A_1)\subset A_3$. Accordingly we obtain the transitivity. Next we assume the second condition in the proposition. Here e_1R/e_1J is e_2R/e_2J^2-projective and e_2R/e_2J^2 is e_1R/e_1J^3-projective. Hence e_1R/e_1J is e_1R/e_1J^3-projective by the transitivity. Therefore $e_1J=e_1J^2$, and hence $e_1J=0$.

From Proposition 1 we have

THEOREM 3. *Let R be a perfect ring. Then transitivity of relative projectives over finitely generated modules holds if and only if either R is semi-simple or $R/J(R)$ is a simple ring.*

We shall study the above problem for almost relative projectives. In this case the transitivity is the converse of (B). Let M_0, M_1 and M_2 be finitely generated R-modules. The property of transitivity of almost relative projectives is

(B_1) *If M_0 is almost M_1-projective and M_1 is almost M_2-projective, then M_0 is always almost M_2-projective.*

LEMMA 4. *Let R be any ring and $\{e_i\}_{i=1}^m$ a set of local idempotents. Put $P=\sum_{i=1}^{m}{}^{\oplus} e_iR$ and $\bar P=\sum^{\oplus} e_iR/e_iJ^n$ for a fixed integer n. Assume that every submodule of $\bar P$ is standard. Then every submodule A of P which contains $\sum^{\oplus} e_iJ^n$ is also standard in P.*

Proof. Put $\bar e_i\bar R=e_iR/e_iJ^n$ and $\bar A=A/(\sum^{\oplus} e_iJ^n)$. Then there exists a direct decomposition of $\bar P:=\sum^{\oplus}\bar P_i$ such that $\bar A=\sum^{\oplus}(\bar P_i\cap\bar A)$ and $\bar P_i\approx\bar e_i\bar R$. Since $\mathrm{End}_R(\bar e_i\bar R)$ is a local ring, we may assume that $\bar P_1=\bar e_1\bar R(\bar g_1)$, where $\bar g_1:\bar e_1\bar R\to\sum_{j\neq 1}^{\oplus}\bar e_j\bar R$. Then

$$\bar P=\bar P_1\oplus\bar e_2\bar R\oplus\cdots\oplus\bar e_m\bar R=\bar P_1\oplus\bar P_2\oplus\cdots\oplus\bar P_m.$$

Considering the projection of $\bar P$ onto $\sum_{j\geqslant 2}^{\oplus}\bar e_j\bar R$ in the above we may assume that $\bar P_2=\bar e_2\bar R(\bar g_2)$ with $\bar g_2:\bar e_2\bar R\to\bar P_1\oplus\sum_{k\geqslant 3}^{\oplus}\bar e_k\bar R$. Hence we obtain inductively $\bar g_i:\bar e_i\bar R\to\sum_{k<i}^{\oplus}\bar P_k\oplus\sum_{k'>i}^{\oplus}\bar e_{k'}\bar R$ such that $\bar P_i=\bar e_i\bar R(\bar g_i)$ and $\bar P=\sum_{k<i}^{\oplus}\bar P_k\oplus\sum_{k'>i}^{\oplus}\bar e_{k'}\bar R$. By induction we can show that $\bar g_i$ is liftable to an element $g_i:e_iR\to\sum_{k\leqq i}^{\oplus}e_kR(g_k)\oplus\sum_{k'>i}^{\oplus}e_{k'}R$ and $P=\sum_{k\leqq i}^{\oplus}e_iR(g_i)^{\oplus}\sum_{k'>i}^{\oplus}e_{k'}R$. Therefore we have $P=\sum_{i=1}^{m}{}^{\oplus} e_iR(g_i)$ and $\bar{e}_i\bar{R}(g_i)=\bar{P}_i$ for all i. Since $A\supset\sum^{\oplus}e_jJ^n=\sum^{\oplus}e_iR(g_i)J^n$, $A\supset\sum^{\oplus}(e_iR(g_i)\cap A)\supset\sum^{\oplus}e_iR(g_i)J^n$. As a consequence $A=\sum^{\oplus}(e_iR(g_i)\cap A)$.

THEOREM 4. *Let R be any ring such that R/J is artinian, and let e be a fixed local idempotent. Then the following are equivalent.*

(1) (B_1) *holds on the set of local modules of the form eR/A, where $eR\supset A\supset eJ^n$ for some n.*

(2) *Any two local modules of the form eR/A are mutually almost relative projective, where A is as in* (1).

(3) *eR/N is uniserial with respect to submodules A/N with $|eR/A|<\infty$ and $(eJ)^2\subset N$, i.e., any simple sub-factor module eJ^i/eJ^{i+1} of eR/N except eR/eJ and* $\mathrm{Soc}(eR/N)$ *(if it exists) is not isomorphic to eR/eJ, where $N=\bigcap_n(eJ^n)$.*

Further let $\boldsymbol{M}'(e)$ be the set $\{M\}$ given before Theorem 2 such that M is a homomorphic image of $(eR/eJ^n)^{(m)}$ for some integers n and m. Then the following are equivalent.

(4) (B_1) *holds on $\boldsymbol{M}'(e)$.*

(5) *Any two R-modules in $\boldsymbol{M}'(e)$ are mutually almost relative projective.*

(6) *eR/N is unserial with respect to A/N as in* (3) *and for any n every submodule of $eR\oplus eR$ which contains $eJ^n\oplus eJ^n$ is standard.*

Proof. (2)$\Rightarrow$(1). This is trivial.

(1)$\Rightarrow$(3). Assume $eJ^i\neq eJ^{i+1}$. Let A be a submodule such that $eJ^{i+1}\subset A\subset eJ^i$. Then $eReA\subset eJ^i$ and $eJeeReA\subset eJeeJ^i\subset eJ^{i+1}\subset A$. Hence eR/A and $eR/eReA$ are mutually almost relative projective by [**5**, Proposition 2]. Hence eR/A is (almost) eR/A-projective by (1), and so A is characteristic. Let B be another submodule such that $eJ^{i+1}\subset B\subset eJ^i$. Then since A, $A+B$ and B are characteristic, $A\subset A+B$ and $eJe(A+B)\subset eJ^{i+1}\subset B$, eR/A is (almost) $eR/(A+B)$-projective and $eR/(A+B)$ is almost eR/B-projective. Hence eR/A is almost eR/B-projective and so $eR/A\oplus eR/B$ has LPSM. Therefore $A\subset B$ or $B\subset A$ by Lemma 1. Accordingly eJ^i/eJ^{i+1} is simple. Hence eR/N is uniserial with respect to A/N. Since eR/eJ^i is almost eR/eJ^{i+1}-projective for any i, we know that eR/eJ is almost eR/eJ^n-projective for all n by (B_1). Hence $eJeeJ\subset eJ^n$, and so $(eJ)^2\subset N$.

(3)$\Rightarrow$(2). This is clear from [**5**, Proposition 2].

(5)$\Rightarrow$(4). This is clear.

(4)$\Rightarrow$(6). We assume (B_1). Then we know from the first equivalence that eR/N is uniserial with respect to A/N, and eR/eJ is almost eR/N-projective by (3), since $N\subset eJ$ are characteristic. Moreover eR/N is M'-projective from [**1**, p. 22, Exercise 4] for any M' in $\boldsymbol{M}'(e)$. Hence by (B_1) eR/eJ is almost M'-projective. Let $eJ^n\subset A_i\subset B_i$ be submodules in eR, $i=1,2$ such that $B_1/A_1\approx B_2/A_2$ via g. First we shall show that g is liftable to an element in $\mathrm{Hom}_R(eR/eJ^n, eR/eJ^n)$ (cf. the proof (2)$\Rightarrow$(1) in Theorem 2). If $B_1=eR$, g is easily liftable to an element in $\mathrm{Hom}_R(eR/eJ^n, eR/eJ^n)$. Hence we assume $B_i\subset eJ$. Put $M=(eR\oplus eR)/B_1(g)B_2\in\boldsymbol{M}'(e)$, where $B_1(g)B_2=\{b_1+b_2\in B_1\oplus B_2 \mid g(b_1+A_1)=b_2+A_2\}$. Then eR/eJ is almost M-projective from the above. Therefore M is decomposable by Lemma 3 and hence g is liftable to an element in $\mathrm{Hom}_R(eR/A_1, eR/A_2)$ or $\mathrm{Hom}_R(eR/A_2, eR/A_1)$ by [**14**, Lemma 2.1] (cf. [**7**, p. 526, Remark]). As a consequence g is liftable to an element in $\mathrm{Hom}_R(eR/eJ^n, eR/eJ^n)$, since g is given by an element in eRe.

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Thus every submodule of $eR/eJ^n \oplus eR/eJ^n$ is standard by [**8**, Lemma 5], and we obtain (6) from Lemma 4.

(6)$\Rightarrow$(5). Assume that eR/N is uniserial as in (6). Then eR/A_1 is almost eR/A_2-projective for any $A_i \subset eR$ with $|eR/A_i| < \infty$ by (2). Further every module in $\boldsymbol{M}'(e)$ is a direct sum of local modules eR/A by [**8**, Lemma 5]. Hence (5) holds by [**9**, Theorem] (note that every module in $\boldsymbol{M}'(e)$ is of finite length).

Assume that R is semi-perfect and that (1) in Theorem 4 holds for any local modules. Then the above proof (1)$\Rightarrow$(3) shows that the set of right ideals $\{A \subset eR \mid |eR/A| < \infty\}$ is uniserial. Hence eR/eJ^i is almost eR/eJ^j-projective for any i and j. Further since eR/eJ^i is (almost) fR/fJ^{i+1}-projective (f is a primitive idempotent not isomorphic to e), any two local modules with finite length are mutually almost relative projective. Hence from [**8**, Theorem 4] and the proof of Theorem 4 we can get

PROPOSITION 2. *Let R be a semi-perfect ring. Then the following are equivalent*:

(1) (B_1) *holds for any local modules*,

(2) *R is a right Nakayama ring with radical square-zero.*

Further the following are equivalent:

(3) (B_1) *holds for any finitely generated R-modules*,

(4) *R is a two-sided Nakayama ring with $J^2 = 0$.*

4. Transitivity on relative injectives. We shall explore here the dual results to the previous section. We can dually define the concepts of transitivity of (almost) relative injectives and ($B_1^{\#}$). Let S be a simple R-module. By $\boldsymbol{M}(S)$ we denote the set of R-modules M such that $\mathrm{Soc}(M) \approx S^{(n(M))}$ with $n(M) < \infty$; by $\boldsymbol{M}'(S)$ we denote the set of M in $\boldsymbol{M}(S)$ such that $|M| < \infty$.

PROPOSITION $1^{\#}$. *Let R be any ring and S_1 a simple R-module. Then the transitivity of relative injectives on $\boldsymbol{M}(S_1)$ holds. Assume further that R is a right semi-artinian ring. In this case, if the transitivity on $\boldsymbol{M}(S_1) \cup \boldsymbol{M}(S_2)$ holds, then S_1 and S_2 are injective, where S_2 is a simple module not isomorphic to S_1.*

THEOREM $3^{\#}$. *Let R be a perfect ring. Then the transitivity of relative injectives over the modules with finite Goldie dimension holds if and only if either R is semi-simple or $R/J(R)$ is a simple ring.*

We can obtain the dual result to Proposition 2, which we skip. Finally we observe the dual to Theorem 4.

THEOREM $4^{\#}$. *Let R be any ring and S a fixed simple R-module. Then the following are equivalent.*

(1) ($B_1^{\#}$) *holds on the set of all submodules with finite length in $E(S)$.*

(2) *Any two submodules of finite length in $E(S)$ are mutually almost relative injective.*

(3) *$E' = \bigcup_n \mathrm{Soc}_n(E)$ is a uniserial module such that any simple sub-factor module of E' except $E'/J(E')$ (if it exists) and $\mathrm{Soc}(E)$ is not isomorphic to $\mathrm{Soc}(E)$. (Here $\{\mathrm{Soc}_n(E)\}$ is the lower Loewy series of E.)*

Further the following are equivalent.

(4) ($B_1^{\#}$) *holds on $\boldsymbol{M}'(S)$.*

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(5) *Any two modules in* $M'(S)$ *are mutually almost relative injective.*
(6) E' *is a uniserial module as in* (3) *and every submodule of* $E' \oplus E'$ *is standard.*

Proof. First we note that if $\mathrm{Soc}_n(E)$ is uniserial for all n, and if a submodule A in E' is not contained in $\mathrm{Soc}_n(E)$ for any n, then $A = E'$, provided $E' \neq \mathrm{Soc}_t(E)$ for all t. Since $A \not\subset \mathrm{Soc}_n(E)$ for any n, there exists a in $A \backslash \mathrm{Soc}_n(E)$. A being a submodule in E', a is an element in $\mathrm{Soc}_m(E)$ for some m, and hence $m > n$. We may assume $a \notin \mathrm{Soc}_{m-1}(E)$. Since $\mathrm{Soc}_m(E)$ is uniserial, $\mathrm{Soc}_n(E) \subset \mathrm{Soc}_m(E) = aR \subset A$. Therefore $A \supset \mathrm{Soc}_n(E)$. Now we prove the theorem. This is dual to Theorem 4. Hence we shall show only (4)$\Rightarrow$(6). Put $E = E(S)$. Then $E' = \bigcup_n \mathrm{Soc}_n(E)$ is a uniserial module as in (3) from the first equivalence. Set $E^* = E_1 \oplus E_2$ with $E_i = E'$ and denote the projection of E^* onto E_i by θ_i. Let B^* be a submodule of E^*, and put $B_i = \theta_i(B^*)$ and $A_i = E_i \cap B^*$. Then $g: B_1/A_1 \approx B_2/A_2$ and $B^* = B_1(g)B_2$ (cf. [**8**]). If $A_1 = 0$, then g is liftable to an element $\tilde{g}$ in $\mathrm{End}_R(E')$, since E' is characteristic in E. Hence $E^* = E_1 \oplus E_2(\tilde{g}) \supset 0 \oplus B_2(g) = B^*$. Therefore we assume $A_i \neq 0$ for $i = 1, 2$. In the dual manner to the proof of Theorem 4, we can show from the first equivalence that S is almost B^*-injective. On the other hand, since we may assume $A_i \neq 0$ and $B_i \neq A_i$, B^* is decomposable by Lemma $3^{\#}$. Further since $\mathrm{Soc}(B^*) = S \oplus S$, $B^* = D_1 \oplus D_2$ and the D_i are isomorphic to submodules in E' for the D_i are uniform. Hence they are uniserial. Assume $B_1 = E'$ and $|E'| = \infty$. Then $|D_1| = \infty$ or $|D_2| = \infty$. We may assume that $|D_1| = \infty$. Since D_1 is uniform, $\theta_i \mid D_1$ is a monomorphism for $i = 1$ or 2, say $i = 1$. Since $|D_1| = \infty$, $\theta_1 \mid D_1$ is an isomorphism from the initial remark. Putting $h = \theta_2(\theta_1 \mid E')^{-1}: E' \to E'$, we obtain $E^* = E'(h) \oplus E'$ and $D_1 = E'(h) \subset B^*$. Hence $B^* = E'(h) \oplus B^* \cap E'$ and $B^* \cap E' \subset E'$. Therefore B^* is standard. Finally we assume $|B_i| < \infty$ for $i = 1, 2$. Then $B' = B \oplus B_2 \supset B^*$. Let j_i be the projection of B' onto B_i. We may suppose $|B_1| \geqq |B_2|$. Since $B^* = B_1(g)B_2$, $\pi_1(B^*) = B_1$, and hence we assume $\pi_1(D_1) = B_1$. On the other hand, since D_1 is uniform, D_1 is monomorphic to a submodule of B_1 or B_2, i.e. $|D_1| \leqq |B_1|$. Hence $\pi_1 \mid D_1$ is an isomorphism. Put $h = \pi_2\pi_1^{-1} \mid B_1 : B_1 \to B_2$. Then $B' = B_1(h) \oplus B_2$ and $B_1(h) = D_1$. As a consequence $B^* = B_1(h) \oplus B^* \cap B_2$. Since E is injective and E' is characteristic, we obtain an extension h' of h in $\mathrm{End}_R(E')$. Hence $E^* = E_1(h') \oplus E_2 \supset B_1(h') \oplus B^* \cap E_2 \supset B_1(h) \oplus B^* \cap B_2 = B^*$, i.e. B^* is standard in E^*.

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Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyosi-Ku
Osaka 558
Japan