EXISTENCE OF SOLUTIONS OF THIRD-ORDER FUNCTIONAL PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract

In this paper some existence results for third-order differential equations with nonlinear boundary value conditions are derived. Functional dependence in the data is allowed. In the proofs we use the method of upper and lower solutions, Schauder's fixed point theorem and results from Cabada and Heikkilä on third-order differential equations with linear and nonfunctional initial-boundary value conditions.

1. Introduction

Third-order equations arise in a large number of physical and technological processes, such as the deflection of a curved beam with a constant or varying cross-section, three layer beams and electromagnetic waves or gravity-driven flows; see [1, 9, 19] for details. In studying most of the considered problems, the authors reduced them to some related first- and/or second-order equations. The existence results follow, among other techniques, by applying degree theory, monotone iterative techniques or lower and upper solutions to the equivalent problems, see for instance [1, 5, 19]. In other cases, the third-order problem is approached directly by using Green's functions and comparison principles. In consequence, the method of lower and upper solutions is developed for the particular equation considered in each situation. In this direction, the reader can see the papers [3, 4, 12] where periodic boundary value conditions are considered, [13–17] in which three-point boundary conditions are studied, and [5, 7, 10, 20] in which two-point boundary conditions are studied.

Nonlinear boundary conditions, coupled with the method of lower and upper solu-

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tions, were considered by Wang in [18], where the problem

$$
\begin{cases}
y''' = f(t, y, y', y''), \quad y(a) = y_1, \\
h(y'(a), y''(a)) = 0, \\
g(y(b), y'(b), y''(b)) = 0,
\end{cases}
$$

(and the symmetric boundary conditions) was treated; and Chen, who obtained in [8] existence results for the problem

$$
\begin{cases}
u''' = f(t, u, u', u''), \\
u'(0) = g_1(u(0)), \quad u''(0) = g_2(u(0)), \quad u''(1) = g_3(u(1)).
\end{cases}
$$

In this paper we study the nonlinear third-order functional initial-boundary value problem (IBVP)

$$
\frac{d}{dt} \varphi((\mu \cdot u')')(t) = f(t, u) \quad \text{for a.e. } t \in [t_0, t_1], \tag{1.1}
$$

$$
L_1(u(t_0), u(t_1), u'(t_0), u'(t_1), u) = 0, \tag{1.2}
$$

$$
L_2(u(t_0), u(t_1)) = 0, \tag{1.3}
$$

$$
(\mu \cdot u')'(t_0) = c_2. \tag{1.4}
$$

Here $c_2, t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, are given. In what follows we denote $J = [t_0, t_1]$. Here $C^j(J)$ will be the set of those real-valued functions u whose j-th derivatives are continuous in J. By $AC(J)$ we denote the set of those real-valued functions which are absolutely continuous in J and by $L^1(J)$ we denote the set of Lebesgue measurable functions such that $\int_J |f(x)| dx < \infty$. Here $L^\infty(J)$ will be the set of Lebesgue measurable functions that are essentially bounded in J. The functions μ, φ, f, L_1 and L_2 are assumed to satisfy the following conditions:

$(\varphi\mu)$ $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and $\mu : J \to (0, \infty)$ is continuous.

(f1) $f : J \times C(J) \to \mathbb{R}, f(\cdot, v)$ is measurable for all fixed $v \in C(J)$, and for each $R > 0$ there exists a function $m_R \in L^1(J)$ such that $|f(t, v)| \leq m_R(t)$ for a.e. $t \in J$ and for all $v \in C(J)$ satisfying $\| v \|_\infty \leq R$.

(f2) $f(t, v_n) \to f(t, v)$ for a.e. $t \in J$ whenever $v_n \to v$ in $C(J)$.

(L) $L_1 \in C(\mathbb{R}^4 \times C(J), \mathbb{R})$ is increasing in the third variable, decreasing in the fourth and increasing in the fifth one. $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and is decreasing with respect to its first variable.

A function u is said to be a *solution* of problem (1.1)–(1.4) if it satisfies these four equations, and belongs to the set

$$
Y = \{u \in C^1(J) \mid \varphi \circ (\mu \cdot u')' \in AC(J)\}, \tag{1.5}
$$

where $\varphi \circ (\mu \cdot u')'$ denotes the composition of φ and $(\mu \cdot u')'$. We say that $u \in Y$ is a *lower solution* for problem (1.1)–(1.4) if $L_2(u(t_0), \cdot)$ is injective in $\mathbb{R}$ and u satisfies the following inequalities:

$$\begin{cases} \dfrac{d}{dt}\varphi((\mu \cdot u')')(t) \geq f(t, u) \quad \text{for a.e. } t \in J, \\ L_1(u(t_0), u(t_1), u'(t_0), u'(t_1), u) \geq 0 = L_2(u(t_0), u(t_1)), \quad (\mu \cdot u')'(t_0) \geq c_2, \end{cases}$$

and an *upper solution* if the reversed inequalities hold.

For any pair of continuous functions $v, w : J \to \mathbb{R}$ such that $v \leq w$ in J, we denote $[v, w] = \{x \in C(J); v(t) \leq x(t) \leq w(t), \text{ for all } t \in J\}$.

Assuming the existence of a lower solution α and an upper solution β such that $\alpha \leq \beta$ in J, we prove the existence of solutions of problem (1.1)–(1.4) lying between α and β. Before proving this existence result in Section 3, we derive in Section 2 some existence results for related nonlinear IBVPs for which functional dependence on the right-hand side of the four equations is allowed. In this case the existence of lower and upper solutions is not assumed. These results generalise some of those obtained by the authors in [6], where similar results were obtained when functional dependence was not allowed in the boundary conditions. Finally, in Section 4, we present an example where our main existence result is applied to a particular case and deduce an existence result for the physical model of the deformation of an elastic beam.

2. First existence results

In this section we study the solvability of the functional IBVP

$$\begin{cases} \dfrac{d}{dt}\varphi((\mu \cdot u')')(t) = g(t, u, u'(t)) \quad \text{for a.e. } t \in J, \\ a_0 u(t_0) - b_0 u'(t_0) = C_0(u), \\ a_1 u(t_1) + b_1 u'(t_1) = C_1(u), \\ \qquad\qquad (\mu \cdot u')'(t_0) = C_2(u), \end{cases} \tag{2.1}$$

in the set Y defined in (1.5). We assume that $C(J)$ is ordered pointwise and normed by the maximum norm and impose the following hypotheses to functionals C_i:

(C1) $C_i : C(J) \to \mathbb{R}$ is continuous for each $i = 0, 1, 2$.
(C2) There exists $\bar{K} > 0$ such that $|C_i(u)| \leq \bar{K}$ for all $u \in C(J)$ and $i = 0, 1, 2$.

The constants a_i and b_i satisfy the hypothesis

(A) $a_0, a_1, b_0, b_1 \in \mathbb{R}_+, a_0 a_1 + a_0 b_1 + a_1 b_0 > 0$.

The function $g : J \times C(J) \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following conditions:

(g1) $g(\cdot, v, x)$ is measurable and $|g(\cdot, v, x)| \leq m \in L^1(J)$ for all $v \in C(J)$ and $x \in \mathbb{R}$.

(g2) $g(t, v_n, x_n) \to g(t, v, x)$ for a.e. $t \in J$ whenever $v_n \to v$ in $C(J)$ and $x_n \to x$.

Denote

$$
\begin{cases}
y_0(t) = \displaystyle\int_{t_0}^{t} \frac{a_0}{\mu(s)}\, ds + \frac{b_0}{\mu(t_0)}, \quad y_1(t) = \int_{t}^{t_1} \frac{a_1}{\mu(s)}\, ds + \frac{b_1}{\mu(t_1)}, \quad t \in J, \\[3mm]
D = \displaystyle\int_{t_0}^{t_1} \frac{a_0 a_1}{\mu(s)}\, ds + \frac{a_0 b_1}{\mu(t_1)} + \frac{a_1 b_0}{\mu(t_0)}, \\[3mm]
k(t, s) = \begin{cases} y_1(t) y_0(s)/D, & t_0 \leq s \leq t, \\ y_0(t) y_1(s)/D, & t \leq s \leq t_1, \end{cases}
\end{cases}
\tag{2.2}
$$

and

$$
\begin{cases}
z_0(t) = \max\{y_0(t), a_0/\mu(t)\}, \quad z_1(t) = \max\{y_1(t), a_1/\mu(t)\}, \quad t \in J, \\[3mm]
l(t, s) = \begin{cases} z_1(t) y_0(s)/D, & t_0 \leq s \leq t, \\ z_0(t) y_1(s)/D, & t \leq s \leq t_1, \end{cases} \\[3mm]
M(t) = \displaystyle\max_{u \in C(J)} \left\{ \left| \varphi^{-1}\left(\varphi(C_2(u)) + \int_{t_0}^{t} m(x)\, dx \right) \right|, \right. \\[3mm]
\qquad\qquad \left. \left| \varphi^{-1}\left(\varphi(C_2(u)) - \int_{t_0}^{t} m(x)\, dx \right) \right| \right\}, \quad t \in J,
\end{cases}
\tag{2.3}
$$

and define a function $b \in C(J)$ by

$$
b(t) = \max_{u \in C(J)} \left\{ \frac{|C_0(u)| z_1(t) + |C_1(u)| z_0(t)}{D} \right\} + \int_{t_0}^{t_1} \ell(t, s) M(s)\, ds, \quad t \in J. \tag{2.4}
$$

Clearly, if (A) and $(\varphi\mu)$ hold then $D > 0$ and y_0, z_0, y_1, z_1, k and l are nonnegative-valued functions.

We shall next prove an existence result for the nonlinear IBVP (2.1) by using Schauder's fixed point theorem. The arguments are similar to those used in [6, Theorem 4.1], where C_0, C_1 and C_2 are constant functions and $g \equiv g(x, u, u'(x), (\mu \cdot u')'(x))$.

In the proof of the above mentioned existence result we apply the following result of [6].

PROPOSITION 2.1. *If the hypotheses $(\varphi\mu)$ and (A) hold, if f is Lebesgue integrable, and $c_0, c_1, c_2 \in \mathbb{R}$, then the IBVP*

$$
\begin{cases}
\dfrac{d}{dt} \varphi((\mu \cdot u')'(t)) = f(t), \quad \text{for a.e. } t \in J, \\[3mm]
a_0 u(t_0) - b_0 u'(t_0) = c_0, \quad a_1 u(t_1) + b_1 u'(t_1) = c_1, \quad (\mu \cdot u')'(t_0) = c_2,
\end{cases}
\tag{2.5}
$$

has a unique solution u in Y, and can be represented as

$$u(t) = \frac{c_0 y_1(t) + c_1 y_0(t)}{D} + \int_{t_0}^{t_1} k(t,s) h(s)\, ds, \quad t \in J,$$

where y_0, y_1, D and k are given in (2.2) and

$$h(s) = -\varphi^{-1}\left(\varphi(c_2) + \int_{t_0}^{s} f(x)\, dx\right), \quad s \in J.$$

THEOREM 2.1. *Assume that the hypotheses ($\varphi\mu$), (C1), (C2), (A), (g1) and (g2) hold, and let b be defined by (2.4). Then problem (2.1) has a solution in the set $Y \cap B$, where Y is given in (1.5) and*

$$B = \left\{ u \in C^1(J) \mid \max\{|u(t)|, |u'(t)|\} \le b(t),\ t \in J \right\}. \tag{2.6}$$

PROOF. Conditions (g1) and (g2) imply that the function $\sigma : J \to \mathbb{R}$, defined by $\sigma(t) = g(t, u, u'(t))$, is Lebesgue integrable for each $u \in B$. Thus, from Proposition 2.1, we see that $u \in Y$ is a solution of problem (2.1) if and only if

$$u(t) = \frac{C_0(u) y_1(t) + C_1(u) y_0(t)}{D} + \int_{t_0}^{t_1} k(t,s) g_u(s)\, ds, \quad t \in J, \tag{2.7}$$

where

$$g_u(s) = -\varphi^{-1}\left(\varphi(C_2(u)) + \int_{t_0}^{s} g(x, u, u'(x))\, dx\right). \tag{2.8}$$

To prove that (2.7) admits a solution, we define a mapping F on B by

$$Fu(t) = \frac{C_0(u) y_1(t) + C_1(u) y_0(t)}{D} + \int_{t_0}^{t_1} k(t,s) g_u(s)\, ds, \quad t \in J. \tag{2.9}$$

It follows, by differentiation, that for a.e. $t \in J$

$$(Fu)'(t) = \frac{-a_1}{\mu(t) D}\left(C_0(u) + \int_{t_0}^{t} y_0(s) g_u(s)\, ds\right)$$
$$+ \frac{a_0}{\mu(t) D}\left(C_1(u) + \int_{t}^{t_1} y_1(s) g_u(s)\, ds\right). \tag{2.10}$$

Since the right-hand side of (2.10) is continuous, it holds for all $t \in J$. Thus it follows from (2.10) that $Fu \in C^1(J)$, and that $\mu \cdot (Fu)' \in AC(J)$. Consequently, (2.10) and (2.2) imply that

$$\frac{d}{dt}\left(\mu \cdot (Fu)'\right)(t) = -g_u(t) \quad \text{for a.e. } t \in J. \tag{2.11}$$

Since the right-hand side of the above equation is continuous, it holds for all $t \in J$, and it can be rewritten as

$$\varphi((\mu \cdot (Fu)')')(t) = \varphi(C_2(u)) + \int_{t_0}^{t} g(x, u, u'(x)) \, dx, \quad t \in J, \qquad (2.12)$$

which implies that $Fu \in Y$.

From condition (g1) and from the definition of function F we conclude by (2.3), (2.4) and (2.6) that $F[B] \subseteq B$.

Define a Banach norm in $C^1(J)$ by

$$\|u\| = \max_{t \in J} \{|u(t)|, |u'(t)|\}. \qquad (2.13)$$

Obviously, B is a closed and convex subset of $C^1(J)$. To prove that F is continuous in B, assume that u_n, $u \in B$, $n \in \mathbb{N}$. Denoting by $k_0 = \max\{l(t, s) \mid t, s \in J\}$, it follows from (2.9) and (2.10) that for each $t \in J$,

$$|(Fu_n)'(t) - (Fu)'(t)| \le k_0 \int_{t_0}^{t_1} |g_{u_n}(s) - g_u(s)| \, ds$$
$$+ \frac{a_1}{\mu(t)D} |C_0(u_n) - C_0(u)| + \frac{a_0}{\mu(t)D} |C_1(u_n) - C_1(u)|$$

and

$$|Fu_n(t) - Fu(t)| \le k_0 \int_{t_0}^{t_1} |g_{u_n}(s) - g_u(s)| \, ds$$
$$+ \frac{|C_0(u_n) - C_0(u)|y_1(t) + |C_1(u_n) - C_1(u)|y_0(t)}{D}.$$

The above inequalities, conditions (C1) and (C2) together with the hypothesis (g2) and the dominated convergence theorem imply that $\|Fu_n - Fu\| \to 0$ as $\|u_n - u\| \to 0$. Thus F is continuous in B.

Equation (2.4) together with (2.9) and (2.10) imply that the set $\{Fu \mid u \in B\}$ is uniformly bounded and equicontinuous in $C(J)$. On the other hand, since $\mu \cdot (Fu)' \in AC(J)$, $Fu \in Y$ and μ is a continuous and positive function, from (2.11) we obtain that

$$(Fu)'(t) = \frac{1}{\mu(t)} \left(\mu(t_0)(Fu)'(t_0) - \int_{t_0}^{t} g_u(s) \, ds \right), \quad t \in J.$$

Now, from (2.10), conditions (g1), (g2), (C1) and (C2) and the definition of g_u, we conclude that the functions $\{(Fu)', u \in B\}$ form a uniformly bounded and equicontinuous set in $C(J)$.

These results ensure by Ascoli-Arzela's theorem that given any sequence $\{u_n\}$ of B there is a subsequence of $\{Fu_n\}$ which converges with respect to the norm given by (2.13).

The above proof implies that F is a compact self-mapping in a closed and convex subset B of $C^1(J)$, whence F has by Schauder's theorem a fixed point u in B. This and (2.9) ensure that u is a solution of the integral equation (2.7), that is, u is a solution of (2.1) in $Y \cap B$.

3. Main existence results

This section is devoted to the study of the existence of solutions of problem (1.1)–(1.4) in the presence of a pair of ordered lower and upper solutions. Our main result reads as follows.

THEOREM 3.1. *Assume that conditions $(\varphi\mu)$, (f1), (f2) and (L) hold. If problem (1.1)–(1.4) has a lower solution α and an upper solution β such that $\alpha \leq \beta$ in J, and the following inequality holds*:

$$f(t, \beta) \leq f(t, v) \leq f(t, \alpha), \quad \text{for a.e. } t \in J \text{ and all } v \in [\alpha, \beta], \tag{3.1}$$

then problem (1.1)–(1.4) has at least one solution in the sector $[\alpha, \beta]$.

PROOF. Consider the following truncated problem:

$$\begin{cases} \dfrac{d}{dt} \varphi\left((\mu \cdot u')'\right)(t) = f(t, p(u)) & \text{for a.e. } t \in J, \\ u(t_0) = A(u), \quad u(t_1) = B(u), \quad (\mu \cdot u')'(t_0) = c_2, \end{cases} \tag{3.2}$$

where

$$p(v)(t) = \max\{\alpha(t), \min\{v(t), \beta(t)\}\} \quad \text{for all } t \in J,$$
$$A(v) = p(v(t_0) + L_1(v(t_0), v(t_1), v'(t_0), v'(t_1), v))(t_0) \quad \text{and}$$
$$B(v) = p(v(t_1) - L_2(v(t_0), v(t_1)))(t_1), \quad v \in C^1(J).$$

Obviously A and B satisfy (C1) and (C2) and, using the continuity of the truncated function p, we deduce from (f1) and (f2) that conditions (g1) and (g2) hold for $f(t, p(u))$. Thus Theorem 2.1 implies that the modified problem (3.2) has at least one solution u.

Since every solution of (3.2) is given by the expression (2.7) (with obvious notation), then the definition of α, the positivity of functions y_0, y_1 and k and hypothesis $(\varphi\mu)$ together with (3.1), and the definition of A and B, imply that every solution u of (3.2) satisfies for all $t \in J$ the following inequality:

$$u(t) = \frac{A(u)y_1(t) + B(u)y_0(t)}{D} - \int_{t_0}^{t_1} k(t, s)\varphi^{-1}\left(\varphi(c_2) + \int_{t_0}^{s} f(x, p(u))\, dx\right) ds$$

$$\geq \frac{\alpha(t_0)y_1(t) + \alpha(t_1)y_0(t)}{D}$$
$$- \int_{t_0}^{t_1} k(t,s)\varphi^{-1}\left(\varphi((\mu \cdot \alpha')'(t_0)) + \int_{t_0}^{s} f(x,\alpha)\,dx\right)ds$$
$$:= a(t).$$

Note that, by Proposition 2.1, the function a is the unique solution of the problem

$$\begin{cases} \dfrac{d}{dt}\varphi((\mu \cdot u')'(t)) = f(t,\alpha), & \text{for a.e. } t \in J, \\ u(t_0) = \alpha(t_0), \quad u(t_1) = \alpha(t_1), \quad (\mu \cdot u')'(t_0) = (\mu \cdot \alpha')'(t_0). \end{cases}$$

Consequently, $(\mu \cdot \alpha')' \geq (\mu \cdot a')'$ in J. Since $\alpha(t_0) - a(t_0) = \alpha(t_1) - a(t_1) = 0$, there is $\bar{t} \in J$ such that $\alpha'(\bar{t}) = a'(\bar{t})$. Thus we conclude that $\alpha - a$ is decreasing in $[t_0, \bar{t}]$ and increasing in $[\bar{t}, t_1]$. In particular, $\alpha \leq a$, so that $\alpha \leq u$ in J.

In an analogous way we prove that $u \leq \beta$ in J.

Finally, we prove that every solution of (3.2) satisfies the boundary conditions (1.2)–(1.3), and thus the proof is concluded.

First, note that if $u(t_1) - L_2(u(t_0), u(t_1)) < \alpha(t_1)$, the definition of B implies that $u(t_1) = \alpha(t_1)$. Now using condition (L) and the fact that $u \in [\alpha, \beta]$, we arrive at a contradiction:

$$\alpha(t_1) > \alpha(t_1) - L_2(u(t_0), \alpha(t_1)) \geq \alpha(t_1) - L_2(\alpha(t_0), \alpha(t_1)) = \alpha(t_1).$$

Analogously if $u(t_1) - L_2(u(t_0), u(t_1)) > \beta(t_1)$ we have $u(t_1) = \beta(t_1)$ and a contradiction is reached similarly.

Now $\alpha(t_1) \leq u(t_1) - L_2(u(t_0), u(t_1)) \leq \beta(t_1)$ and hence $L_2(u(t_0), u(t_1)) = 0$.

To prove that $L_1(u(t_0), u(t_1), u'(t_0), u'(t_1), u) = 0$ it is enough to show that

$$\alpha(t_0) \leq u(t_0) + L_1(u(t_0), u(t_1), u'(t_0), u'(t_1), u) \leq \beta(t_0).$$

If $u(t_0) + L_1(u(t_0), u(t_1), u'(t_0), u'(t_1), u) < \alpha(t_0)$ then $u(t_0) = \alpha(t_0)$ and then $0 = L_2(u(t_0), u(t_1)) = L_2(\alpha(t_0), \alpha(t_1))$. Now, since $L_2(\alpha(t_0), \cdot)$ is injective and by the definition of a lower solution, we have that $u(t_1) = \alpha(t_1)$.

As a consequence, $u - \alpha$ is nonnegative on J and attains its minimum in t_0 and t_1, thus $u'(t_1) \leq \alpha'(t_1)$ and $u'(t_0) \geq \alpha'(t_0)$. Using the definition of a lower solution and the properties of L_1 we obtain a contradiction:

$$\alpha(t_0) > \alpha(t_0) + L_1(\alpha(t_0), \alpha(t_1), u'(t_0), u'(t_1), u)$$
$$\geq \alpha(t_0) + L_1(\alpha(t_0), \alpha(t_1), \alpha'(t_0), \alpha'(t_1), \alpha) \geq \alpha(t_0).$$

It can be analogously proven that

$$u(t_0) + L_1(u(t_0), u(t_1), u'(t_0), u'(t_1), u) \leq \beta(t_0).$$

Then the result has been proved.

REMARK 3.1. We can also consider boundary conditions of the type

$$L_1(u(t_0), u'(t_0), u) = 0 = L_2(u, u(t_1)),$$

where $L_1(x, y, \xi)$ is increasing in y and ξ, and $L_2(\eta, z)$ is decreasing in η. Moreover α satisfies $L_1(\alpha(t_0), \alpha'(t_0), \alpha) \geq 0 \geq L_2(\alpha, \alpha(t_1))$ and β the reversed inequalities. We only need to adapt the last part of the above proof to this new situation to conclude analogous existence results for this problem.

As a corollary of Theorem 3.1 we obtain the following existence result for periodic IBVPs.

COROLLARY 3.1. *Assume that conditions* $(\varphi\mu)$, (f1) *and* (f2) *hold, and that* (3.1) *holds for* α, $\beta \in Y$ *satisfying* $\alpha \leq \beta$ *in* J *and*

$$\begin{cases} \dfrac{d}{dt} \varphi((\mu \cdot \alpha')')(t) \geq f(t, \alpha) & \textit{for a.e. } x \in J, \\ \alpha(t_0) = \alpha(t_1), \quad \alpha'(t_0) \geq \alpha'(t_1), \quad (\mu \cdot \alpha')'(t_0) \geq c_2, \end{cases}$$

and

$$\begin{cases} \dfrac{d}{dt} \varphi((\mu \cdot \beta')')(t) \leq f(t, \beta) & \textit{for a.e. } x \in J, \\ \beta(t_0) = \beta(t_1), \quad \beta'(t_0) \leq \beta'(t_1), \quad (\mu \cdot \beta')'(t_0) \leq c_2. \end{cases}$$

Then there exists a solution $u \in [\alpha, \beta]$ *of the problem*

$$\begin{cases} \dfrac{d}{dt} \varphi((\mu \cdot u')')(t) = f(t, u) & \textit{for a.e. } x \in J, \\ u(t_0) = u(t_1), \quad u'(t_0) = u'(t_1), \quad (\mu \cdot u')'(t_0) = c_2. \end{cases} \tag{3.3}$$

Taking Remark 3.1 into account we obtain existence results for initial-nonlocal multipoint boundary value problems.

COROLLARY 3.2. *Let* $d_i \geq 0$, $h_j \geq 0$ *and* p_j, $q_i \in [t_0, t_1]$ *for* $i = 1, \dots, n$, $j = 1, \dots, m$. *Assume that conditions* (f1) *and* (f2) *hold, that inequality* (3.1) *is satisfied for* α, $\beta \in Y$, *such that* $\alpha \leq \beta$ *in* J, *and that*

$$\begin{cases} \dfrac{d}{dt} \varphi((\mu \cdot \alpha')')(t) \geq f(t, \alpha) & \textit{for a.e. } x \in J, \\ \alpha(t_0) \leq \displaystyle\sum_{i=1}^{n} d_i \alpha(q_i), \quad \alpha(t_1) \leq \displaystyle\sum_{j=1}^{m} h_j \alpha(p_j), \quad (\mu \cdot \alpha')'(t_0) \geq c_2, \end{cases}$$

and

$$\begin{cases} \dfrac{d}{dt} \varphi((\mu \cdot \beta')')(t) \leq f(t, \beta) & \textit{for a.e. } x \in J, \\ \beta(t_0) \geq \displaystyle\sum_{i=1}^{n} d_i \beta(q_i), \quad \beta(t_1) \geq \displaystyle\sum_{j=1}^{m} h_j \beta(p_j), \quad (\mu \cdot \beta')'(t_0) \leq c_2. \end{cases}$$

Then there exists a solution $u \in [\alpha, \beta]$ of the problem

$$\begin{cases} \dfrac{d}{dt}\varphi((\mu \cdot u')')(t) = f(t, u) & \text{for a.e. } x \in J, \\ u(t_0) = \displaystyle\sum_{i=1}^{n} d_i u(q_i), \quad u(t_1) = \sum_{j=1}^{m} h_j u(p_j), \quad (\mu \cdot u')'(t_0) = c_2. \end{cases}$$

Problem (3.3) has been considered in [7] with $\mu \equiv 1$. Nonlocal conditions (for second-order problems) have been studied, for instance in [2].

4. An example and applications to real phenomena

In this section we present, in a first moment, an example where the existence result Theorem 3.1 is applied. With this example we try to illustrate what kind of problems we can study with these techniques. The result is the following.

EXAMPLE 4.1. For all real constants $A > 0$, $B \le 0$, $M \ge 1$ and $p > 1$, the following third-order functional problem has at least one positive solution in Y:

$$\frac{d}{dt}\varphi_p((\mu \cdot u')')(t) = -\min\{|u(t)|, t \in [\pi, 2\pi]\} \text{ for a.e. } t \in [0, 2\pi], \tag{4.1}$$

$$u(0) = \max\{u(t), \ t \in [0, \pi]\}/M, \tag{4.2}$$

$$u(2\pi) = A, \tag{4.3}$$

$$(\mu \cdot u')'(0) = B. \tag{4.4}$$

Here $\varphi_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}$, $\mu(t) = t + 1$ for all $t \in [0, 2\pi]$.

SOLUTION. Defining

$$f(t, u) = -\min\{|u(t)|, t \in [\pi, 2\pi]\},$$
$$L_1(u(0), u'(0), u(2\pi), u'(2\pi), u) = \max\{u(t), t \in [0, \pi]\}/M - u(0)$$

and

$$L_2(u(0), u(2\pi)) = u(2\pi) - A,$$

it is immediate to verify that conditions $(\varphi\mu)$, (f1), (f2) and (L) hold.

Now let $\alpha(t) = K \log(t + 1)$, with $K = A/\log(2\pi + 1)$. It is clear that α is a lower solution for this problem.

For each $C \ge 0$, we define $\beta_C(t) = A + C((2\pi + 1)^2 - (t + 1)^2)$. Obviously, $\beta_C \in Y$, and $f(t, \beta_C) = -A$ for every $C \ge 0$. Thus the inequality

$$\frac{d}{dt}\varphi((\mu \cdot \beta_C')')(t) \le f(t, \beta_C) \quad \text{for a.e. } t \in J = [0, 2\pi]$$

holds if and only if

$$C \geq \frac{A^{1/(p-1)}}{4(p-1)^{1/(p-1)}}.$$

Clearly, $L_1(\beta_C(0), \beta'_C(0), \beta_C(2\pi), \beta'_C(2\pi), \beta_C) \leq 0 = L_2(\beta_C(0), \beta_C(2\pi))$.

Finally, since $(\mu \cdot \beta'_C)'(0) = -4 C$, we see that β_C is an upper solution of problem (4.1)–(4.4) if and only if

$$C \geq \frac{1}{4} \max \left\{ \frac{A^{1/(p-1)}}{(p-1)^{1/(p-1)}}, -B \right\}. \tag{4.5}$$

Consequently, since $\alpha \leq \beta_C$ for all $C \geq 0$, and since the function f satisfies condition (3.1) with $\beta = \beta_C$, it follows from Theorem 3.1 that problem (4.1)–(4.4) has at least one solution in the sector $[\alpha, \beta_C]$, with C given in (4.5).

After this example, in which we expose the applicability of our results to abstract and general problems, we present an application of the given results to problems involving real-world phenomena.

It is well known, see [11] and references therein, that the deformations of an elastic beam are described by the fourth-order equation

$$u''''(t) + g(t) u(t) = h(t), \quad \text{for a.e. } t \in (0, 1), \tag{4.6}$$

with g and h in $L^1(0, 1)$.

Assuming that the right endpoint of the beam is fixed or cantilevered and the left one is free, then the following conditions must be satisfied:

$$u''(0) = u'''(0) = 0 \quad \text{and} \quad u(1) = u'(1) = 0. \tag{4.7}$$

It is not difficult to verify that if $u \in \{x \in C^3([0, 1]), x''' \in A C([0, 1])\}$ is a solution of problem (4.6)–(4.7) then u' is a solution of the third-order integro-differential equation

$$\begin{cases} y'''(t) = g(t) \displaystyle\int_t^1 y(s)\, ds + h(t), & \text{for a.e. } t \in (0, 1); \\ y(1) = y'(0) = y''(0) = 0. \end{cases} \tag{4.8}$$

Now, assuming that g and h are in $L^\infty([0, 1])$ and that $g(t) \leq 0$ for a.e. $t \in [0, 1]$, we conclude that problem (4.8) has at least one solution.

To see this it is enough to define φ as the identity, $\mu \equiv 1, f (t, u) = g(t) \int_t^1 u(s)\, ds + h(t), L_1(x, y, z, p, u) = z, L_2(x, y) = y$ and $c_2 = 0$ and consider for every $C > 0$, $\alpha_C(t) = C(t^3 - 1)$ and $\beta_C(t) = -C(t^3 - 1)$. It is easy to verify that for C large enough the hypotheses of Theorem 3.1 are fulfilled. Consequently, there is a solution y of problem (4.8) lying between α_C and β_C. Now $u(t) = \int_1^t y(s)\, ds$ is a solution of (4.6)–(4.7).

Acknowledgements

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