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ABELIAN GROUPS QUASI-INJECTIVE OVER THEIR ENDOMORPHISM RINGS

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1. Introduction. L. Fuchs has posed the problem of identifying those abelian groups that can serve as the additive structure of an injective module over some ring [**1**, p. 179], and in particular of identifying those abelian groups which are injective as modules over their endomorphism rings [**1**, p. 112]. Richman and Walker have recently answered the latter question, generalized in a non-trivial way [**7**], and have shown that the groups in question are of a rather restricted structure.

In this paper we consider abelian groups which are quasi-injective over their endomorphism rings. We show that divisible groups are quasi-injective as are direct sums of cyclic p-groups. Quasi-injectivity of certain direct sums (products) is characterized in terms of the summands (factors). In general it seems that the answer to the question of whether or not a group G is quasi-njective over its endomorphism ring E depends on how big $\mathrm{Hom}_E(H, G)$ is, with H a fully invariant subgroup of G.

All groups under discussion are abelian and our notation is standard. We write $E(G)$ for the endomorphism ring of G when the dependence on G must be displayed; otherwise, we write simply E. We say that G is quasi-injective if G is a quasi-injective E-module when viewed as an E-module in the natural way.

2. Generalities. In this section we collect a few general facts, starting with

PROPOSITION 2.1. *Suppose H is a fully invariant subgroup of the quasi-injective group G and that restriction is a ring epimorphism of $E(G)$ onto $E(H)$. Then H is quasi-injective over $E(H)$.*

Proof. Let $f: K \rightarrow H$ be an $E(H)$-map where K is fully invariant in H. Clearly K is an $E(G)$-submodule of H and f is an $E(G)$-map. By hypothesis on G, there is an α in the center of $E(G)$ such that $f = \alpha|K$. Since restriction is an epimorphism of $E(G)$ onto $E(H)$, $\alpha|H$ lies in the center of $E(H)$. Then $f = \alpha|K = (\alpha|H)|K$ as required.

The Proposition above implies that the divisible subgroup of a quasi-injective group is quasi-injective (however, see Theorem 3.2 below) as are the primary components of a quasi-injective torsion group. More precisely,

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COROLLARY 2.2. *If H is a fully invariant direct summand of a quasi-injective group, then H is quasi-injective.*

PROPOSITION 2.3. *Suppose G is the direct sum of fully invariant subgroups G_α. Then G is quasi-injective if and only if each G_α is quasi-injective.*

Proof. If G is quasi-injective, so is each G_α by Corollary 2.2. Thus assume the G_α are quasi-injective, H is fully invariant in G, and $f: H \rightarrow G$ is an E-map. Then $H = \sum H \cap G_\alpha$ and $f_\alpha = f \,|H \cap G_\alpha$ is clearly an $E(G_\alpha)$-map of the fully invariant subgroup $H \cap G_\alpha$ of G_α into G_α. By hypothesis f_α extends to an $E(G_\alpha)$-map of G_α into itself, say $\bar{f}_\alpha$. The $\bar{f}_\alpha$ then define a unique map $\bar{f}$ of G into itself, which clearly extends f and commutes with all endomorphisms of G.

COROLLARY 2.4. *A torsion group is quasi-injective if and only if each of its primary components is quasi-injective.*

Suppose now that $G = \prod G_\alpha$ where each G_α is fully invariant in G. If G is quasi-injective, then Corollary 2.2 shows that each G_α is quasi-injective. Moreover, the standard proof of the fact that products of injective modules are injective can be easily modified to show that, if $G = \prod G_\alpha$, G_α is fully invariant in G, and each G_α is quasi-injective, then G is quasi-injective provided we know that the projections of G onto the G_α are E-maps. A condition for this is given in the following lemma (cf. also [**7**, Lemma 4]). Our terminology reflects the fact that we are identifying the G_α and their direct sum $\sum G_\alpha$ with the obvious sub-groups of $G = \prod G_\alpha$.

LEMMA 2.5. *Suppose $G = \prod G_\alpha$ with each G_α fully invariant in G and put $\bar{G} = G/\sum G_\alpha$. Then the projection λ_α of G onto G_α is an E-map if and only if* $\mathrm{Hom}_E(\bar{G}, G_\alpha) = 0$.

Proof. For each α we have $G = G_\alpha \oplus \prod_{\beta\neq\alpha} G_\beta$ and the projection onto $\prod_{\beta\neq\alpha} G_\beta$ is $\rho_\alpha = 1 - \lambda_\alpha$. For $\eta \in E$ then

$$\lambda_\alpha\eta = \lambda_\alpha\eta(\lambda_\alpha + \rho_\alpha) = \lambda_\alpha\eta\lambda_\alpha + \lambda_\alpha\eta\rho_\alpha.$$

Since $G_\alpha = \lambda_\alpha G$ is fully invariant in G, $\eta\lambda_\alpha G \leqq \lambda_\alpha G = G_\alpha$ so $\lambda_\alpha\eta\lambda_\alpha = \eta\lambda_\alpha$. Thus

$$\lambda_\alpha\eta = \eta\lambda_\alpha + \lambda_\alpha\eta\rho_\alpha.$$

Each G_β being fully invariant in G, $\eta\rho_\alpha G_\beta \leqq G_\beta$, so for $\beta \neq \alpha$, $\lambda_\alpha\eta\rho_\alpha G_\beta = 0$. Clearly $\rho_\alpha G_\alpha = 0$ so $\lambda_\alpha\eta\rho_\alpha$ annihilates $\sum G_\alpha$. Therefore there is a map $\zeta \in \mathrm{Hom}(\bar{G}, G_\alpha)$ such that $\lambda_\alpha\eta\rho_\alpha = \zeta\nu$, where $\nu: G \rightarrow \bar{G}$ is the natural map. This proves half of the statement, for if $\zeta = 0$ then $\lambda_\alpha\eta = \eta\lambda_\alpha$. On the other hand, if $\zeta \in \mathrm{Hom}(\bar{G}, G_\alpha)$, $\zeta \neq 0$, then $\zeta\nu$ is a non-zero endomorphism of G. Since $\zeta\nu G \leqq G_\alpha$, $\lambda_\alpha\zeta\nu = \zeta\nu$. Since $\nu = \nu(\lambda_\alpha + \rho_\alpha)$ and $\nu\lambda_\alpha = 0$, $\nu = \nu\rho_\alpha$. Thus $0 \neq \zeta\nu = \lambda_\alpha(\zeta\nu)\rho_\alpha$ so, as above, λ_α does not commute with $\zeta\nu$. This proves the lemma.

As remarked above, the usual proof for injective modules together with Lemma 2.5 now yield

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PROPOSITION 2.6. *Suppose* $G = \prod_\alpha G_\alpha$ *with each* G_α *fully invariant in* G *and* $\operatorname{Hom}(\bar{G}, G_\alpha) = 0$ *for each* α. *Then* G *is quasi-injective if and only if each* G_α *is quasi-injective.*

3. Divisible groups. Divisible groups are quasi-injective almost by accident, since their fully invariant subgroups are so easily identifiable. We first reduce to the torsion case with

LEMMA 3.1. *If* G *is divisible and* H *is fully invariant in* G, *then either* $H = G$ *or* H *is a fully invariant subgroup of the torsion subgroup* T *of* G.

Proof. If H is contained in T then H is clearly fully invariant in T. Suppose then that $H_0 = H + T \neq T$. We show that $H = G$. Since every endomorphism of G/T is induced by an endomorphism of G, and since H_0 is a fully invariant subgroup of G properly containing T, H_0/T is a non-zero fully invariant subgroup of the torsion free divisible group G/T. Thus $H_0/T = G/T$, or $H_0 = G$, and the natural map ν of G onto G/T induces an epimorphism of H onto G/T. Now choose an arbitrary summand $Z(p^\infty)$ of T. Let π be a projection of G/T onto a copy of Q and let ρ be a map of Q onto the chosen $Z(p^\infty)$. Then $\rho\pi\nu$ is an endomorphism of G which maps H onto $Z(p^\infty)$. Since H is fully invariant in G, this yields $Z(p^\infty) \leqq H$. Since $Z(p^\infty)$ was an arbitrary summand of T, $T \leqq H$. Then $G = H + T = H$ as required.

THEOREM 3.2. *Every divisible group* G *is quasi-injective over its endomorphism ring.*

Proof. Let G be divisible with torsion subgroup T, fully invariant subgroup H, and let $f \in \operatorname{Hom}_{E(G)}(H, G)$. If $H = G$, there is nothing to prove. Otherwise, by the lemma above, H is a fully invariant subgroup of T. Since every endomorphism of T is induced by an endomorphism of G we have, with a slight abuse of language, $f \in \operatorname{Hom}_{E(T)}(H, T)$. It suffices now to show that T is quasi-injective over $E(T)$ and by Corollary 2.4 we may assume T is a p-group. Since T is Z-injective, restriction of elements of $E(T)$ to act in H is an epimorphism of $E(T)$ onto $E(H)$. Since the only fully invariant subgroups of a divisible p-group T are T itself, 0, and the p^n-layers $T[p^n] = \{x \in T | p^n x = 0\}$, $\operatorname{Hom}_Z(H, T) = E(H)$. Now clearly with these identifications, $\operatorname{Hom}_{E(T)}(H, T)$ is contained in the center of $E(H)$. But the center of $E(H)$ is Z/p^nZ if H is $T[p^n]$ (and, otherwise, $H = T$) so each map in the center of $E(H)$ is induced by multiplication by some p-adic integer, i.e., by some element of the center of $E(T)$.

For the next corollary, we use the term *homogeneous* p-*group* to denote a p-group which is the direct sum of copies of Z/p^nZ for some fixed n.

COROLLARY 3.3. *Homogeneous* p-*groups are quasi-injective.*

Proof. Such a group is fully invariant in its divisible hull so the statement follows from Theorem 3.2 and Proposition 2.1.

We remark that this will be generalized in the next section.

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4. Direct sums of finite groups. In this section we show that direct sums of finite groups are quasi-injective over their endomorphism rings. In particular this covers the cases of bounded p-groups as well as countable p-groups without elements of infinite height. By Corollary 2.4 we may assume the groups in question are p-groups. The result we are after will follow from more general considerations in which we only assume that our groups have no elements of infinite height.

Let G be a p-group with $G^1 = 0$, with fully invariant subgroup H, and with basic subgroup B. We want to estimate the size of $\mathrm{Hom}_E(H, G)$, where E is the endomorphism ring of G. Write $B = \sum B_n$, where $B_n = \sum [b_{n\alpha}]$ is a direct sum of cyclic groups of order p^n, and $G_n = p^n G + \sum_{k>n} B_k$. Then

$$G = B_1 \oplus \ldots \oplus B_n \oplus G_n$$

for each n, so

$$H = B_1 \cap H \oplus \ldots \oplus B_n \cap H \oplus G_n \cap H.$$

Our immeidate object is to show that for $f \in \mathrm{Hom}_E(H, G)$, we have $f(B_j \cap H) \leqq B_j \cap H$ for each j.

Since $B_j = \sum [b_{j\alpha}]$, we have

$$B_j \cap H = \sum [b_{j\alpha}] \cap H = \sum [p^{n_{j\alpha}} b_{j\alpha}]$$

for some integers $n_{j\alpha}$, $0 \leqq n_{j\alpha} < j$. For each pair of indices α, β, $[b_{j\alpha}]$ and $[b_{j\beta}]$ are isomorphic summands of G so there is an endomorphism ρ of G such that $\rho(b_{j\alpha}) = b_{j\beta}$. Then

$$p^{n_{j\alpha}} b_{j\beta} = p(p^{n_{j\alpha}} b_{j\beta}) \in [b_{j\beta}] \cap H = [p^{n_{j\beta}} b_{j\beta}]$$

so that $n_{j\alpha} \geqq n_{j\beta}$. By symmetry, $n_{j\alpha} = n_{j\beta}$ and we conclude that $B_j \cap H = p^{n_j} B_j$, n_j being the common value of the $n_{j\alpha}$.

Any $f \in \mathrm{Hom}_E(H, G)$ commutes in particular with the projections of G onto its direct summands. Therefore $f(B_j \cap H) \leqq B_j$ for every j. By the remarks above, $p^{j-n_j} G_j \cap H = 0$ so $f(B_j \cap H) \leqq B_j[p^{j-n_j}]$. But

$$B[^{j-n_j}] = p^{n_j} B_j = B_j \cap H$$

and $f(B_j \cap H) \leqq B_j \cap H$ as we wished to show.

Now let $x \in H$ and write $x = x_1 + \ldots + x_n + y_n$, $x_j \in B_j$, $y_n \in G_n$. Then $f(x) = \sum f(x_j) + f(y_n)$ with $f(x_j) \in B_j \cap H$ and $f(y_n) \in G_n$. Since this is true of each n, at least $f(x)$ belongs to the closure of H in the p-adic topology on G. However, from Kaplansky's characterization of the fully invariant subgroups of p-groups without elements of infinite height [**3**], it is easy to see that such subgroups are closed. Thus $f(H) \leqq H$ and we have proved

PROPOSITION 4.1. *Let G be a p-group with no elements of infinite height and let H be fully invariant in G. Then* $\mathrm{Hom}_E(H, G) = \mathrm{Hom}_E(H, H)$.

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To get our main theorem we remark that it follows from results in [**6**] that if G has no elements of infinite height, B is basic in G and H is B-large, then $\mathrm{Hom}_E(H, H)$ is simply the center of $E(H)$. Thus by Proposition 4.1 each map in $\mathrm{Hom}_E(H, G)$ is induced by multiplication by some p-adic integer, an operation defined on G as well. Finally, if G is a direct sum of cyclic p-groups, every fully invariant subgroup of G is B-large with B taken to be G itself. This proves

THEOREM 4.2. *Direct sums of cyclic p-groups are quasi-injective over their* endomorphism rings.

Free abelian groups are not quasi-injective. In fact if G is torsion-free and quasi-injective, then G must be divisible since the E-map φ, defined by $\varphi(nx) = x$ on the fully invariant subgroup nG of G, must extend to a map of G to G. We conclude with the following somewhat miscellaneous result.

THEOREM 4.3. *Suppose G is a p-group ($p \neq 2$) which is quasi-injective. Then for each ordinal β for which $G/p^\beta G$ is totally projective, $p^\beta G$ is quasi-injective.*

Proof. $H = p^\beta G$ is a fully invariant subgroup of G. Suppose K is a fully invariant subgroup of H which, by transitivity, is a fully invariant subgroup of G. Suppose $f: K \to H$ is an $E(H)$-homomorphism. f is also an E-homomorphism and so there exists an E-endomorphism f' which extends f. If $f^* = f'|H$, then f^* is in $E(H)$. To show f^* is an $E(H)$-homomorphism, let $\alpha \in E(H)$ and $h \in H$. From [**2**, Corollaries 4.3 and 4.5] we can write $\alpha = \Pi + \Psi$ where Π, Ψ are in the automorphism group of H. From [**2**, Corollary 4.4], Π and Ψ can be extended to automorphisms Π' and Ψ' of G. Let $\alpha^* = \Pi' + \Psi'$ which belongs to E. Then

$$f^*(\alpha(h)) = f^*(\alpha^*(h)) = f'(\alpha^*(h)) = \alpha^*(f'(h)) = \alpha^*(f^*(h)) = \alpha(f^*(h)).$$

This shows that H is quasi-injective over $\mathrm{E}(H)$.

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