

RAMANUJAN CAYLEY GRAPHS OF FROBENIUS GROUPS

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Abstract

We determine a bound for the valency in a family of dihedrants of twice odd prime orders which guarantees that the Cayley graphs are Ramanujan graphs. We take two families of Cayley graphs with the underlying dihedral group of order $2p$: one is the family of all Cayley graphs and the other is the family of normal ones. In the normal case, which is easier, we discuss the problem for a wider class of groups, the Frobenius groups. The result for the family of all Cayley graphs is similar to that for circulants: the prime p is ‘exceptional’ if and only if it is represented by one of six specific quadratic polynomials.

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1. Introduction

A k -regular graph X with standard assumptions is called *Ramanujan* if its largest nontrivial eigenvalue (in the sense of absolute value) is not greater than the Ramanujan bound $2\sqrt{k-1}$. The Ramanujan property of a graph means that the associated Ihara zeta function (formulated in [7], [8]) satisfies the ‘Riemann hypothesis’, which enables us to have a good estimate for the number of prime cycles in the graph (see, for example, [9]). See also [6] for further relations between this property and various mathematical objects.

We considered the following problem in our previous paper [5]. Let G be a finite group and \mathcal{S} a set of Cayley subsets of G , and put

$$\mathcal{X} = \mathcal{X}_{G,\mathcal{S}} = \{X(S) \mid S \in \mathcal{S}\},$$

where $X(S)$ is the Cayley graph of G with respect to the Cayley subset $S \in \mathcal{S}$. Letting $\mathcal{L} = \{|G| - |S| \mid S \in \mathcal{S}\}$ be the set of ‘covalencies’ of graphs in \mathcal{X} , we write $\mathcal{X} = \bigsqcup_{l \in \mathcal{L}} \mathcal{X}_l$, where $\mathcal{X}_l = \{X(S) \in \mathcal{X} \mid |G| - |S| = l\}$. Notice that $\mathcal{X}_1 = \{K_{|G|}\}$ if $1 \in \mathcal{L}$, where $K_{|G|} = X(G \setminus \{1\})$ is the complete graph with $|G|$ -vertices. According to [1],

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some neighbours of $K_{|G|}$ are expected to be Ramanujan. We want to estimate them precisely, that is, to determine the bound

$$\hat{l}_{G,S} = \max\{l \in \mathcal{L} \mid X \in \mathcal{X}_k \text{ is Ramanujan for } 1 \leq k \leq l\}$$

of edge-removal in S which preserves the Ramanujan property from the complete graph $K_{|G|}$.

Previously, we have discussed this problem when \mathcal{X} is the family consisting of all circulants of odd prime orders. In this paper, we treat the cases when \mathcal{X} is a family of dihedrants of twice odd prime orders. Specifically, we consider two families of Cayley graphs with the underlying group D_{2p} : the dihedral group of order $2p$, whose Cayley subsets S are the set \mathcal{S}_A of all Cayley subsets and the set \mathcal{S}_N of all normal ones. Here, we call a Cayley subset normal if it is a union of conjugacy classes of G . The normal case is the easier to discuss, since the spectra of the normal Cayley graphs are given by an explicit formula (see Lemma 2.1). We discuss the problem, in this case, for a wider class of groups, the Frobenius groups. On the other hand, it is hard to handle the family for all Cayley subsets in general. However, we can study the family of all dihedrants in detail, since their eigenvalues can be explicitly written down. We prove the following result which is similar to the one for circulants obtained in [5].

THEOREM 1.1 (Theorem 4.5). *In the family consisting of all dihedrants of twice odd prime orders $2p$, the prime p is ‘exceptional’ if and only if it is of the form of one of six specific quadratic polynomials.*

The classical Hardy–Littlewood conjecture asserts that every quadratic polynomial represents an infinite number of primes under some standard conditions. So our result implies that there are an infinite number of exceptional primes if and only if the Hardy–Littlewood conjecture is true for at least one of these six quadratic polynomials.

Throughout the paper, we denote the set of all odd primes by \mathbb{P} and the cyclic group of order m by $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{Z}_{\geq 1}$.

2. Preliminaries

2.1. Cayley graphs and their spectra. Let X be a k -regular finite graph with m vertices which is undirected, connected and simple. The adjacency matrix A_X of X is the symmetric matrix of size m whose entry is one if the corresponding pair of vertices are connected by an edge and zero otherwise. We call the eigenvalues of A_X the eigenvalues (or the spectra) of X . The set $\Lambda(X)$ of all eigenvalues of X is given by

$$\Lambda(X) = \{\lambda_i \mid k = \lambda_0 > \lambda_1 \geq \dots \geq \lambda_{m-1} \geq -k\}.$$

Let $\mu(X)$ be the largest nontrivial eigenvalue of X in the sense of absolute value

$$\mu(X) = \max\{|\lambda| \mid \lambda \in \Lambda(X), |\lambda| \neq k\}.$$

The graph X is called a Ramanujan graph if $\mu(X) \leq 2\sqrt{k-1}$. Here the constant $2\sqrt{k-1}$ is often called the Ramanujan bound for X and is denoted by $\text{RB}(X)$.

Let G be a finite group with the identity element 1 and S be a Cayley subset of G , that is, a symmetric generating subset of G without 1 . We denote by $X(S)$ the *Cayley graph* of G with respect to the Cayley subset S . This is the undirected, connected and simple $|S|$ -regular graph with the vertex set G and the edge set $\{(x, y) \in G^2 \mid x^{-1}y \in S\}$. In what follows, for a Cayley subset S , we write $\Lambda(S) = \Lambda(X(S))$, $\mu(S) = \mu(X(S))$, $\text{RB}(S) = \text{RB}(X(S))$ and so on. It is well known that the spectra of Cayley graphs are described by the irreducible characters of the underlying group, as follows.

LEMMA 2.1 [2]. *Let $X = X(S)$ be a Cayley graph of a finite group G with respect to a Cayley subset S and let A_X be its adjacency matrix.*

- (1) $A_X = \sum_{s \in S} R_s$, where R_s is the matrix of the right multiplication by $s \in G$ in the group algebra $\mathbb{C}G$.
- (2) Let χ_1, \dots, χ_h be all of the irreducible characters of G with $\deg \chi_i = n_i$. Then $\Lambda(S) = \{\lambda_{i,j} \mid 1 \leq i \leq h, 1 \leq j \leq n_i\}$, where the multiplicity of $\lambda_{i,j}$ is n_i and $\{\lambda_{i,j}\}_{1 \leq j \leq n_i}$ is determined by the n_i equations

$$\lambda_{i,1}^t + \dots + \lambda_{i,n_i}^t = \sum_{s_1, \dots, s_t \in S} \chi_i \left(\prod_{l=1}^t s_l \right), \quad 1 \leq t \leq n_i.$$

In particular, if S is normal, that is, a union of conjugacy classes of G , then $\lambda_{i,1} = \dots = \lambda_{i,n_i}$. Therefore $\Lambda(S) = \{\lambda_i \mid 1 \leq i \leq h\}$, where the multiplicity of λ_i is n_i^2 and

$$\lambda_i = \frac{1}{n_i} \sum_{s \in S} \chi_i(s).$$

2.2. Two formulations for a Ramanujan boundary problem. Let G be a finite group and let \mathcal{S} be a set consisting of Cayley subsets of G . For $S \in \mathcal{S}$, we define $l(S) = |G| - |S| = |G \setminus S|$ and call it a *covalency* of $X(S)$. Letting $\mathcal{L} = \{l(S) \mid S \in \mathcal{S}\}$, we write $\mathcal{S} = \bigsqcup_{l \in \mathcal{L}} \mathcal{S}_l$, where $\mathcal{S}_l = \{S \in \mathcal{S} \mid l(S) = l\}$. We consider the following bound for the covalency which guarantees the Ramanujan property:

$$\hat{l} = \hat{l}(G) = \max\{l \in \mathcal{L} \mid X(S) \text{ is Ramanujan for all } S \in \mathcal{S}_k \ (1 \leq k \leq l)\}.$$

In view of Lemma 2.1, we will take the following two subsets as \mathcal{S} : the set \mathcal{S}_N of all normal Cayley subsets of G and the set \mathcal{S}_A of all Cayley subsets of G . These two sets are the same if G is abelian. We write \hat{l}_N and \hat{l}_A for the bounds corresponding to these two cases. In general, \hat{l}_N is easier to evaluate than \hat{l}_A , because the spectra for $X(S)$ with $S \in \mathcal{S}_N$ have a closed formula, as in Lemma 2.1, and we can find the trivial lower bound l_0 for \hat{l}_N based on the following lemma.

LEMMA 2.2. *For $\mathcal{S} = \mathcal{S}_N$, put $l_0 = \max\{l \in \mathcal{L} \mid l \leq 2(\sqrt{|G|} - 1)\}$. Then $l_0 \leq \hat{l}_N$.*

PROOF. Take $S \in \mathcal{S}_N$ with $l(S) \leq \frac{1}{2}|G|$. It is enough to see that $X(S)$ is Ramanujan if $l(S) \leq 2(\sqrt{|G|} - 1)$. For any nontrivial irreducible character χ_i of G ,

$$\lambda_i = \frac{1}{n_i} \sum_{s \in S} \chi_i(s) = -\frac{1}{n_i} \sum_{s \in G \setminus S} \chi_i(s)$$

TABLE 1. The character table of the Frobenius group $G = N \rtimes H$.

	1	$x_i (1 \leq i \leq k)$	$y_j (1 \leq j \leq h)$
1	1	1	1
$\chi_\alpha (1 \leq \alpha \leq h)$	$\chi_\alpha(1)$	$\chi_\alpha(1)$	$\chi_\alpha(y_j)$
$\phi_\beta (1 \leq \beta \leq k)$	$ H \psi_\beta(1)$	$\sum_{z \in H} \psi_\beta(x^z)$	0

from the orthogonality. Since $|\chi_i(s)| \leq n_i$, this shows that $|\lambda_i| \leq \min\{|S|, l(S)\} = l(S)$. Hence, if $l(S) \leq \text{RB}(S) = 2\sqrt{|G| - l(S)} - 1$, or, equivalently, $l(S) \leq 2(\sqrt{|G|} - 1)$, then $X(S)$ is Ramanujan. Note that $2(\sqrt{|G|} - 1) \leq \frac{1}{2}|G|$ for any G . \square

3. The normal cases

In this section, we discuss the determination of \hat{l}_N for the normal dihedrants of order $2p$ with $p \in \mathbb{P}$, that is, the normal Cayley graphs of the dihedral groups D_{2p} . To do this, we study a wider class of graphs, the Cayley graphs of the Frobenius groups. We refer to [3] for the Frobenius groups. For a group G and $x, y \in G$, we write $x^y = y^{-1}xy$ and $\text{Conj}_G(x)$ for the conjugacy class of $x \in G$ in G . Let $c(G)$ denote the number of conjugacy classes in G . Throughout this section, we drop the subscript N on S_N for brevity.

3.1. Spectra of normal Frobenius graphs. We recall the character table of the Frobenius group $G = N \rtimes H$, where N and H are subgroups of G called the Frobenius kernel and complement, respectively. Notice that $r = (|N| - 1)/|H|$ is a positive integer. It is known that a set of representatives of the conjugacy classes of G can be taken as $\{1\} \sqcup \{x_i\}_{i=1}^k \sqcup \{y_j\}_{j=1}^h$, where $x_i \in N (1 \leq i \leq k)$ and $y_j \in H (1 \leq j \leq h)$ with $k = (c(N) - 1)/|H|$ and $h = c(H) - 1$. Notice that $\text{Conj}_G(x_i) \subset N$ and $\text{Conj}_G(y_j) \subset G \setminus N$ and that

$$|\text{Conj}_G(x_i)| = |\text{Conj}_N(x_i)| |H|, \quad |\text{Conj}_G(y_j)| = |\text{Conj}_H(y_j)| |N|.$$

The irreducible characters of G are given as follows. Since $H \simeq G/N$, a nontrivial irreducible character of H corresponds to a character of G whose kernel contains N . We write these as $\chi_\alpha (1 \leq \alpha \leq h)$. Moreover, for a nontrivial irreducible character ψ_β of N , its induced character is again an irreducible character of G . We write these as $\phi_\beta = \text{Ind}(\psi_\beta) (1 \leq \beta \leq k)$. Notice that

$$\phi_\beta(x) = \frac{1}{|N|} \sum_{\substack{y \in G \\ x^y \in N}} \psi_\beta(x^y) = \sum_{z \in H} \psi_\beta(x^z).$$

It is known that these, together with the trivial character **1**, exhaust all irreducible characters of G . The character table of G is as shown in Table 1.

Let us calculate the eigenvalues of normal Cayley graphs of Frobenius groups. For subsets $X \subset \{x_i\}_{i=1}^k$ and $Y \subset \{y_j\}_{j=1}^h$, we put $S_{X,Y} = S_X \sqcup S_Y$, where

$$S_X = \bigsqcup_{x \in X} \text{Conj}_G(x), \quad S_Y = \bigsqcup_{y \in Y} \text{Conj}_G(y).$$

We say that $X \subset \{x_i\}_{i=1}^k$ (respectively, $Y \subset \{y_j\}_{j=1}^h$) is *symmetric* if S_X (respectively, S_Y) is symmetric. It is clear that $S_{X,Y}$ is symmetric if and only if both X and Y are symmetric. Further,

$$\mathcal{S} \subset \{S_{X,Y} \mid \text{both } X \subset \{x_i\}_{i=1}^k \text{ and } Y \subset \{y_j\}_{j=1}^h \text{ are symmetric}\}.$$

Notice that, if $S_{X,Y}$ on the right-hand side satisfies $|S_{X,Y}| > \frac{1}{2}|G|$, then it generates G and hence is in \mathcal{S} and, moreover, $Y \neq \emptyset$ because, otherwise, $|S_{X,Y}| = |S_{X,\emptyset}| < |N| - 1$, which contradicts $\frac{1}{2}|G| \geq |N|$. This means that, in the determination of l_0 and \hat{l}_N , we may assume that $S \in \mathcal{S}$ is always of the form of $S = S_{X,Y}$ for some symmetric subsets $X \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y \subset \{y_j\}_{j=1}^h$.

The following lemma is a consequence of Lemma 2.1 and the character table of G .

LEMMA 3.1. For $X \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y \subset \{y_j\}_{j=1}^h$,

$$\Lambda(S_{X,Y}) = \{\lambda_1\} \cup \{\lambda_{\chi_\alpha}\}_{\alpha=1}^h \cup \{\lambda_{\phi_\beta}\}_{\beta=1}^k,$$

where

$$\lambda_1 = |S_{X,Y}| = |H| \sum_{x \in X} |\text{Conj}_N(x)| + |N| \sum_{y \in Y} |\text{Conj}_H(y)|, \tag{3.1}$$

$$\lambda_{\chi_\alpha} = |H| \sum_{x \in X} |\text{Conj}_N(x)| + \frac{|N|}{\chi_\alpha(1)} \sum_{y \in Y} \chi_\alpha(y) |\text{Conj}_H(y)|,$$

$$\lambda_{\phi_\beta} = \frac{1}{\psi_\beta(1)} \sum_{x \in X} \sum_{z \in H} \psi_\beta(x^z) |\text{Conj}_N(x)|$$

have the multiplicities $1, \chi_\alpha(1)^2$ and $|H|^2 \psi_\beta(1)^2$, respectively.

3.2. The boundary in the normal cases. To determine l_0 and \hat{l}_N , we first describe the set $\mathcal{L} = \{l(S) \mid S \in \mathcal{S}\}$. For symmetric subsets $X \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y \subset \{y_j\}_{j=1}^h$, set

$$a_X = r - \sum_{x \in X} |\text{Conj}_N(x)|, \quad b_Y = |H| - 1 - \sum_{y \in Y} |\text{Conj}_H(y)|.$$

Then $0 \leq a_X \leq r, 0 \leq b_Y < |H| - 1$ and, from (3.1),

$$l(S_{X,Y}) = |G| - |S_{X,Y}| = 1 + a_X |H| + b_Y |N|. \tag{3.2}$$

LEMMA 3.2. Let $X, X' \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y, Y' \subset \{y_j\}_{j=1}^h$ be symmetric subsets. Then $l(S_{X,Y}) = l(S_{X',Y'})$ if and only if $(a_X, b_Y) = (a_{X'}, b_{Y'})$.

PROOF. From (3.2), $l(S_{X,Y}) = l(S_{X',Y'})$ is equivalent to $(a_X - a_{X'})|H| + (b_Y - b_{Y'})|N| = 0$. Since $(a_X - a_{X'})|H| < |N|$, it follows that $a_X = a_{X'}$ and hence $b_Y = b_{Y'}$. \square

Put $l(a, b) = 1 + a|H| + b|N|$. From Lemma 3.2, we write $\mathcal{S} = \bigsqcup_{a \in A, b \in B} \mathcal{S}_{l(a,b)}$, where

$$A = \{a_X \mid X \subset \{x_i\}_{i=1}^k \text{ is symmetric}\}, \quad B = \{b_Y \mid \emptyset \neq Y \subset \{y_j\}_{j=1}^h \text{ is symmetric}\}.$$

We arrange the elements of A and B in ascending order, respectively: that is,

$$A = \{a_i \mid a_1 < a_2 < \dots < a_m\}, \quad B = \{b_i \mid b_1 < b_2 < \dots < b_n\},$$

with $m = |A|$ and $n = |B|$. Here we observe that $a_1 = 0$ and $a_m = r$, which, respectively, correspond to the cases $X = \{x_i\}_{i=1}^k$ and $X = \emptyset$. Similarly, $b_1 = 0$, which corresponds to the case $Y = \{y_j\}_{j=1}^h$, and $b_n < |H| - 1$ because $Y \neq \emptyset$. Moreover, when $h \geq 2$, since the centre of H is nontrivial, there exists $y' \in \{y_j\}_{j=1}^h$ such that $\text{Conj}_H(y') = \{y'\}$. This implies that $b_2 = 1$ if $\{y'\}$ is not symmetric (that is, $y'^2 \neq 1$) and $b_2 = 2$ otherwise. The relations among $l(a, b)$ for $a \in A$ and $b \in B$ are

$$\begin{aligned} 1 &= l(a_1, b_1) < l(a_2, b_1) < \dots < l(a_m, b_1) = (b_1 + 1)|N| = |N| \\ &< b_2|N| + 1 = l(a_1, b_2) < l(a_2, b_2) < \dots < l(a_m, b_2) = (b_2 + 1)|N| \\ &< b_3|N| + 1 = l(a_1, b_3) < l(a_2, b_3) < \dots < l(a_m, b_3) = (b_3 + 1)|N| \\ &\vdots \\ &< b_n|N| + 1 = l(a_1, b_n) < l(a_2, b_n) < \dots < l(a_m, b_n) = (b_n + 1)|N|. \end{aligned}$$

The following is the main result in this section.

THEOREM 3.3. *Assume that $r = (|N| - 1)/|H| \geq 4$.*

- (1) *There exists $1 \leq i_0 < m$ such that $l_0 = l(a_{i_0}, b_1) < |N|$.*
- (2) $\hat{l}_N = l_0$.

PROOF. Under the condition $r \geq 4$, we have $|H| \leq \frac{1}{4}(|N| - 1) < \frac{1}{4}|N|$ and hence $2(\sqrt{|G|} - 1) < 2\sqrt{|N||H|} < 2\sqrt{|N| \cdot \frac{1}{4}|N|} = |N|$. Therefore, the first assertion follows from the definition of l_0 .

To prove the second assertion, it is enough to show that there exists $S \in \mathcal{S}_{l(a_{i_0+1}, b_1)}$ such that $X(S)$ is not Ramanujan. Actually, take any $S = S_{X,Y} \in \mathcal{S}_{l(a_{i_0+1}, b_1)}$. Then, since $Y = \{y_j\}_{j=1}^h$, Lemma 3.1 gives

$$\begin{aligned} \lambda_{\chi_\alpha}(S_{X,Y}) &= |H| \sum_{x \in X} |\text{Conj}_N(x)| + \frac{|N|}{\chi_\alpha(1)} \sum_{i=1}^h \chi_\alpha(y_i) |\text{Conj}_H(y_i)| \\ &= |H| \left(\frac{|N| - 1}{|H|} - a_{i_0+1} \right) + \frac{|N|}{\chi_\alpha(1)} (-\chi_\alpha(1)) \\ &= -(1 + a_{i_0+1}|H|) = -l(a_{i_0+1}, b_1) = -l(S_{X,Y}). \end{aligned}$$

Here, the second equality follows from the orthogonality of characters together with the fact that χ_α is regarded as a nontrivial irreducible character of H . This implies that $|\lambda_{\chi_\alpha}(S_{X,Y})| = l(S_{X,Y}) > l_0$ and hence $|\lambda_{\chi_\alpha}(S_{X,Y})| > \text{RB}(S)$, by the definition of l_0 . This completes the proof. □

We remark that, since $l(a_{i_0}, b_1) \leq 2(\sqrt{|G|} - 1) < l(a_{i_0+1}, b_1)$ with $l(a_i, b_1) = 1 + a_i|H|$, a_{i_0} can be expressed as

$$a_{i_0} = \max \left\{ a \in A \mid a \leq \frac{2\sqrt{|N||H|} - 3}{|H|} \right\}. \tag{3.3}$$

We apply the above theorem to dihedrants for

$$D_{2p} = \mathbb{Z}_p \rtimes \mathbb{Z}_2 = \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle,$$

with $p \in \mathbb{P}$. This is a Frobenius group with $r = (p - 1)/2$. We can take representatives of the conjugacy classes of D_{2p} as $\{1\} \sqcup \{x^v\}_{v=1}^{(p-1)/2} \sqcup \{y\}$. Since all the conjugacy classes $\text{Conj}_{D_{2p}}(x^v) = \{x^v, x^{-v}\}$ for $1 \leq v \leq \frac{1}{2}(p - 1)$ and $\text{Conj}_{D_{2p}}(y) = \{x^a y \mid 0 \leq a \leq p - 1\}$ are symmetric, $A = \{i \mid 0 \leq i \leq \frac{1}{2}(p - 1)\}$ and $B = \{0\}$. Now, from Theorem 3.3 together with (3.3), we obtain the following result (noticing that $r \geq 4$ if $p \geq 11$).

COROLLARY 3.4. *Let $G = D_{2p}$, where $p \in \mathbb{P}$. If $p \geq 11$, then*

$$\hat{l}_N = l_0 = 2 \lfloor \sqrt{2p} - \frac{1}{2} \rfloor - 1,$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x .

REMARK 3.5. There are several examples of Frobenius groups with $r = (|N| - 1)/|H| \leq 3$ for which the claims in Theorem 3.3 do not hold. For example, consider the dihedral groups D_{2p} for $p = 3, 5, 7$, which correspond to the cases $r = 1, 2, 3$, respectively. In these cases, $(m, n) = (\frac{1}{2}(p + 1), 1)$ and we can check that $l_0 = l(a_{(p-1)/2}, b_1) = p - 2$. Moreover, $\hat{l}_N = l(a_{(p+1)/2}, b_1) = p$ because the corresponding Cayley graph is $X(S_{\emptyset, \{y\}})$, which is Ramanujan because $\Lambda(S_{\emptyset, \{y\}}) = \{\pm p, 0\}$.

4. The cases with all Cayley subsets

The determination of \hat{l}_A is much more difficult than that of \hat{l}_N because of Lemma 2.1. In this section, we study this problem for the case of dihedrants. Throughout this section, we drop the subscript A on S_A , the set of all Cayley subsets of D_{2p} .

4.1. Initial results. Let us consider the dihedrant $X(S)$ of D_{2p} with respect to $S \in \mathcal{S}$. Divide D_{2p} into two parts as $D_{2p} = D_1 \sqcup D_2$, where $D_1 = \{1, x, x^2, \dots, x^{p-1}\}$ and $D_2 = \{y, xy, x^2y, \dots, x^{p-1}y\}$. According to this decomposition, we write $S = S_1 \sqcup S_2$ and $l(S) = l_1(S) + l_2(S)$, where $S_i = S \cap D_i$ and $l_i(S) = |D_i \setminus S_i| = p - |S_i|$ for $i = 1, 2$. Since any subset of D_2 is symmetric because the order of any element in D_2 is two, S is symmetric if and only if S_1 is. This implies that $|S_1|$ is always even and hence $l_1(S)$ is odd. Define $z_j = z_j(S), w_j = w_j(S) \in \mathbb{C}$ ($0 \leq j \leq p - 1$) by $z_0 = |S_1| + |S_2|, w_0 = |S_1| - |S_2|$ and

$$z_j = \sum_{x^a \in S_1} \omega^{ja} = - \sum_{x^a \in D_1 \setminus S_1} \omega^{ja}, \quad w_j = \sum_{x^a y \in S_2} \omega^{ja} = - \sum_{x^a y \in D_2 \setminus S_2} \omega^{ja} \tag{4.1}$$

for $1 \leq j \leq p - 1$. Here, $\omega = e^{2\pi i/p}$. Note that $z_j \in \mathbb{R}$ because S_1 is symmetric. Then the eigenvalues of $X(S)$ are described by using z_j and w_j from Lemma 2.1.

LEMMA 4.1.

- (1) $\Lambda(S) = \{\mu_j^{(+)}, \mu_j^{(-)} \mid 0 \leq j \leq p - 1\}$, where $\mu_j^{(\pm)} = z_j \pm |w_j|$.
- (2) Let $|\mu_j| = \max\{|\mu_j^{(+)}, |\mu_j^{(-)}|\}$. Then $|\mu_j| = |z_j| + |w_j|$.

Now a lower bound of \hat{l}_A follows from the trivial estimate for the eigenvalues.

LEMMA 4.2. For $p \geq 29$, we have $\hat{l}_A \geq \lfloor 2\sqrt{2p} \rfloor - 2$.

PROOF. We first remark that $\mu_0^{(+)} = |S_1| + |S_2| = |S|$ is the largest eigenvalue of $X(S)$ and hence $\mu(S) = \max\{|\mu| \mid \mu \in \Lambda(S), |\mu| \neq \mu_0^{(+)}\} = \max\{|\mu_0^{(-)}|, |\mu_1|, \dots, |\mu_{p-1}|\}$.

Assume that $l(S) \leq \frac{1}{2}p$, which implies that $l_i(S) \leq \frac{1}{2}p$ for $i = 1, 2$. Then, from (4.1), for $1 \leq j \leq p - 1$, we see that $|\mu_j| = |z_j| + |w_j| \leq \min\{|S_1|, l_1(S)\} + \min\{|S_2|, l_2(S)\} \leq l_1(S) + l_2(S) = l(S)$. Moreover, $\mu_0^{(-)} = 2|S_1| - 2p + l(S) \leq l(S)$. Thus $\mu(S) \leq l(S)$. Therefore, if $l(S) \leq \text{RB}(S) = 2\sqrt{2p - l(S) - 1}$ or, equivalently, $l(S) \leq \lfloor 2\sqrt{2p} \rfloor - 2$, then $X(S)$ is Ramanujan. Notice that $2\sqrt{2p} - 2 \leq \frac{1}{2}p$ when $p \geq 29$. \square

From this lemma, together with the result in the previous section, we can narrow the candidates for \hat{l}_A down to at most two.

THEOREM 4.3. Assume that $p \geq 29$ and put $\hat{l}_N = 2\lfloor \sqrt{2p} - \frac{1}{2} \rfloor - 1$, as in Corollary 3.4.

- (1) If $\lfloor 2\sqrt{2p} \rfloor$ is even, then $\hat{l}_A = \hat{l}_N + 1$.
- (2) If $\lfloor 2\sqrt{2p} \rfloor$ is odd, then $\hat{l}_A = \hat{l}_N$ or $\hat{l}_A = \hat{l}_N + 1$.

PROOF. We first remark that, for $\alpha \in \mathbb{R}$,

$$2\lfloor \alpha - \frac{1}{2} \rfloor - 1 = \begin{cases} \lfloor 2\alpha \rfloor - 2 - 1 & \text{if } 0 \leq \alpha - \lfloor \alpha \rfloor < \frac{1}{2} \text{ or, equivalently, } \lfloor 2\alpha \rfloor \text{ is even,} \\ \lfloor 2\alpha \rfloor - 2 & \text{if } \frac{1}{2} \leq \alpha - \lfloor \alpha \rfloor < 1 \text{ or, equivalently, } \lfloor 2\alpha \rfloor \text{ is odd.} \end{cases}$$

Using this formula with $\alpha = \sqrt{2p}$, we see that $\lfloor 2\sqrt{2p} \rfloor - 2$ coincides with $\hat{l}_N + 1$ (respectively, \hat{l}_N) if $\lfloor 2\sqrt{2p} \rfloor$ is even (respectively, odd) and hence, from Lemma 4.2, $\hat{l}_A \geq \hat{l}_N + 1$ (respectively, $\hat{l}_A \geq \hat{l}_N$). The results follow because $\hat{l}_A < l_1 = \hat{l}_N + 2$, where $l_1 = \min\{l \in \mathcal{L}_N \mid l > l_0\}$ with $\mathcal{L}_N = \{l(S) \mid S \in \mathcal{S}_N\}$. \square

From this theorem, $\hat{l}_A = \hat{l}_N + \varepsilon$ with $\varepsilon = \varepsilon_p \in \{0, 1\}$. Similarly to the case of circulants [5], we call p *exceptional* if $\lfloor 2\sqrt{2p} \rfloor$ is odd and $\varepsilon = 1$ and *ordinary* otherwise.

4.2. A characterisation of exceptional primes. Assume that $\lfloor 2\sqrt{2p} \rfloor$ is odd. For $l \in \mathcal{L}$, let $\mu(l) = \max\{\mu(S) \mid S \in \mathcal{S}_l\}$ and $\text{RB}(l) = \text{RB}(S) = 2\sqrt{2p - l - 1}$ for $S \in \mathcal{S}_l$. From the definition, p is exceptional if and only if $\mu(\hat{l}_N + 1) \leq \text{RB}(\hat{l}_N + 1)$. To study this inequality, we first construct $S \in \mathcal{S}_{\hat{l}_N + 1}$ such that $\mu(\hat{l}_N + 1) = \mu(S)$.

For $l \in \mathcal{L}$, let $L(l) = \{(l_1, l_2) \in \mathbb{Z}_{\geq 0}^2 \mid l_1 + l_2 = l, l_1 \text{ odd}\}$. Further, for $(l_1, l_2) \in L(l)$, define $S^{(l_1, l_2)} = S_1^{(l_1)} \sqcup S_2^{(l_2)} \in \mathcal{S}_l$ by $S_1^{(l_1)} = D_1 \setminus \{1, x^{\pm 1}, x^{\pm 2}, \dots, x^{\pm(l_1-1)/2}\}$ and $S_2^{(l_2)} = D_2 \setminus \{y, xy, x^2y, \dots, x^{l_2-1}y\}$. One sees that $l_i(S^{(l_1, l_2)}) = l_i$ for $i = 1, 2$ and

$$z_j = \sum_{h=-(l_1-1)/2}^{(l_1-1)/2} \omega^{hj} = \frac{\sin \pi j l_1 / p}{\sin \pi j / p}, \quad w_j = \sum_{h=0}^{l_2-1} \omega^{hj} = \omega^{j(l_2-1)/2} \frac{\sin \pi j l_2 / p}{\sin \pi j / p},$$

and hence $|\mu_j| = |\mu_j(l_1, l_2)|$ can be written as

$$|\mu_j| = |z_j| + |w_j| = \frac{\sin \pi j l_1 / p}{\sin \pi j / p} + \frac{\sin \pi j l_2 / p}{\sin \pi j / p} = 2 \frac{\sin \pi j l / 2p}{\sin \pi j / p} \cos \frac{\pi j |l_1 - l_2|}{2p}.$$

Now let us write $\hat{l}_N = \lfloor 2\sqrt{2p} \rfloor - 2$ as $\hat{l}_N = 4k + r$ for some $k \geq 0$ and $r \in \{1, 3\}$.

LEMMA 4.4. *Let $(\check{l}_1, \check{l}_2) = (\frac{1}{2}(\hat{l}_N + 1), \frac{1}{2}(\hat{l}_N + 1))$ if $r = 1$ and $(\frac{1}{2}(\hat{l}_N + 3), \frac{1}{2}(\hat{l}_N - 1))$ otherwise. Then,*

$$\mu(\hat{l}_N + 1) = \mu(S^{(\check{l}_1, \check{l}_2)}) = |\mu_1(\check{l}_1, \check{l}_2)|.$$

PROOF. Consider when $|l_1 - l_2|$ takes its minimum under the condition that $l_1 + l_2 = \hat{l}_N + 1$ and l_1 is odd. Note that, since p is prime, $|\mu_1(\check{l}_1, \check{l}_2)|$ is the maximum among $|\mu_j(\check{l}_1, \check{l}_2)|$ for $1 \leq j \leq p - 1$. □

When $\hat{l}_N = 4k + r$, we see that $p \in I_{r,k} \cap \mathbb{P}$, where

$$\begin{aligned} I_{r,k} &= \{t \in \mathbb{R} \mid \lfloor 2\sqrt{2t} \rfloor - 2 = 4k + r\} \\ &= [2k^2 + (r + 2)k + \frac{1}{8}(r + 2)^2, 2k^2 + (r + 3)k + \frac{1}{8}(r + 3)^2). \end{aligned}$$

This means that p can be expressed as $p = f_{r,c_r}(k)$ for some integers $k \geq 3$ and $c_r \in \mathbb{Z}$ with $-k + 2 \leq c_1 \leq 1$, if $r = 1$, and $-k + 4 \leq c_3 \leq 4$, otherwise. Here,

$$f_{r,c_r}(t) = 2t^2 + (r + 3)t + c_r.$$

Let $I_r = \bigsqcup_{k \geq 3} I_{r,k} \cap \mathbb{P}$ and $C_r = \{r - 4, r - 2, r\}$. Moreover, for $c_r \in C_r$, define an integer $k_{r,c_r} \geq 3$ by $(k_{1,-3}, k_{1,-1}, k_{1,1}) = (5, 3, 3)$ and $(k_{3,-1}, k_{3,1}, k_{3,3}) = (7, 3, 3)$. We now obtain the following theorem, which gives a characterisation for exceptional primes.

THEOREM 4.5. *A prime $p \in I_r$ with $p \geq 29$ is exceptional if and only if it is of the form of $p = f_{r,c_r}(k)$ for some $c_r \in C_r$ and $k \geq k_{r,c_r}$.*

PROOF. To clarify when the inequality $\mu(\hat{l}_N + 1) = |\mu_1(\check{l}_1, \check{l}_2)| \leq \text{RB}(\hat{l}_N + 1)$ holds, we introduce an interpolation function $d_r(t)$ of the difference between $|\mu_1(\check{l}_1, \check{l}_2)|$ and $\text{RB}(\hat{l}_N + 1)$ on $I_{r,k}$. Set

$$d_r(t) = 2 \frac{\sin \pi(4k + r + 1)/2t}{\sin \pi/t} \cos \frac{\pi(r - 1)}{2t} - 2\sqrt{2t - 4k - r - 2}.$$

One can see that $d_r(t)$ is monotonically decreasing on $I_{r,k}$ for sufficiently large k . Moreover, at $t = p = f_{r,c_r}(k) \in I_{r,k} \cap \mathbb{P}$,

$$d_r(p) = \frac{3(r + 3)^2 - 24c_r - 16\pi^2}{24} k^{-1} + O(k^{-2})$$

as $k \rightarrow \infty$ because

$$\begin{aligned} |\mu_1(\check{l}_1, \check{l}_2)| &= 2 \frac{\sin \pi(4k + r + 1)/(2(2k^2 + (r + 3)k + c_r))}{\sin \pi/(2k^2 + (r + 3)k + c_r)} \cos \frac{\pi(r - 2)}{2(2k^2 + (r + 3)k + c_r)} \\ &= 4k + r + 1 - \frac{2\pi^2}{3} \frac{1}{k} + O(k^{-2}), \\ \text{RB}(\hat{l}_N + 1) &= 2\sqrt{2(2k^2 + (r + 3)k + c_r) - 4k - r - 2} \\ &= 4k + r + 1 - \frac{(r + 3)^2 - 8c_r}{8} \frac{1}{k} + O(k^{-2}). \end{aligned}$$

This shows $d_r(p) < 0$ for sufficiently large k if and only if $3(r + 3)^2 - 24c_r - 16\pi^2 < 0$, that is, if $c_r \in C_r$ (note that c_r should be odd because p is). In fact, for each $r \in \{1, 3\}$ and $c_r \in C_r$, we can check that the inequality $d_r(p) < 0$ with $p = f_{r,c_r}(k)$ holds if and only if $k \geq k_{r,c_r}$. \square

For $r \in \{1, 3\}$ and $c_r \in C_r$, let $J_{r,c_r} = \{f_{r,c_r}(k) \in I_r \mid k \geq k_{r,c_r}\}$, that is, let J_{r,c_r} be the set of exceptional primes p of the form $p = f_{r,c_r}(k)$. The classical Hardy–Littlewood conjecture [4] asserts that if a quadratic polynomial $f(t) = at^2 + bt + c$ with $a, b, c \in \mathbb{Z}$ satisfies the conditions that $a > 0$, $(a, b, c) = 1$, $a + b$ and c are not both even and $D = b^2 - 4ac$ is not a square, then there are an infinite number of primes represented by $f(t)$ and, moreover, that $\pi(f; x) = \#\{k \leq x \mid f(k) \in \mathbb{P}\}$ has the asymptotic behaviour

$$\pi(f; x) \sim \frac{C(f)}{2} \frac{x}{\log x}, \quad C(f) = 2 \prod_{p \geq 3} \left(1 - \frac{\left(\frac{D}{p}\right)}{p-1}\right), \tag{4.2}$$

as $x \rightarrow \infty$. Here, $C(f)$ is called the Hardy–Littlewood constant and (D/p) is the Legendre symbol. From Theorem 4.5, the existence of an infinite number of exceptional primes is related to this conjecture.

COROLLARY 4.6. *There are an infinite number of exceptional primes if the Hardy–Littlewood conjecture is true for at least one of $f_{r,c_r}(t)$ with $r \in \{1, 3\}$ and $c_r \in C_r$.*

REMARK 4.7. From (4.2), we can expect that $\pi(f_{r,c_r}; x) \sim \frac{1}{2}C(f_{r,c_r})x/\log x$, where

$$\frac{C(f_{r,c_r})}{2} = \prod_{p \geq 3} \left(1 - \frac{c'_r}{p-1}\right) = \begin{cases} \left\{ \begin{array}{ll} 0.671043\dots, & r = 1, c_1 = -3, \\ 1.03556\dots, & r = 1, c_1 = -1, \\ 1.84998\dots, & r = 1, c_1 = 1, \end{array} \right. \\ \left\{ \begin{array}{ll} 1.14801\dots, & r = 3, c_3 = -1, \\ 0.757353\dots, & r = 3, c_3 = 1, \\ 1.38332\dots, & r = 3, c_3 = 3, \end{array} \right. \end{cases}$$

with $c'_1 = 4 - 2c_1$ and $c'_3 = 9 - 2c_3$.

REMARK 4.8. The existence of an infinite number of ordinary primes can be verified as follows. Let $J = \bigsqcup_{r \in \{1,3\}} \bigsqcup_{c_r \in C_r} J_{r,c_r}$. For a positive integer a , let

$$J_{r,c_r}(a) = \{n \in \mathbb{Z}_{\geq 0} \mid 0 \leq n \leq a - 1 \text{ and } n \equiv f_{r,c_r}(k) \pmod{a} \text{ for some } 0 \leq k \leq a - 1\}$$

and $J(a) = \bigcup_{r \in \{1,3\}} \bigcup_{c_r \in C_r} J_{r,c_r}(a)$. If we can find $b \in \{0, 1, 2, \dots, a - 1\} \setminus J(a)$ satisfying $(a, b) = 1$, then $\{at + b \mid t \in \mathbb{Z}\} \cap J = \emptyset$ and, from the Dirichlet theorem for primes in arithmetic progressions, there are an infinite number of primes in $\{at + b \mid t \in \mathbb{Z}\}$. These are ordinary primes, by Theorem 4.5. To achieve this purpose, one may take, for example, $(a, b) = (29, 4), (35, 8)$ or $(40, 33)$.

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