J. Austral. Math. Soc. (Series A) **57** (1994), 170–178

MODULES WITH FINITE SPANNING DIMENSION

BHAVANARI SATYANARAYANA

(Received 17 January 1990; revised 1 May 1992)

Communicated by B. J. Gardner

Abstract

It is well known that if M is a module with finite spanning dimension, then one can talk of $\mathrm{Sd}(K)$, the spanning dimension of K only when K is a supplement submodule in M. In this paper we extend this concept to general submodules and obtained some important results. We characterize the set of all supplement submodules of the module $R/(x)$ over R where R is a Euclidean domain and $x \in R$. Moreover, it is proved that the number of distinct supplements in $R/(x)$ is 2^k and $\mathrm{Sd}(R/(x)) = k$ where k is the number of distinct nonassociate prime factors of x.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 16A34, 16A53 and 16A64.

Introduction

Let R be a (not necessarily commutative) ring with unity. Throughout this paper by a module we mean a unital left R-module. M stands for a module with finite spanning dimension and A, B stand for submodules of M. We write f.s.d. for finite spanning dimension. For fundamental definitions and results we refer to [1] - [5]. We now list the following results from the literature which are used frequently.

LEMMA 0.1. (i) *If A is a non-small submodule of M then there exist two submodules S and K such that S is a supplement of A, $K \subseteq A$, K is a supplement of S and also K and S are mutual supplements.*

(ii) *If S is a submodule such that $S + A = M$ then S is a supplement of A if and only if $S \cap A$ is small in S.*

(iii) *Every supplement of M has* f.s.d.

(iv) *If $S^* \subseteq S \subseteq M$, and S is a supplement then S^* is a supplement in M if and only if S^* is a supplement in S.*

1. Preliminary Results

DEFINITION. Suppose A is non-small in M. B is said to be an S-supplement for A if $B \subseteq A$ and there exist a supplement X of A such that B is a supplement of X. In this case, we also say that B is an S-supplement for A with X in M.

LEMMA 1.1. *For a non-small submodule S of M, the following conditions are equivalent:*

(i) *There exists a submodule H of M such that S is a supplement of H in M;*

(ii) *For any submodule A of M such that $A \subseteq S$ we have that A is small in M if and only if A is small in S.*

PROOF. (i) implies (ii). Suppose A is non-small. Now A is small in S because $A + B = S$ implies $A + B + H = S + H = M$. This in turn implies $B + H = M$, which implies $B = S$. The other part is clear.

(ii) implies (i). Since S is non-small, by Lemma 0.1, there exist two submodules S' and H of M such that H is a supplement of S, S' is a supplement of H and $S' \subseteq S$. Now S' is an S-supplement for S with H. By [3, Lemma 3(ii)], S' is a supplement of $H \cap S$ in S. Since H is a supplement of S we have $H \cap S$ is small in H and so it is small in M. Therefore $H \cap S$ is small in S, which shows that $S' = S$.

THEOREM 1.2. *Suppose A is a non-small submodule and B, B' are two S-supplements for A with X, X' respectively. Then*

(i) $\mathrm{Sd}(B) = \mathrm{Sd}(B')$

(ii) *B is a maximal supplement contained in A.*

PROOF. (i) By [3, Lemma 4] we have $\mathrm{Sd}(B) + \mathrm{Sd}(X) = \mathrm{Sd}(M) = \mathrm{Sd}(B') + \mathrm{Sd}(X')$ and $\mathrm{Sd}(X) = \mathrm{Sd}(M/A) = \mathrm{Sd}(X')$, which shows that $\mathrm{Sd}(B) = \mathrm{Sd}(B')$.

(ii) Let A' be a supplement such that $B \subseteq A' \subseteq A$. Since $A \cap X$ is small in M, so is $A' \cap X$. By Lemma 1.1, $A' \cap X$ is small in A'. Now $A' + X \supseteq B + X = M$ shows that A' is a supplement of X and hence $B = A'$.

DEFINITION. A set of hollow submodules $\{H_i \mid 1 \leq i \leq k\}$ is said to satisfy *Property (S)* if there exists a supplement X of $H_1 + \cdots + H_k$ such that the sum $M = H_1 + \cdots + H_k + X$ is non-redundant. In this case, we also say that H_i, $1 \leq i \leq k$ satisfy Property (S) with X.

EXAMPLE. If A and B are mutual supplements and H_i, $1 \leq i \leq k$ are hollow submodules such that $A = H_1 + \cdots + H_k$ and the sum is non-redundant then H_i, $1 \leq i \leq k$ satisfy Property (S) with B.

LEMMA 1.3. *If* $H_i, 1 \le i \le k$ *satisfy* Property (S) *with* B *then* $k \le \mathrm{Sd}(M)$. *Moreover,* $k + \mathrm{Sd}(B) = \mathrm{Sd}(M)$.

PROOF. If $\mathrm{Sd}(B) = t$ then there exist hollow submodules B_i, $1 \le i \le t$ such that $B = B_1 + \cdots + B_t$ and the sum is non-redundant. Clearly the sum $M = H_1 + \cdots + H_k + B_1 + \cdots + B_t$ is non-redundant and so $\mathrm{Sd}(M) = k + t \ge k$.

LEMMA 1.4. *Let* H *be a non-small submodule and* H_i, $1 \le i \le k$ *satisfy* Property (S) *with* B *such that* $H_i \subseteq H, 1 \le i \le k$.

(a) *If* B *is not a supplement of* H *then there exists* $H_{k+1} \subseteq H \cap B$ *such that* H_i, $1 \le i \le k+1$ *satisfy Property (S).*

(b) *The following two conditions are equivalent:*

(i) B *is a supplement of* H;

(ii) $\{H_i \mid 1 \le i \le k\}$ *is a maximal set of hollow submodules that satisfy* Property (S) *and* $H_i \subseteq H, 1 \le i \le k$.

PROOF. (a) If B is not a supplement of H then there exist a minimal hollow non-small submodule $H_{k+1} \subseteq B \cap H$. If Y is a supplement of H_{k+1} in B then Y and H_{k+1} are mutual supplements in B. Now H_i, $1 \le i \le k+1$ satisfy Property (S) with Y.

(b) (i) implies (ii). By Lemma 1.3, $k + \mathrm{Sd}(B) = \mathrm{Sd}(M)$. If $\{H_i \mid 1 \le i \le k\}$ is not maximal with the required properties then by Lemma 1.3, we can find a maximal set $\{H_1, \ldots, H_m\}$ of hollow submodules for some $m > k$ such that H_i, $1 \le i \le m$ satisfy Property (S) with X and $H_i \subseteq H$, $1 \le i \le m$. By Part (a), X is a supplement of H and by Lemma 1.3, $\mathrm{Sd}(X) + m = \mathrm{Sd}(M)$. Since B and X are supplements of H by [3, Lemma 4] we have $\mathrm{Sd}(B) = \mathrm{Sd}(M/H) = \mathrm{Sd}(X)$. Therefore $\mathrm{Sd}(M) - k = \mathrm{Sd}(B) = \mathrm{Sd}(X) = \mathrm{Sd}(M) - m$ which implies $k = m$, a contradiction. (ii) implies (i) follows from Part (a).

2. Spanning Dimension of a Submodule

DEFINITION. For a submodule A of M we define $\mathrm{Sd}_M(A)$ as follows: If A is small then $\mathrm{Sd}_M(A) = 0$ and if A is non-small then $\mathrm{Sd}_M(A) = \mathrm{Sd}(B)$ where B is an S-supplement for A.

THEOREM 2.1. *Suppose* A *and* B *are non-small submodules of* M *such that* $A + B = M$. *Let* Y *be a supplement of* A *such that* $Y \subseteq B$, *and* X *be an* S-*supplement for* A *with* Y. *Then the following conditions* (i) *to* (iii) *hold:*

(i) $\mathrm{Sd}(M) = \mathrm{Sd}(M/A) + \mathrm{Sd}_M(A)$;

(ii) $\mathrm{Sd}_M(A) + \mathrm{Sd}(Y) = \mathrm{Sd}(M)$;

(iii) *If* $X \cap B$ *is small in* M *then* $\mathrm{Sd}_M(A) + \mathrm{Sd}_M(B) = \mathrm{Sd}(M)$.
Moreover, if $A \cap B$ *is small then the following* (iv) *to* (vi) *hold:*
(iv) $\mathrm{Sd}_M(A) + \mathrm{Sd}_M(B) = \mathrm{Sd}(M)$;
(v) $\mathrm{Sd}_M(A) = \mathrm{Sd}(M/B)$;
(vi) $\mathrm{Sd}(M/A) + \mathrm{Sd}(M/B) = \mathrm{Sd}(M)$.

PROOF. By [3, Lemma 4(i), 4(iv)] we have $\mathrm{Sd}(Y) = \mathrm{Sd}(M/A)$ and $\mathrm{Sd}(X) + \mathrm{Sd}(Y) = \mathrm{Sd}(M)$. Since $\mathrm{Sd}(X) = \mathrm{Sd}_M(A)$ we have $\mathrm{Sd}_M(A) + \mathrm{Sd}_M(Y) = \mathrm{Sd}(M)$ and $\mathrm{Sd}(M) = \mathrm{Sd}(M/A) + \mathrm{Sd}_M(A)$. This completes the proof of (i) and (ii).

By Lemma 1.1, $X \cap B$ is small in X and so X is a supplement of B. By (ii), $\mathrm{Sd}_M(B) + \mathrm{Sd}(X) = \mathrm{Sd}(M)$ and so we have (iii).

Equation (iv) follows from (iii), and (v) follows [3, Lemma 4(i)].

Equation (vi) follows from (iv) and (v).

LEMMA 2.2. *If at least one of* A *and* B *is small then* $\mathrm{Sd}_M(A) + \mathrm{Sd}_M(B) = \mathrm{Sd}_M(A + B) + \mathrm{Sd}_M(A \cap B)$.

PROOF. If both A and B are small then the proof is clear. Suppose A is small and B is non-small. It is enough to show $\mathrm{Sd}_M(B) = \mathrm{Sd}_M(A + B)$. Let S be an S-supplement for $A + B$ with H. Since A is small, $A + B + H = M$ implies $B + H = M$. Clearly H is a supplement of B. Let S' be an S-supplement for B with H. Now we have

$$\mathrm{Sd}_M(B) = \mathrm{Sd}(S') = \mathrm{Sd}(M) - \mathrm{Sd}(H) = \mathrm{Sd}(S) = \mathrm{Sd}_M(A + B).$$

LEMMA 2.3. *Suppose* H *and* H' *are two non-small submodules of* M.
(i) $\mathrm{Sd}_M(H) = m$ *where* $m = \max\{k \mid k = \mathrm{Sd}(S)$ *for some supplement* S *in* M *such that* $S \subseteq H\}$.
(ii) *If* $H \subseteq H'$ *then* $\mathrm{Sd}_M(H) \leq \mathrm{Sd}_M(H')$.

PROOF. Clearly $\mathrm{Sd}_M(H) \leq m$. Let S be a supplement such that $S \subseteq H$ and $\mathrm{Sd}(S) = k$. Let Y be a supplement of S. Now by Lemma 0.1(ii) and Lemma 1.1, we have that S and Y are mutual supplements. There exist hollow submodules H_i, $1 \leq i \leq k$ such that $S = H_1 + \ldots + H_k$, the sum is non-redundant and H_i, $1 \leq i \leq k$ satisfy Property (S) with Y. Now we can find a maximal set $\{H_i \mid 1 \leq i \leq n\}$ for some $n \geq k$ satisfying Property (S) with X and each $H_i \subseteq H$. By Lemma 1.4, X is a supplement of H. Let S^* be an S-supplement for H with X. Now by [3, Lemma 4] and Lemma 1.3, we have $\mathrm{Sd}(S^*) + \mathrm{Sd}(X) = \mathrm{Sd}(M) = n + \mathrm{Sd}(X)$ which implies $\mathrm{Sd}_M(H) = \mathrm{Sd}(S^*) = n \geq k$. Now (ii) follows from (i).

THEOREM 2.4. *If* S *and* S' *are two supplements in* M *such that* $S + S'$ *is a supplement then* $\mathrm{Sd}(S) + \mathrm{Sd}(S') = \mathrm{Sd}(S + S') + \mathrm{Sd}_M(S \cap S')$.

https://doi.org/10.1017/S1446788700037502 Published online by Cambridge University Press

PROOF. If $S \cap S'$ is small in M then by Lemma 1.1, $S \cap S'$ is small in both S and S', and by Theorem 2.1(iv), $\mathrm{Sd}(S + S') = \mathrm{Sd}(S) + \mathrm{Sd}(S')$ and so the result follows. Now suppose $S \cap S'$ is non-small. Let L be a supplement of $S \cap S'$ and S^* be a supplement of L such that $S^* \subseteq S \cap S'$. Now $\mathrm{Sd}(S^*) = \mathrm{Sd}_M(S \cap S')$. Since S^* is a supplement of L we have that $S^* \cap L$ is small in S^*. Now $M = S^* + L$ implies

$$S = M \cap S = (S^* + L) \cap S = S^* + L \cap S.$$

Since $S^* \cap (L \cap S) = S^* \cap L$ is small in S^* we have that S^* is a supplement of $L \cap S$ in S. Let X be a supplement of S^* such that $X \subseteq L \cap S$. Now X is an S-supplement for $L \cap S$ with S^* in S. In a similar way, there exists a submodule X' which is an S-supplement for $L \cap S'$ with S^* in S'. Then it is clear that $\mathrm{Sd}(S) = \mathrm{Sd}(S^*) + \mathrm{Sd}(X)$, $\mathrm{Sd}(S') = \mathrm{Sd}(S^*) + \mathrm{Sd}(X')$ and $\mathrm{Sd}(S + S') = \mathrm{Sd}(S^*) + \mathrm{Sd}(X) + \mathrm{Sd}(X')$. Now the result follows since $\mathrm{Sd}_M(S \cap S') = \mathrm{Sd}(S^*)$.

3. Modules with a Property (B)

Now we state

PROPERTY (B). *If X and Y are two submodules of M and S, S' are S-supplements for X, Y respectively then $S + S'$ is an S-supplement for $X + Y$.*

NOTE. (i) By Theorem 4.3(iii), $R/(x)$ satisfies Property (B) where R is a Euclidean domain and $x \in R$.

(ii) If M satisfies Property (B) then the sum of two supplements in M is a supplement.

THEOREM 3.1. *If M satisfies* Property (B) *then* $\mathrm{Sd}_M(A) + \mathrm{Sd}_M(B) = \mathrm{Sd}_M(A + B) + \mathrm{Sd}_M(A \cap B)$.

PROOF. If at least one of A and B is small then the result follows from Lemma 2.2. Suppose both A and B are non-small. If $A \cap B$ is small then for any S-supplements X, X' for A, B respectively we have $\mathrm{Sd}_M(A \cap B) = 0 = \mathrm{Sd}_M(X \cap X')$ and hence the proof follows from Theorem 2.4. We now suppose $A \cap B$ is non-small. Let S^* be an S-supplement for $A \cap B$, K a supplement of A, S a supplement of $K + S^*$ such that $S \subseteq A$, K' a supplement of B and S' a supplement of $K' + S^*$ such that $S' \subseteq B$. Since $(S + S^*) \cap K \subseteq A \cap K$ we have $(S + S^*) \cap K$ is small. By Property (B) and Lemma 1.1, we have $(S + S^*) \cap K$ is small in $S + S^*$ and hence $S + S^*$ is an S-supplement for A with K. Similarly $S^* + S'$ is an S-supplement for B with K'. By Property (B) we have

$\mathrm{Sd}_M(A+B) = \mathrm{Sd}(S+S^*+S'+S^*)$. Since $S^* \subseteq ((S+S^*)\cap(S'+S^*)) \subseteq \mathrm{Sd}_M(A\cap B)$ we have

$$\mathrm{Sd}(S^*) \le \mathrm{Sd}_M((S+S^*)\cap(S'+S^*)) \le \mathrm{Sd}_M(A\cap B) = \mathrm{Sd}(S^*).$$

By using Theorem 2.4, we have

$$\begin{aligned}\mathrm{Sd}_M(A+B) &= \mathrm{Sd}(S+S^*+S'+S^*)\\ &= \mathrm{Sd}(S+S^*) + \mathrm{Sd}(S'+S^*) - \mathrm{Sd}_M((S+S^*)\cap(S'+S^*))\\ &= \mathrm{Sd}_M(A) + \mathrm{Sd}_M(B) - \mathrm{Sd}_M(A\cap B).\end{aligned}$$

REMARK. Here Z stands for the ring of integers and Z_n for the set of integers modulo n.

(i) In general, the sum of two supplements of a module with f.s.d.may not be a supplement. For example, consider $M = Z_4 \oplus Z_4$ over Z. $S = Z_4 \oplus (o)$ and $S^* = (1,2)Z_4$ are two supplements of $(o) \oplus Z_4$ where $S + S^* = Z_4 \oplus 2Z_4$ is not a supplement. For one more example, consider $M = Z \oplus Z$ over Z. Then $S = Z \oplus (o)$, $S^* = (1,2)Z$ are supplements of $(o) \oplus Z$, $(o,1)Z$ respectively, but $S + S^* = Z \oplus 2Z$ is not a supplement. Let us verify these facts in the following:

Firstly we verify that $S^* = (1,2)Z$ is a supplement of $A = (o,1)Z$. Note that $(1,o) = (1,2) - (o,1)2 \in S^* + A$ implies $M = S^* + A$. Let B be any proper submodule of S^*. Then $B = (a, 2a)Z$ for some $a \in Z$ such that $a \ge 2$. Supposing $B + A = M$, $(1,o) \in M = B + A$ implies $(1,o) = (a,2a)x + (o,1)y$ for some $x, y \in Z$, implying $ax = 1$ and finally $x = 1/a$, contradicting the fact that $a \ge 2$ and $1/a = x \in Z$. Hence $B + A \ne M$ for any proper submodule B of S^*. This shows that S^* is a supplement of A.

Next, we see that $X = Z \oplus 2Z$ is not a supplement. Suppose on the contrary that X is a supplement of H in M. Then $(o,1) \in M = X + H$ implies $(o,1) = (a,b) + (x,y)$ for some $(a,b) \in X$ and $(x,y) \in H$ implies $1 = b + y$.Thus y is an odd number since b is even. So there exist $(x,y) \in H$, $y \ne 0$, y being odd. Since $(-x,-y)$ is also in H we may take $y > o$. Let $y^* = \min\{y \mid y > o$, there exists $x \in Z$ such that $(x,y) \in H$ and y is an odd number $\}$. There exists x^* such that $(x^*, y^*) \in H$. If $y^* = 1$ then $(o,1) = (-x^*, o) + (x^*, y^*) \in (Z \oplus o) + H$. This implies $o \oplus Z \subseteq (Z \oplus o) + H$, and so $M \subseteq (Z \oplus o) + H$, contradicting the fact that X is a supplement of H. Therefore $y^* > 1$. which implies $y^* \ge 3$. Thus $(o,1) = (x^*, y^*+1) - (x^*, y^*) \in (Z \oplus (y^*+1)Z) + H$, Now as in the above steps we get $M = (Z \oplus (y^*+1)Z) + H$, a contradiction, since X is a supplement of H and $Z \oplus (y^*+1)Z \subsetneq X$. Hence X cannot be a supplement.

(ii) We know that if S is a supplement then $\mathrm{Sd}(M) = \mathrm{Sd}(S)$ implies $M = S$. But for a general submodule this condition fails. For example, consider $M = Z_{24}$ over Z.

Since $M = (8) + (3)$, the sum of two hollow submodules, we have $\mathrm{Sd}(M) = 2$. If $H = (2)$ then $H = (8) + (6)$ and so $\mathrm{Sd}(H) = 2$. Thus we have $\mathrm{Sd}(M) = \mathrm{Sd}(H)$ and $M \neq H$.

4. Supplements in Modules over Euclidean Domains

Throughout this section R stands for a Euclidean domain and (y) stands for the ideal generated by $y \in R$. When we are considering (y) we may assume that $y = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ where p_i, $1 \leq i \leq k$ are non-associate primes and $a_i \geq 1$ for each i. We let x denote a fixed element of R with $x = p_1^{a_1} \dots p_k^{a_k}$, where p_i, $1 \leq i \leq k$ are non-associate primes and each $a_i \geq 1$. Let f denote some permutation on $\{1, 2, \dots, k\}$ and write $b_i = a_{f(i)}$, $1 \leq i \leq k$. We write $q_i = p_{f(1)}^{b_1} \cdots p_{f(i)}^{b_i}$ and $t_i = p_{f(i)}^{b_i} \cdots p_{f(k)}^{b_k}$ for $1 \leq i \leq k$. We define s_i, $1 \leq i \leq k$ as $s_1 = t_2$, $s_k = q_{k-1}$ and $s_i = q_{i-1}.t_{i+1}$ for $2 \leq i \leq k-1$. We write $A_i = (s_i)$, $B_i = (t_i)$ and $C_i = (q_i)$. Π denotes the canonical mapping from R to $R/(x)$ and $\mathrm{IC}(x)$ denotes the set of all ideals of R containing (x).

LEMMA 4.1. (i) *If* $H \in \mathrm{IC}(x)$ *then* $H = (y)$ *where* $y = p_1^{y_1} \dots p_k^{y_k}$ *with* $0 \leq y_i \leq a_i$ *for* $1 \leq i \leq k$.

(ii) $\Pi(C_m)$ *is a supplement of* $\Pi(B_{m+1})$.

(iii) $\Pi(A_m)$ *is hollow for* $1 \leq m \leq k$.

(iv) $\Pi(A_1) + \cdots + \Pi(A_k) = \Pi(R)$ *and* $\mathrm{Sd}(\Pi(R)) = k$.

(v) *There is no* $H \in \mathrm{IC}(x)$ *such that* $B_m \subsetneqq H \subsetneqq B_{m+1}$ *and* H *is a supplement in* $\Pi(R)$.

PROOF. (i) is clear.

(ii) Clearly $\Pi(C_m) + \Pi(B_{m+1}) = \Pi(R)$. Let $A \in \mathrm{IC}(x)$ be such that $\Pi(A) + \Pi(B_{m+1}) = \Pi(R)$ and $A \subseteq C_m$. Suppose $A = (s')$ where $s' = p_{f(1)}^{h_1} \cdots p_{f(k)}^{h_k}$ and $0 \leq h_i \leq b_i$, $1 \leq i \leq k$. Since $A \subseteq C_m$ we have q_m divides s' and hence $h_i = b_i$, $1 \leq i \leq m$. If $A \neq C_m$ then $h_q \neq 0$ for some $m+1 \leq q \leq k$. Now $A + B_{m+1} \subseteq (p_{f(q)}^{h_q}) \neq R$, a contradiction to $\Pi(A) + \Pi(B_{m+1}) = \Pi(R)$.

(iii) A straightforward verification.

(iv) Since $p_{f(1)}^{b_1}$ and $p_{f(2)}^{b_2}$ are relatively prime there exist $a, b \in R$ such that $1 = a.p_{f(1)}^{b_1} + b.p_{f(2)}^{b_2}$ and so $t_3 = as_2 + bs_1 \in A_2 + A_1$, which shows that $A_1 + A_2 = B_3$. Similarly we can verify $A_1 + A_2 + A_3 = B_4, \dots, A_1 + \cdots + A_{k-1} = B_k$ and $A_1 + \cdots + A_k = B_k + A_k = R$. Thus $\Pi(A_1) + \cdots + \Pi(A_k) = \Pi(R)$. No $\Pi(A_i)$ can be deleted from this sum because $A_j \subseteq (p_{f(i)}^{b_i})$ for $j \neq i$ and $1 \leq j \leq k$. By (iii) each $\Pi(A_i)$ is hollow and by [1, Theorem 3.1], $\mathrm{Sd}(\Pi(R)) = k$.

(v) Since for any module M, $\mathrm{Sd}(M) = \max\{k \mid$ there exists a proper chain $S_k \supset S_{k-1} \supset \dots \supset S_0 = (0)$ of supplements in $M\}$, we have $\Pi(R) \supset \Pi(B_k) \supset \dots \supset \Pi(B_1) = (0)$ is a maximal proper chain of supplements in $\Pi(R)$ and so the result follows.

One can easily verify the following two theorems using the simple techniques developed already.

THEOREM 4.2. (i) *If* $A \in \mathrm{IC}(x)$ *and* $A = (p_{f(1)}^{h_1} \cdots p_{f(m)}^{h_m})$ *for some* $1 \le m \le k$ *and* $h_i \ne 0$, $1 \le i \le m$ *then* $\Pi(C_m)$ *is the unique supplement for* $\Pi(A)$. *Moreover, if* $\Pi(A)$ *is a supplement then* $A = C_m$.

(ii) *The set of all non-zero supplements in* $\Pi(R)$ *is precisely* $\{\Pi(Y) \mid Y = (y)$, $y = p_{g(1)}^{y_1} \cdots p_{g(m)}^{y_m}$ *where* g *is a permutation on* $\{1, 2, \ldots, k\}$, $1 \le m \le k$ *and* $y_i = a_{g(i)}$, $1 \le i \le m\}$.

(iii) $\Pi(C_m) = \Pi(A_{m+1}) + \ldots + \Pi(A_k)$ *and* $\mathrm{Sd}(\Pi(C_m)) = k - m$.

(iv) *If* $\Pi(A)$ *is a supplement with* $\mathrm{Sd}(\Pi(A)) = k - m > 0$ *and* $A = (y) \in \mathrm{IC}(x)$ *then the number of non-associate prime factors of* y *is* m.

(v) *The number of distinct supplements* $\Pi(A)$ *with* $A \in \mathrm{IC}(x)$ *and* $\mathrm{Sd}(\Pi(A)) = k - m$, *is* ${}^kC_m$. *The total number of distinct supplements in* $\Pi(R)$ *is* 2^k.

THEOREM 4.3. *Let* $A, B, S, S' \in \mathrm{IC}(x)$ *such that* $\Pi(S)$ *and* $\Pi(S')$ *are supplements in* $\Pi(R)$. *Then the following hold:*

(i) $\Pi(S) + \Pi(S')$ *is a supplement with*

$$\mathrm{Sd}(\Pi(S) + \Pi(S')) \ge \max\{\mathrm{Sd}(\Pi(S)), \mathrm{Sd}(\Pi(S'))\}.$$

(ii) $\Pi(S) \cap \Pi(S')$ *is a supplement with*

$$\mathrm{Sd}(\Pi(S) \cap \Pi(S')) \le \min\{\mathrm{Sd}(\Pi(S)), \mathrm{Sd}(\Pi(S'))\}.$$

(iii) *If* $\Pi(S)$, $\Pi(S')$ *are* S*-supplements for* $\Pi(A)$, $\Pi(B)$ *respectively then* $\Pi(S) + \Pi(S')$ *is an* S*-supplement for* $\Pi(A) + \Pi(B)$.

5. Acknowledgements

The author wishes to thank Professor D. Ramakotaiah, the Vice-Chancellor of Nagarjuna University and Professor R. Sridharan, T.I.F.R. for their encouragement. He wishes to express his thanks to Dr Amit Roy and Dr U. Balakrishnan of T.I.F.R., for their valuable discussions, and to the authorities of T.I.F.R., Bombay for providing him facilities to be there as a Visiting Fellow in May 1989 in which period the paper was written. Also he thanks the referee for valuable suggestions.

References

[1] P. Fleury, 'A note on dualizing Goldie dimension', *Canad. Math. Bull.* **17** (1974), 511–517.

[2] I. N. Herstein, *Topics in algebra*, 2nd edition (Vikas Publishing House, 1983).
[3] Bh. Satyanarayana, 'On modules with finite spanning dimension', *Proc. Japan Acad. Ser. A Math. Sci.* **61A** (1985), 23–25.
[4] ———, 'On modules with FSD and a property (P)', in: *Proc. Ramanujan Centennial International Conference, Annamalai Nagar, 15–18 December* (1987) pp. 137–140.
[5] ———, 'A note on E-direct and S-inverse systems', *Proc. Japan Acad. Ser. A Math. Sci.* **64A** (1988), 292–295.

Department of Mathematics
Nagarjuna University
Nagarjuna Nagar - 522 510
Andra Pradesh
India