LETTERS TO THE EDITOR

ON CONDITIONAL ORNSTEIN–UHLENBECK PROCESSES

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Abstract

It is well known that the law of a Brownian motion started from $x > 0$ and conditioned never to hit 0 is identical with the law of a three-dimensional Bessel process started from $x$. Here we show that a similar description is valid for all linear Ornstein–Uhlenbeck Brownian motions. Further, using the same techniques, it is seen that we may construct a non-stationary Ornstein–Uhlenbeck process from a stationary one.

EXCESSIVE TRANSFORMS

Let $X = \{X_t, t \geq 0\}$ be a linear, canonical Ornstein–Uhlenbeck process with the drift parameter $\gamma \in \mathbb{R}$, i.e. $X$ is a diffusion with the generator

$$A = \frac{1}{2}D^2 - \gamma xD,$$

where $D = d/dx$. Introduce $Y(t) = (X_1^2(t) + X_2^2(t) + X_3^2(t))^1$, where $X_i$, $i = 1, 2, 3$, are three independent copies of $X$. It is well known that $Y = \{Y_t, t \geq 0\}$ is a time-homogeneous diffusion and has the generator

$$g = \frac{1}{2}D^2 + \left(\frac{1}{x} - \gamma x\right)D, \quad x > 0,$$

(see e.g. [4]).

**Theorem** 1. For $\alpha > 0$ let $Y, Y_0 = \alpha$, be a diffusion with the generator $g$, and $\tilde{X}, \tilde{X}_0 = \alpha$, an Ornstein–Uhlenbeck process killed when it (eventually) hits 0.

(i) For $\gamma \geq 0$ let $\tilde{Y}$ be the process obtained from $Y$ by killing it exponentially with rate $\gamma$ and conditioning it to hit 0. Then $\tilde{X}$ and $\tilde{Y}$ are identical in law.

(ii) For $\gamma \leq 0$ let $\tilde{X}$ be the process obtained from $\tilde{X}$ by killing it exponentially with rate $-\gamma$ and conditioning it to drift out to $\infty$. Then $\tilde{X}$ and $\tilde{Y}$ are identical in law.

*Remark.* The case $\gamma = 0$, i.e. there is no killing, is included in both (i) and (ii) above, and was first, perhaps, treated by McKean (see [2], and also [1], [5]). He interpreted this kind of conditioning as an $h$-transform of a three-dimensional Bessel process in the case (i) and of a killed Brownian motion in the case (ii), and explained excursions of Brownian motion in these terms. The theorem above may be considered as a generaliza-

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tion to this result of McKeans. However, the proper generalization is simply that:

the Ornstein-Uhlenbeck process conditioned not to hit 0 is a diffusion with the generator $s^{-1} \mathcal{A}(s \cdot)$, where $s$ is the scale function for $X$.

**Proof.** As we remarked above, the conditioning described in the theorem should be interpreted as an excessive transform of the diffusion in question. We refer to [3], pp. 299–308 for details.

In the case $\gamma > 0$ we note that $h(x) = 1/x$ is the (essentially) unique decreasing and strictly positive solution to $\mathcal{A}u = \gamma u$. Therefore $\tilde{Y}$ is a diffusion with the transition semigroup

$$
\tilde{P}_t f = \frac{1}{h} e^{-\gamma t} P_t (hf),
$$

where $P$ is the semigroup of $Y$. Using the definition of infinitesimal generator it is seen that $\tilde{Y}$ has the generator

$$
\mathcal{A}^\gamma f = \frac{1}{h} (g - \gamma)(hf).
$$

On $C^2[0, \infty)$ we have $\mathcal{A}^\gamma f = \mathcal{A} f$, and the boundary condition is killing. This completes the proof of the case (i).

In the case $\gamma < 0$ we observe that $g(x) = x$ solves $\mathcal{A}u = -\gamma u$. The function $g$ is increasing and strictly positive on $(0, \infty)$ and satisfies the boundary condition of killing, i.e. $\lim_{x \to 0} g(x) = g(x) = 0$. Therefore $\tilde{X}$ is a diffusion with the generator

$$
\mathcal{A}^\gamma f = \frac{1}{g} (\mathcal{A} + \gamma)(gf).
$$

On $C^2[0, \infty)$ we have $\mathcal{A}^\gamma f = \mathcal{A}^\gamma f$, and the proof is complete.

**Theorem 2.** For $\gamma > 0$ let $X^+$ and $X^-$ be two Ornstein-Uhlenbeck processes with the drift parameters $\gamma$ and $-\gamma$, respectively. Denote with $\tilde{X}$ the process obtained from $X^+$ by killing it exponentially with rate $\gamma$ and conditioning it to drift out to $+\infty$ with probability $\frac{1}{2}$ and to $-\infty$ also with probability $\frac{1}{2}$. Then $\tilde{X}$ and $X^-$ are identical in law.

**Proof.** It is well known that the positive, strictly increasing and decreasing solutions $\phi^+$ and $\phi^-$, respectively, to the equation $\mathcal{A}u = \gamma u$ ($\gamma > 0$) are $\phi^+(x) = \phi^-(x) = \exp (\gamma x^2/2) D_{-1}(\sqrt{2} \gamma x)$, where

$$
D_v(x) = \frac{\exp (-x^2/4)}{\Gamma(-v)} \int_0^x t^{-v-1} \exp (-xt - t^2/2) \, dt
$$

is the parabolic cylinder function of order $v$. Hence the conditioning is realized with the function

$$
h(x) = \phi^+(x) + \phi^-(x) = C \cdot \exp (\gamma x^2),
$$

where $C$ is a constant, and the proof may now be concluded as above.

**Remark.** Denote by $\mathbb{P}_0^+$ and $\mathbb{P}_0^-$ the probability measures associated with $X^+$, $X^+_0 = 0$, and $X^-$, $X^-_0 = 0$, respectively. Of course, $\mathbb{P}_0^+$ and $\mathbb{P}_0^-$, when restricted to the $\sigma$-field $\mathcal{F}_t = \sigma(\omega_s : s \leq t)$, are equivalent. From the proof above it is seen that

$$
\frac{d\mathbb{P}_0^-}{d\mathbb{P}_0^+} \bigg|_{\mathcal{F}_t} = \exp (-\gamma \cdot t + \gamma \cdot \omega_t^2) \quad \mathbb{P}_0^+\text{-a.s.};
$$

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a result which is also easily deduced from the general (Cameron–Martin–Girsanov) expression for Radon–Nikodym derivative of two diffusion processes.

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References